

DIRECTED GRAPHS DEFINED BY PRIME DIVISORS OF POLYNOMIAL VALUES

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Communicated by Gábor Farkas

(Received 16 May 2026; accepted 4 September 2026)

Abstract. Let $\mathcal{L}$ be the class of all nonconstant polynomials $f(x) = Ax^2 + Bx + C \in \mathbb{Z}[x]$, which are not of the forms Ax^2, Bx . We consider the directed graph $\mathcal{G}_f$ whose vertex set is $\mathcal{P}$ and $p \rightarrow q$ iff $q \mid f(p), p \neq q$. Let $\mathbb{L}_f(p)$ be the set of simple directed paths starting at p and $\ell_f(p) = \sup\{|L| : L \in \mathbb{L}_f(p)\}$, where $|L|$ is the number of vertices in L . We prove $\sup_{p \in \mathcal{P}} \ell_f(p) = \infty$. We study linear polynomials $f(x) = Bx + C \in \mathbb{Z}[x], BC \neq 0$, in connection with Cunningham chains. For $f(x) = x^2 + 1$, we explicitly construct a simple directed path with 130 vertices starting from 2, hence $\ell_{x^2+1}(2) \geq 130$. It remains open whether $\ell_{x^2+1}(2)$ is finite or infinite. For these topics, we refer to works of C. Frayer [1], A.J. Pollington [6], C. Pomerance [7] and M. van Rossum-Wijsmuller [8].

1. Introduction

Let $\mathcal{P}, \mathbb{N}, \mathbb{Z}$ denote the sets of primes, positive integers and integers, respectively. Let (n, m) denote the greatest common divisor of n and m , where $n, m \in \mathbb{N}$.

Graphs defined by arithmetic relations have attracted considerable attention in recent years. Classical examples include divisor graphs, gcd graphs, prime graphs and other arithmetic graphs, where vertices are integers or primes and edges are determined by divisibility or related arithmetic properties. Such graphs provide a useful connection between graph theory and number theory and have applications in several areas of mathematics. For these topics, we refer to works of C. Frayer [1], A. J. Pollington [6] and C. Pomerance [7].

Key words and phrases: Primes, directed graph, Cunningham chain.

2010 Mathematics Subject Classification: Primary 11N64, 11K65, Secondary 11A25.

<https://doi.org/10.71352/ac.59.040926>

Throughout this paper, let $\mathcal{L}$ be the class of all nonconstant polynomials $f(x) = Ax^2 + Bx + C \in \mathbb{Z}[x]$, which are not of the forms Ax^2, Bx . We note that if $B, C \in \mathbb{Z}, BC \neq 0$, then $f(x) = Bx + C \in \mathcal{L}$.

Let $f(x) \in \mathcal{L}$ and $\Delta = B^2 - 4AC$. We define a directed graph $\mathcal{G}_f$ with vertex set $\mathcal{P}$ and edge $p \rightarrow q$ iff $q \mid f(p), p \neq q$. Let

$$\mathbb{B}_f(p) = \{q \in \mathcal{P} : p \neq q \text{ and } q \mid f(p)\},$$

$$\mathbb{E}_f(p) = \{q \in \mathcal{P} : p \rightarrow \cdots \rightarrow q\}$$

and

$$\mathbb{F}_f = \{q \in \mathcal{P} : \exists x \in \mathbb{Z}, f(x) \equiv 0 \pmod{q}\}.$$

Note $q \in \mathbb{F}_f \iff q \mid f(m)$ for some $m \in \mathbb{Z}$ and $\mathbb{E}_f(p)$ denotes the set of those primes q which can be reached from p by walking on $\mathcal{G}_f$.

Let $\mathbb{L}_f(p)$ be the set of simple directed paths starting at p and $\ell_f(p) = \sup\{|L| : L \in \mathbb{L}_f(p)\}$, where $|L|$ is the number of vertices in L .

In this paper we prove the following result.

Theorem 1.1. *Let $f(x) \in \mathcal{L}$. Then*

$$\sup_{p \in \mathcal{P}} \ell_f(p) = \infty.$$

The proof of the main theorem relies essentially on arithmetic properties specific to quadratic polynomials, in particular on the solvability of quadratic congruences and the quadratic reciprocity law. It is natural to ask whether analogous results remain valid for arbitrary polynomials, which appears to be an interesting topic for future research.

