

SUZUKI-TYPE FIXED POINT THEOREM IN S_N -METRIC SPACES: A STRONG B -METRIC APPROACH

Hoang Thanh Nguyen, Sy Thang Nguyen and
Hoang Anh Kiet Ong
(Da Nang, Vietnam)

Communicated by Bui Minh Phong
(Received 27 February 2026; accepted 19 September 2026)

Abstract. In this paper, we introduce the concept of S_N -metric spaces, which naturally generalizes metric and S -metric spaces. By constructing an induced strong b -metric $R(x, y) = D(x, \dots, x, y)$, we show that every S_N -metric space naturally induces a strong b -metric space with coefficient $q = N - 1$. Based on this structural connection, we first establish a Suzuki-type fixed point theorem in strong b -metric spaces and subsequently derive the corresponding result in S_N -metric spaces as a direct consequence. Non-trivial examples are also provided to illustrate our main result.

1. Introduction and preliminaries

It is known that extensions of Banach's contraction principle have evolved along two directions: one involving the generalization of the underlying space, and the other involving the generalization of the mapping itself. In 1922, Banach [1] proved his famous fixed point theorem on a complete metric space X , where a metric is a function $d : X \times X \rightarrow [0, \infty)$ satisfying positivity, symmetry, and the triangle inequality. Over the years, various generalizations of metric spaces have been introduced. In 1993, Czerwik [3] proposed the concept of a b -metric space by weakening the triangle inequality. Since then, b -metric spaces have been considered as a standard framework for studying fixed point problems [2, 10].

Key words and phrases: Suzuki, fixed point, strong b -metric spaces, S_N -metric spaces.
2010 Mathematics Subject Classification: 47H10, 54H25.
<https://doi.org/10.71352/ac.59.190926>

Another approach to generalizing metric spaces is to interpret the metric as a function measuring the distance among three or more points simultaneously. In 2006 Mustafa and Sims [7] introduced the notion of a G -metric defined by the function $G : X^3 \rightarrow [0, \infty)$ which satisfies a set of axioms that properly generalize the notion of distance. G -metric spaces have proven to be a fertile ground for the development of generalized fixed point theorems.

Subsequently, Sedghi et al. [8] constructed the concept of an S -metric space in 2012. The S -metric, defined as a function $S : X^3 \rightarrow [0, \infty)$, satisfies a distinct set of axioms inspired by but not identical to those of G -metrics. Furthermore, S -metric spaces provide a natural framework in which fixed point theorems have been extensively generalized.

More recently, Khan [4] extended the notion of G -metric to functions of arbitrarily many variables. Specifically, he originally introduced a function $K : X^n \rightarrow [0, \infty)$, ($n \geq 3$), which he denoted as a K -metric, and then further strengthened the symmetry and other axiomatic conditions to define what he calls a G_n -metric in 2021 [5]. Within this framework, Khan has established some fixed point theorems for self-mappings and showed how they extend classical results from standard metric spaces.

In this paper, we propose a new notion of a generalized metric, referred to as the S_N -metric, formulated as a function of N points. This metric serves as a natural extension of both the S -metric and the classical metric; in particular, it coincides with the classical metric when $N = 2$ and reduces to the S -metric when $N = 3$. In comparison with the G_n -metric introduced by Khan [4], the axiomatic structure of the S_N -metric is significantly simpler, yet sufficiently rich to establish concepts such as convergence, Cauchy sequences, and completeness. Moreover, this structure naturally induces an underlying strong b -metric space with parameter $q = N - 1$, providing a powerful framework to derive fixed point results.

In parallel with the generalizations of underlying spaces, fixed point results such as Banach's contraction principle [1] and the Suzuki-type theorem [9, 5] in metric spaces have been extended to these generalized frameworks. In this paper, based on the S_N -metric framework and its induced strong b -metric space with coefficient $q = N - 1$, we first extend the Suzuki-type fixed point theorem to the strong b -metric setting and subsequently derive its consequences in S_N -metric spaces. The classical Suzuki-type fixed point results in both standard metric spaces and S -metric spaces are obtained as particular cases of our main results.

Definition 1.1. Let X be a nonempty set and $N \geq 2$ be a natural number. A function $D : X^N \rightarrow [0, +\infty)$ is said to be an S_N -metric if for all $x_1, x_2, \dots, x_N, a \in X$ the following conditions hold:

- (1) $D(x_1, x_2, \dots, x_N) = 0$ if and only if $x_1 = x_2 = \dots = x_N$;

(2) (Generalized triangle inequality)

$$D(x_1, x_2, \dots, x_N) \leq D(x_1, \dots, x_1, a) + D(x_2, \dots, x_2, a) + \dots + D(x_N, \dots, x_N, a),$$

where each $D(\cdot)$ on the right side has N variables as described;

In this case, (X, D) is called an S_N -metric space.

Example 1.1. Let X be a set with at least N elements and $N \geq 2$. Define $D : X^N \rightarrow [0, \infty)$ by

$$D(x_1, x_2, \dots, x_N) = \begin{cases} 1 & \text{if there exist } i \neq j \text{ with } x_i \neq x_j, \\ 0 & \text{if } x_1 = x_2 = \dots = x_N. \end{cases}$$

It is easy to verify that (X, D) is an S_N -metric space.

Example 1.2. Let $X = \mathbb{R}$ and $N \geq 2$. Define a mapping $D : X^N \rightarrow [0, \infty)$ by

$$D(x_1, x_2, \dots, x_N) = \begin{cases} 0, & \text{if } x_1 = x_2 = \dots = x_N, \\ \max_{1 \leq i < j \leq N} |x_i - x_j|, & \text{otherwise.} \end{cases}$$

Then D satisfies all the axioms of an S_N -metric and (X, D) is an S_N -metric space.

Example 1.3. Let (X, d) be a metric space. Define

$$(1.1) \quad D(x_1, x_2, \dots, x_N) = d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{N-1}, x_N) + d(x_N, x_1)$$

for every $x_1, x_2, \dots, x_N \in X$. Then (X, D) is an S_N -metric space. Indeed, axiom (1) in Definition 1.1 is clear and we need to prove axiom (2). Fix $x_1, \dots, x_N, a \in X$. It is easy to observe that for each $i \in \{1, 2, \dots, N\}$,

$$D(x_i, x_i, \dots, x_i, a) = 2d(x_i, a).$$

Therefore

$$\begin{aligned} D(x_1, x_2, \dots, x_N) &= \sum_{k=1}^{N-1} d(x_k, x_{k+1}) + d(x_N, x_1) \leq \\ &\leq \sum_{k=1}^{N-1} (d(x_k, a) + d(a, x_{k+1})) + (d(x_N, a) + d(a, x_1)) = \\ &= \sum_{k=1}^N 2d(x_k, a) = \sum_{k=1}^N D(x_k, x_k, \dots, x_k, a). \end{aligned}$$

Thus (X, D) is an S_N -metric space.

Definition 1.2. Let (X, D) be an S_N -metric space. The function $R: X \times X \rightarrow [0, +\infty)$ defined by

$$(1.2) \quad R(x, y) = D(x, x, \dots, x, y)$$

is called the induced strong b -metric associated with D .

Proposition 1.4. Let (X, D) be an S_N -metric space. The induced strong b -metric associated with D satisfies the following properties for all $x, y, z \in X$:

- (i) $R(x, y) = 0$ if and only if $x = y$,
- (ii) $R(x, y) = R(y, x)$,
- (iii) $R(x, y) \leq R(x, z) + (N - 1)R(z, y)$.

Proof. By Definition 1.1, we establish the required properties as follows:

- (i) According to axiom (1) of Definition 1.1, $R(x, y) = D(x, x, \dots, x, y) = 0$ if and only if $x = x = \dots = x = y$, which is equivalent to $x = y$.
- (ii) Setting $(x_1, x_2, \dots, x_{N-1}, x_N) = (x, x, \dots, x, y)$ and $a = x$ in axiom (2) of Definition 1.1, we obtain

$$\begin{aligned} R(x, y) &= D(x, x, \dots, x, y) \leq D(x, \dots, x, x) + \dots + D(x, \dots, x, x) + \\ &\quad + D(y, \dots, y, x) = \\ &= 0 + \dots + 0 + D(y, \dots, y, x) = R(y, x). \end{aligned}$$

By interchanging the roles of x and y , we also have $R(y, x) \leq R(x, y)$. Hence, $R(x, y) = R(y, x)$ for all $x, y \in X$.

- (iii) For any $x, y, z \in X$, choosing $(x_1, x_2, \dots, x_{N-1}, x_N) = (y, y, \dots, y, x)$ and replacing a with z in axiom (2) of Definition 1.1 yields

$$\begin{aligned} R(y, x) &= D(y, \dots, y, x) \leq \sum_{i=2}^N D(y, \dots, y, z) + D(x, \dots, x, z) = \\ &= (N - 1)D(y, \dots, y, z) + D(x, \dots, x, z) = \\ &= R(x, z) + (N - 1)R(y, z). \end{aligned}$$

Using the symmetry property (ii), we arrive at

$$R(x, y) \leq R(x, z) + (N - 1)R(z, y).$$

This completes the proof. ■

Consequently, if (X, D) is an S_N -metric space and R is the induced strong b -metric associated with D then $(X, R, N - 1)$ is a strong b -metric space with coefficient $q = N - 1$, as presented in [6]. To ensure the clarity and coherence of this paper, we restate this definition below.

Definition 1.3. . Let X be a nonempty set and $q \geq 1$ be a real number. A function $d: X \times X \rightarrow [0, +\infty)$ is called a strong b -metric if for all $x, y, z \in X$, the following conditions are satisfied:

$$(sb1) \quad d(x, y) = 0 \iff x = y,$$

$$(sb2) \quad d(x, y) = d(y, x),$$

$$(sb3) \quad d(x, z) \leq d(x, y) + qd(y, z).$$

In this case, the triple (X, d, q) is called a strong b -metric space with coefficient q .

From Definition 1.3, the following remark is straightforward to verify.

Remark 1.1. Let (X, d, q) be a strong b -metric space with coefficient $q \geq 1$. Then

$$(1.3) \quad d(x, z) \leq d(x, y) + qd(y, z) \quad \text{for all } x, y, z \in X,$$

is equivalent to

$$(1.4) \quad d(x, z) \leq qd(x, y) + d(y, z) \quad \text{for all } x, y, z \in X.$$

Let (X, D) be an S_N -metric space and R be the induced strong b -metric associated with D . By virtue of the existing concepts and results in strong b -metric spaces [6], the subsequent definitions and propositions arise naturally or can be readily verified.

Definition 1.4. Let (X, D) be an S_N -metric space.

1. A sequence $\{x_n\}$ in X is said to converge to x with respect to the S_N -metric in (X, D) if and only if $R(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ and we denote this by $x_n \xrightarrow{S_N\text{-metric}} x$ or simply $\lim_{n \rightarrow \infty} x_n = x$.
2. A sequence $\{x_n\}$ is called a Cauchy sequence in X if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $R(x_m, x_n) < \varepsilon$.
3. The S_N -metric space (X, D) is said to be complete if every Cauchy sequence in X is convergent in X .

Proposition 1.5. Let $\{x_n\}$ be a Cauchy sequence in the S_N -metric space (X, D) . If a subsequence $\{x_{k_n}\}$ converges to x , then $\{x_n\}$ converges to x .

Proposition 1.6. In an S_N -metric space (X, D) , every convergent sequence is Cauchy.

Proposition 1.7. Let (X, D) be an S_N -metric space. If $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} R(x_n, y_n) = R(x, y)$.

Definition 1.5. Let (X, D) be an S_N -metric space and $T : X \rightarrow X$ be a mapping. Then T is said to be continuous if for every sequence $\{x_n\}$ in X with

$$\lim_{n \rightarrow \infty} x_n = x,$$

we have

$$\lim_{n \rightarrow \infty} Tx_n = Tx.$$

Definition 1.6. Let (X, D) be an S_N -metric space. A mapping $T : X \rightarrow X$ is called a contraction on the S_N -metric space if there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$(1.5) \quad R(Tx, Ty) \leq \alpha R(x, y).$$

Proposition 1.8. In an S_N -metric space (X, D) , if T is a contraction mapping then T is continuous.

2. Main results

In 2008, Suzuki generalized the Banach contraction principle for a broader class of contractions, which subsequently led to a novel characterization of completeness for metric spaces [9].

Theorem 2.1 ([9]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping. Define the function θ by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Suppose there exists $r \in [0, 1)$ such that for all $x, y \in X$,

$$\theta(r)d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq rd(x, y).$$

Then T has a unique fixed point $z \in X$. Furthermore, $\lim_{n \rightarrow \infty} T^n x = z$ for every $x \in X$.

In this section, we generalize this result to S_N -metric spaces. Our approach relies on first proving a Suzuki-type fixed point theorem within the framework of strong b -metric spaces with coefficient $q = N - 1$, from which the desired result for S_N -metric spaces naturally follows. In order to illustrate that our results are non-empty and truly non-trivial, we start by constructing the following example.

Example 2.2. Let $X = [0, 1]$ be equipped with the standard metric $d(x, y) = |x - y|$. We define the mapping $R : X \times X \rightarrow [0, \infty)$ by

$$R(x, y) = d(x, y) + 2[d(x, y)]^2 \quad \text{for all } x, y \in X.$$

Then (X, R) is a complete strong b -metric space with coefficient $q = 5$, but it is not a metric space.

Proof. First, we show that (X, R) is not a standard metric space by showing that the classical triangle inequality is violated. Indeed, choosing $x = 1$, $y = 0$, and $z = 1/2 \in X$, direct calculations yield

$$\begin{aligned} R(1, 0) &= 1 + 2(1)^2 = 3, \\ R(1, 1/2) &= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 = 1, \\ R(1/2, 0) &= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 = 1. \end{aligned}$$

Consequently, we have

$$R(1, 1/2) + R(1/2, 0) = 1 + 1 = 2 < 3 = R(1, 0).$$

Since $R(1, 0) > R(1, 1/2) + R(1/2, 0)$, the standard triangle inequality does not hold. Thus, (X, R) is not a metric space.

Next, we prove that (X, R) is a strong b -metric space with coefficient $q = 5$. By definition of R , $R(x, y) = 0 \iff x = y$ and the symmetry property $R(x, y) = R(y, x)$ are trivially satisfied for all $x, y \in X$. It remains to show that the strong b -metric inequality holds:

$$(2.1) \quad R(x, y) \leq R(x, z) + qR(z, y)$$

for all $x, y, z \in X$ with $q = 5$.

Using the standard triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$, we obtain

$$\begin{aligned} R(x, y) &= d(x, y) + 2[d(x, y)]^2 \leq \\ &\leq (d(x, z) + d(z, y)) + 2(d(x, z) + d(z, y))^2 = \\ &= d(x, z) + d(z, y) + 2[d(x, z)]^2 + 4d(x, z)d(z, y) + 2[d(z, y)]^2 = \\ &= R(x, z) + d(z, y) + 4d(x, z)d(z, y) + 2[d(z, y)]^2. \end{aligned}$$

Since $x, z \in [0, 1]$, we have $d(x, z) \leq 1$. Thus,

$$4d(x, z)d(z, y) \leq 4d(z, y).$$

Substituting this back into the inequality yields

$$\begin{aligned} R(x, y) &\leq R(x, z) + 5d(z, y) + 2[d(z, y)]^2 \leq \\ &\leq R(x, z) + 5d(z, y) + 10[d(z, y)]^2 = \\ &= R(x, z) + 5R(z, y). \end{aligned}$$

Therefore, the strong b -metric inequality (2.1) holds for all $x, y, z \in X$ with $q = 5$.

Finally, we show that (X, R) is complete. Observe that for all $x, y \in X$,

$$d(x, y) \leq R(x, y) \leq 3d(x, y).$$

This implies that the topology induced by the strong b -metric R is equivalent to the topology induced by the standard metric d . Since $X = [0, 1]$ is a closed subset of the complete metric space $(\mathbb{R}, d)$, the space (X, d) is complete, and consequently, (X, R) is also complete. ■

Our strategy for establishing the Suzuki-type fixed point theorem in S_N -metric spaces relies on first proving the result for strong b -metric spaces. To facilitate the upcoming proof, we recall two useful results mentioned in [6].

Lemma 2.1. *Let (X, d, q) be a strong b -metric space with coefficient $q \geq 1$ and $x_0, x_1, \dots, x_{n+1} \in X$ for all $n \in \mathbb{N} \cup \{0\}$. Then, for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$(2.2) \quad d(x_0, x_{n+1}) \leq q \sum_{k=0}^n d(x_k, x_{k+1}).$$

Lemma 2.2. *Let (X, d, q) be a strong b -metric space with coefficient $q \geq 1$. If $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$.*

We now proceed to prove the Suzuki-type fixed point theorem in strong b -metric spaces.

Theorem 2.3. *Let (X, d) be a complete strong b -metric space with coefficient $q \geq 1$, and $T : X \rightarrow X$ be a mapping. Define the function θ by*

$$\theta(r) = \begin{cases} \frac{1}{q}, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2q}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{(1+r)q}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Suppose that there exists $r \in [0, 1)$ such that for all $x, y \in X$,

$$(2.3) \quad \theta(r)d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq rd(x, y).$$

Then, T has a unique fixed point $z \in X$. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to z .

Proof. Since $\theta(r) \leq 1$, we have $\theta(r)d(x, Tx) \leq d(x, Tx)$ for all $x \in X$. Thus, assumption (2.3) implies

$$(2.4) \quad d(Tx, T^2x) \leq rd(x, Tx).$$

Choose any $u \in X$ and define a sequence $\{u_n\}$ in X by $u_0 = u$ and $u_n = T^n u$ for all $n \in \mathbb{N}$. By applying (2.4) inductively, it follows that

$$d(u_n, u_{n+1}) \leq rd(u_{n-1}, u_n) \leq \cdots \leq r^n d(u, Tu),$$

which yields

$$(2.5) \quad d(u_n, u_{n+1}) \leq r^n d(u, Tu), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

For all $n, p \in \mathbb{N}$, by (2.5) and Lemma 2.1, we have

$$\begin{aligned} d(u_n, u_{n+p+1}) &\leq q \sum_{i=0}^p d(u_{n+i}, u_{n+i+1}) \leq \\ &\leq q \sum_{i=0}^p r^{n+i} d(u, Tu) = \\ &= qd(u, Tu) r^n \sum_{i=0}^p r^i \leq \\ &\leq r^n \cdot \frac{qd(u, Tu)}{1-r}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} r^n = 0$, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $r^n < \frac{\varepsilon(1-r)}{qd(u, Tu)+1}$ for all $n \geq n_0$. Hence,

$$d(u_n, u_{n+p+1}) \leq r^n \cdot \frac{qd(u, Tu)}{1-r} < \varepsilon, \quad \forall n \geq n_0.$$

This demonstrates that $\{u_n\}$ is a Cauchy sequence. The completeness of (X, d) guarantees the existence of a point $z \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = z.$$

Now, we prove that

$$(2.6) \quad d(Tx, z) \leq rd(x, z), \quad \forall x \neq z.$$

Indeed, for any $x \neq z$, we have $d(z, x) > 0$. Since $\{u_n\}$ converges to z , Lemma 2.2 ensures that

$$0 < d(z, x) = \lim_{n \rightarrow \infty} d(u_n, x).$$

Let $a = \lim_{n \rightarrow \infty} d(u_n, x) = d(z, x) > 0$. This implies that $d(u_n, x) > \frac{a}{2}$ for sufficiently large n . Moreover, since $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that

$$\theta(r)d(u_{k_n}, Tu_{k_n}) \leq d(u_{k_n}, Tu_{k_n}) < \frac{a}{2} < d(u_{k_n}, x), \quad \forall n \in \mathbb{N}.$$

Utilizing assumption (2.3), we deduce that

$$d(Tu_{k_n}, Tx) \leq rd(u_{k_n}, x), \quad \forall n \in \mathbb{N}.$$

Note that $Tu_{k_n} = u_{k_n+1} \rightarrow z$ as $n \rightarrow \infty$. Taking the limit as $n \rightarrow \infty$ on both sides and applying Lemma 2.2, we obtain

$$d(z, Tx) = \lim_{n \rightarrow \infty} d(Tu_{k_n}, Tx) \leq r \lim_{n \rightarrow \infty} d(u_{k_n}, x) = rd(z, x),$$

thereby verifying (2.6).

Next, we show that

$$(2.7) \quad d(T^n z, z) \leq d(Tz, z), \quad \forall n \in \mathbb{N}.$$

For $n = 1$, this relation holds trivially. Suppose that (2.7) is valid for some $n = l \geq 1$. If $T^l z = z$, then $T^{l+1} z = Tz$, which directly yields $d(T^{l+1} z, z) = d(Tz, z)$. On the other hand, if $T^l z \neq z$, applying (2.6) gives

$$d(T^{l+1} z, z) \leq rd(T^l z, z) \leq rd(Tz, z) < d(Tz, z).$$

Consequently, assertion (2.7) is proved for all $n \in \mathbb{N}$ by induction.

We now analyze the following two cases.

Case I: $0 \leq r \leq 1/\sqrt{2}$.

In this case, we have $\theta(r) \leq \frac{1-r}{r^2q}$. We aim to show that

$$(2.8) \quad d(T^n z, Tz) \leq rd(z, Tz), \quad \forall n \in \mathbb{N}.$$

If $r = 0$, (2.5) implies that $d(T^{n+1}u, T^nu) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. It means that $\{T^nu\}$ is a constant sequence and $\{T^nu\}$ converges to z . Therefore (2.8) holds.

If $r > 0$, (2.8) holds clearly for $n = 1$, and (2.4) implies it holds for $n = 2$. Assume that (2.8) is satisfied for $n = l \geq 2$.

Applying the strong b -metric inequality together with the induction hypothesis, we have

$$d(z, Tz) \leq qd(z, T^l z) + d(T^l z, Tz) \leq qd(z, T^l z) + rd(z, Tz),$$

which leads to $(1-r)d(z, Tz) \leq qd(z, T^l z)$. From (2.5), we have

$$\begin{aligned} \theta(r)d(T^l z, T^{l+1} z) &\leq \frac{1-r}{r^2q}d(T^l z, T^{l+1} z) \leq \\ &\leq \frac{1-r}{r^2q}r^l d(z, Tz) = \\ &= \frac{1-r}{q}r^{l-2}d(z, Tz) \leq \\ &\leq \frac{1-r}{q}d(z, Tz) \leq \\ &\leq d(T^l z, z). \end{aligned}$$

Therefore, assumption (2.3) together with (2.7) implies that

$$d(T^{l+1}z, Tz) \leq rd(T^l z, z) \leq rd(Tz, z) = rd(z, Tz).$$

Thus, (2.8) is verified by induction for all $n \in \mathbb{N}$.

Supposing $Tz \neq z$, claim (2.8) implies $T^n z \neq z$ for all $n \in \mathbb{N}$. Therefore, from (2.6), we obtain

$$d(T^{n+1}z, z) \leq rd(T^n z, z) \leq r^2 d(T^{n-1}z, z) \leq \dots \leq r^n d(Tz, z).$$

Letting $n \rightarrow \infty$ yields $\lim_{n \rightarrow \infty} T^{n+1}z = z$. Hence, by virtue of (2.8) and Lemma 2.2, taking the limit as $n \rightarrow \infty$ gives

$$d(z, Tz) = \lim_{n \rightarrow \infty} d(T^{n+1}z, Tz) \leq rd(z, Tz) < d(z, Tz),$$

which leads to an obvious contradiction $d(z, Tz) < d(z, Tz)$. Thus, we must have $Tz = z$.

Case II: $1/\sqrt{2} \leq r < 1$.

In this case, we have $\theta(r) = \frac{1}{(1+r)^q}$. We claim that there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that

$$(2.9) \quad \theta(r)d(u_{k_n}, Tu_{k_n}) = \frac{1}{q(1+r)}d(u_{k_n}, Tu_{k_n}) \leq d(u_{k_n}, z), \quad \forall n \in \mathbb{N}.$$

Assume for the sake of contradiction that no such subsequence exists. This implies that there are at most finitely many $n \in \mathbb{N}$ satisfying

$$\frac{1}{q(1+r)}d(u_n, Tu_n) \leq d(u_n, z).$$

Hence, there exists an integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{1}{q(1+r)}d(u_n, Tu_n) > d(u_n, z).$$

Since $2n_0 \geq n_0$ and $2n_0 + 1 \geq n_0$, setting $n = n_0$ and applying the strong b -metric inequality along with (2.5), we obtain

$$\begin{aligned} d(u_{2n_0}, u_{2n_0+1}) &\leq d(u_{2n_0}, z) + qd(u_{2n_0+1}, z) < \\ &< \frac{1}{q(1+r)}d(u_{2n_0}, Tu_{2n_0}) + \frac{q}{q(1+r)}d(u_{2n_0+1}, Tu_{2n_0+1}) = \\ &= \frac{1}{q(1+r)}d(u_{2n_0}, u_{2n_0+1}) + \frac{1}{1+r}d(u_{2n_0+1}, u_{2n_0+2}) \leq \\ &\leq \frac{1}{1+r}d(u_{2n_0}, u_{2n_0+1}) + \frac{r}{1+r}d(u_{2n_0}, u_{2n_0+1}) = \\ &= d(u_{2n_0}, u_{2n_0+1}), \end{aligned}$$

which yields the strict inequality $d(u_{2n_0}, u_{2n_0+1}) < d(u_{2n_0}, u_{2n_0+1})$, a contradiction. This completes the proof of (2.9).

Consequently, assumption (2.3) implies that

$$d(Tu_{k_n}, Tz) \leq rd(u_{k_n}, z).$$

It follows from the strong b -metric inequality that

$$d(z, Tz) \leq d(z, Tu_{k_n}) + qd(Tu_{k_n}, Tz) \leq d(z, u_{k_n+1}) + qrd(u_{k_n}, z).$$

Taking the limit as $n \rightarrow \infty$ on both sides, since

$$\lim_{n \rightarrow \infty} u_{k_n} = z \quad \text{and} \quad \lim_{n \rightarrow \infty} u_{k_n+1} = z,$$

we obtain $d(z, Tz) \leq 0$, which forces $z = Tz$.

To conclude the proof, we show the uniqueness of the fixed point z . Suppose there exists another fixed point $z' \neq z$ such that $Tz' = z'$. Then, (2.6) implies $d(z', z) = d(Tz', z) \leq rd(z', z)$, which leads to $r \geq 1$. This contradicts the assumption that $r < 1$. The proof is now complete. ■

As a consequence of the Suzuki-type fixed point theorem for strong b -metric spaces, we obtain the main result for S_N -metric spaces.

Theorem 2.4. *Let (X, D) be a complete S_N -metric space and $T: X \rightarrow X$ be a mapping. Define the function θ_N by*

$$\theta_N(r) = \begin{cases} \frac{1}{N-1}, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2(N-1)}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{(1+r)(N-1)}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Suppose there exists $r \in [0, 1)$ such that for all $x, y \in X$

$$(2.10) \quad \theta_N(r)R(x, Tx) \leq R(x, y) \implies R(Tx, Ty) \leq rR(x, y).$$

Then T has a unique fixed point $z \in X$, and moreover, $\lim_{n \rightarrow \infty} T^n x = z$ for every $x \in X$.

To show that the hypotheses of Theorem 2.4 are non-vacuous, we present the following examples.

Example 2.5. Let (X, R) be the complete strong b -metric space with coefficient $q = 5$ presented in Example 2.2. We define the mapping $D: X^6 \rightarrow [0, \infty)$ by

$$D(x_1, x_2, \dots, x_6) = \frac{1}{5} \sum_{k=1}^5 R(x_k, x_6) \quad \text{for all } x_1, x_2, \dots, x_6 \in X.$$

Then (X, D) is an S_6 -metric space in the sense of Definition 1.1, and satisfies $D(x, \dots, x, y) = R(x, y)$ for all $x, y \in X$.

Proof. First, we verify axiom (1) of Definition 1.1. Since $R(u, v) \geq 0$ for all $u, v \in X$, we have $D(x_1, \dots, x_6) = 0$ if and only if $R(x_k, x_6) = 0$ for all $k \in \{1, 2, 3, 4, 5\}$. Since (X, R) is a strong b -metric space, this is equivalent to $x_1 = x_2 = x_3 = x_4 = x_5 = x_6$.

Next, we check the relation $D(x, \dots, x, y) = R(x, y)$. For any $x, y \in X$, we have

$$D(x, \dots, x, y) = \frac{1}{5} \sum_{k=1}^5 R(x, y) = \frac{1}{5} \cdot 5R(x, y) = R(x, y).$$

Finally, we prove the generalized triangle inequality. For any $x_1, \dots, x_6, a \in X$, using the strong b -metric inequality and Remark 1.1 we have $R(x_k, x_6) \leq 5R(x_k, a) + R(a, x_6)$ for $k = 1, \dots, 5$. Therefore,

$$\begin{aligned} D(x_1, x_2, \dots, x_6) &= \frac{1}{5} \sum_{k=1}^5 R(x_k, x_6) \leq \\ &\leq \frac{1}{5} \sum_{k=1}^5 (5R(x_k, a) + R(a, x_6)) = \\ &= \frac{1}{5} \sum_{k=1}^5 5R(x_k, a) + \frac{1}{5} \cdot 5R(a, x_6) = \\ &= \sum_{k=1}^5 R(x_k, a) + R(a, x_6) = \\ &= \sum_{i=1}^6 D(x_i, \dots, x_i, a). \end{aligned}$$

Hence, axiom (2) of Definition 1.1 holds. Therefore, (X, D) is an S_6 -metric space.

Example 2.6. Let (X, R) be the complete strong b -metric space with coefficient $q = 5$ defined in Example 2.2, where $R(x, y) = |x - y| + 2|x - y|^2$. Let (X, D) be the S_6 -metric space presented in Example 2.5 with $D(x_1, \dots, x_6) = \frac{1}{5} \sum_{k=1}^5 R(x_k, x_6)$, which satisfies $D(x, \dots, x, y) = R(x, y)$.

Define the mapping $T : X \rightarrow X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, 1), \\ \frac{1}{10}, & \text{if } x = 1. \end{cases}$$

Then T is not a Banach contraction on (X, R) in the sense of Definition 1.6, but it satisfies (2.10) of Theorem 2.4 with $N = 6$ and $r = \frac{3}{5} \in [0, 1)$.

Proof. First, we show that T is not a Banach contraction. Consider the sequence $x_n = 1 - \frac{1}{n} \in [0, 1)$ for $n \geq 2$, which converges to $y = 1$ in (X, R) . As

$n \rightarrow \infty$, we have $R(x_n, 1) \rightarrow 0$. However, $T(x_n) = 0$ for all $n \geq 2$, so

$$R(T(x_n), T(1)) = R\left(0, \frac{1}{10}\right) = \frac{1}{10} + 2\left(\frac{1}{10}\right)^2 = \frac{3}{25} > 0.$$

This means that $\lim_{n \rightarrow \infty} R(T(x_n), T(1)) = \frac{3}{25} \neq 0$, implying that T is not continuous at $x = 1$. Consequently, T cannot be a Banach contraction on (X, R) .

Next, we verify that T satisfies (2.10) with $N = 6$ and $r = \frac{3}{5}$. Since $\frac{3}{5} < \frac{\sqrt{5}-1}{2}$, we have $\theta_6\left(\frac{3}{5}\right) = \frac{1}{N-1} = \frac{1}{5}$.

Suppose $x, y \in X$ satisfy $\frac{1}{5}R(x, T(x)) \leq R(x, y)$. We examine all possible cases:

Case 1: $x, y \in [0, 1)$. Here, $T(x) = 0$ and $T(y) = 0$. Thus, $R(T(x), T(y)) = R(0, 0) = 0 \leq \frac{3}{5}R(x, y)$ holds trivially.

Case 2: $x \in [0, 1)$ and $y = 1$. Here, $T(x) = 0$ and $T(1) = \frac{1}{10}$, so

$$R(T(x), T(1)) = R\left(0, \frac{1}{10}\right) = \frac{3}{25}.$$

If $x = 0$, then $R(0, T(0)) = 0$ and $R(0, 1) = 3$. Hence, $R(T(0), T(1)) = \frac{3}{25} \leq \frac{3}{5} \cdot 3 = \frac{9}{5} = \frac{3}{5}R(0, 1)$.

If $x \in (0, 1)$ and $y = 1$. The condition $\frac{1}{5}R(x, T(x)) \leq R(x, 1)$ is equivalent to $\frac{1}{5}(x + 2x^2) \leq 3 - 5x + 2x^2$, which yields $8x^2 - 26x + 15 \geq 0$. Since $x \in (0, 1)$, this holds if and only if $x \in (0, \frac{3}{4}]$. For all $x \in (0, \frac{3}{4}]$, we have $R(x, 1) \geq R(\frac{3}{4}, 1) = \frac{3}{8}$. Consequently,

$$R(T(x), T(1)) = \frac{3}{25} < \frac{9}{40} = \frac{3}{5} \cdot \frac{3}{8} \leq \frac{3}{5}R(x, 1).$$

Case 3: $x = 1$ and $y \in [0, 1)$. We have $T(1) = \frac{1}{10}$, so $R(1, T(1)) = R(1, \frac{1}{10}) = \frac{9}{10} + 2\left(\frac{9}{10}\right)^2 = \frac{63}{25}$. The condition $\frac{1}{5}R(1, T(1)) \leq R(1, y)$ implies

$$R(1, y) \geq \frac{1}{5} \cdot \frac{63}{25} = \frac{63}{125}.$$

Since $T(1) = \frac{1}{10}$ and $T(y) = 0$, we have

$$R(T(1), T(y)) = R\left(\frac{1}{10}, 0\right) = \frac{3}{25}.$$

Therefore,

$$R(T(1), T(y)) = \frac{3}{25} < \frac{3}{5} \cdot \frac{63}{125} \leq \frac{3}{5}R(1, y).$$

Case 4: $x = 1$ and $y = 1$. Then $R(T(1), T(1)) = R(\frac{1}{10}, \frac{1}{10}) = 0 \leq \frac{3}{5}R(1, 1)$.

Thus, condition (2.10) holds for all $x, y \in X$ with $r = \frac{3}{5}$. ■

References

- [1] **Banach, S.**, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, **3** (1) (1922), 133–181.
<https://doi.org/10.4064/fm-3-1-133-181>
- [2] **Cobzaş, S. and S. Czerwik**, The completion of generalized b -metric spaces and fixed points, *Fixed Point Theory*, **21** (1) (2020), 133–150.
<https://doi.org/10.24193/fpt-ro.2020.1.10>
- [3] **Czerwik, S.**, Contraction mappings in b -metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1) (1993), 5–11.
- [4] **Khan, K.A.**, Generalized n -metric spaces and fixed point theorems, arXiv preprint (2018)
<https://arxiv.org/abs/1809.08997>
<https://doi.org/10.48550/arXiv.1809.08997>
- [5] **Khan, K.A.**, Banach and Suzuki-type fixed point theorems in Generalized n -metric spaces with an application, arXiv preprint (2021)
<https://arxiv.org/abs/2108.05194>
<https://doi.org/10.48550/arXiv.2108.05194>
- [6] **Kirk, W.A. and N. Shahzad**, *Fixed Point Theory in Distance Spaces*, Springer, Cham, 2014.
<https://doi.org/10.1007/978-3-319-10927-5>
- [7] **Mustafa, Z. and B. Sims**, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis*, **7** (2) (2006), 289–297.
- [8] **Sedghi, S., N. Shobe and A. Aliouche**, A generalization of fixed point theorems in S -metric spaces, *Matematički Vesnik*, **64** (249) (2012), 258–266.
- [9] **Suzuki, T.**, A generalized Banach contraction principle that characterizes metric completeness, *Proceedings of the American Mathematical Society*, **136** (5) (2008), 1861–1869.
<https://doi.org/10.1090/S0002-9939-07-09055-7>
- [10] **Suzuki, T.**, Basic inequality on a b -metric space and its applications, *Journal of Inequalities and Applications*, **2017** (2017), Article 256.
<https://doi.org/10.1186/s13660-017-1528-3>

Hoang Thanh Nguyen

<https://orcid.org/0009-0008-0464-9129>

University of Science and Education – The University of Da Nang

Faculty of Mathematics and Informatics

Da Nang City

Vietnam

nhthanh@ued.udn.vn

Sy Thang Nguyen

University of Science and Education – The University of Da Nang

Faculty of Mathematics and Informatics

Da Nang City

Vietnam

`nsthang3004@gmail.com`

Hoang Anh Kiet Ong

<https://orcid.org/0009-0008-6148-2396>

University of Science and Education – The University of Da Nang

Faculty of Mathematics and Informatics

Da Nang City

Vietnam

`anhkiet.maths@gmail.com`