2. Fundamental lemma

Lemma 2.1. *Let $f(x) = Ax^2 + Bx + C \in \mathcal{L}$ be a polynomial with $T := |AC\Delta| \neq 0$. If*

$$p \in \mathcal{P}, (p, 2T) = 1 \text{ and } p \in \mathbb{F}_f,$$

then there are infinitely many primes P such that

$$P \in \mathcal{P}, (P, 2T) = 1, P \in \mathbb{F}_f \text{ and } P \rightarrow p.$$

Proof. Let

$$(2.1) \quad \Delta = \epsilon \cdot 2^\alpha \cdot \Delta_1, \quad (\Delta_1, 2) = 1,$$

where $\epsilon \in \{-1, 1\}$, $\alpha \in \mathbb{N}_0$ and

$$\Delta_1 = \begin{cases} 1 & \text{if } s = 0, \\ \prod_{i=1}^s \pi_i^{\alpha_i} & \text{if } s \geq 1, \end{cases}$$

with $s \geq 0$, $\pi_1, \dots, \pi_s$ distinct odd primes and $\alpha_i \in \mathbb{N}$.

Assume that the conditions of Lemma 2.1 are satisfied, i.e. there is a prime $p \in \mathcal{P}$ such that $(p, 2T) = 1$ and $p \in \mathbb{F}_f$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$(2.2) \quad f(n_0) \equiv 0 \pmod{p}.$$

If $p \mid n_0$, then $0 \equiv f(n_0) = An_0^2 + Bn_0 + C \equiv C \pmod{p}$, which contradicts $(p, 2|AC\Delta|) = (p, 2T) = 1$. Therefore

$$(2.3) \quad (n_0, p) = 1.$$

It follows from the condition $(p, 2T) = 1$ that

$$(2.4) \quad \exists t_0 \in \mathbb{N}, q_0 = pt_0 + n_0 \in \mathbb{N} \text{ and } q_0 \equiv 1 \pmod{8T}.$$

Then it follows from (2.2)–(2.4) that

$$(2.5) \quad \begin{aligned} f(8Tpt + q_0) &\equiv f(q_0) \equiv f(pt_0 + n_0) \equiv \\ &\equiv f(n_0) \equiv 0 \pmod{p} \text{ for every } t \in \mathbb{N} \end{aligned}$$

and

$$(2.6) \quad (8Tp, q_0) = (p, q_0) = (p, pt_0 + n_0) = (p, n_0) = 1.$$

By Dirichlet's theorem on primes in arithmetic progressions, there exist infinitely many integers $t \in \mathbb{N}$ such that $8Tpt + q_0 \in \mathcal{P}$. For those such primes $P = 8Tpt + q_0$, we have

$$P \equiv q_0 \equiv 1 \pmod{8T},$$

consequently

$$(2.7) \quad P \equiv 1 \pmod{8} \text{ and } P \equiv 1 \pmod{\pi_i} \quad (i = 1, \dots, s).$$

Using the properties of the Legendre symbol together with the law of quadratic reciprocity, we obtain from (2.7) that

$$\left(\frac{\epsilon}{P}\right) = 1, \quad \left(\frac{2}{P}\right) = 1,$$

and

$$\left(\frac{\pi_i}{P}\right) = (-1)^{\frac{\pi_i-1}{2} \cdot \frac{P-1}{2}} \left(\frac{P}{\pi_i}\right) = \left(\frac{1}{\pi_i}\right) = 1, \quad (i = 1, 2, \dots, s) \text{ if } \Delta_1 > 1.$$

Consequently

$$(2.8) \quad \left(\frac{\Delta}{P}\right) = 1,$$

because

$$\left(\frac{\Delta}{P}\right) = \begin{cases} \left(\frac{\varepsilon}{P}\right) \left(\frac{2}{P}\right)^\alpha = 1, & \text{if } \Delta_1 = 1, \\ \left(\frac{\varepsilon}{P}\right) \left(\frac{2}{P}\right)^\alpha \left(\frac{\pi_1}{P}\right)^{\alpha_1} \cdots \left(\frac{\pi_s}{P}\right)^{\alpha_s} = 1, & \text{if } \Delta_1 > 1. \end{cases}$$

We now complete the proof of Lemma 2.1.

For the primes $P = 8Tpt + q_0$, we have from (2.2), (2.4) and (2.5) that $P > p$, $f(P) \equiv f(n_0) \equiv 0 \pmod{p}$, that is $p|f(P)$, $P \rightarrow p$.

Finally, we shall prove that $P \in \mathbb{F}_f$.

Since

$$(P, 2T) = (8Tpt + q_0, 2T) = (q_0, 2T) = 1,$$

we have

$$f(x) \equiv 0 \pmod{P} \iff (2Ax + B)^2 \equiv \Delta \pmod{P}.$$

Since $T = |AC\Delta|$ and $(P, 2T) = 1$, we have $(P, 2A) = 1$. By using (2.8), there exists $u \in \mathbb{N}$ such that $u^2 \equiv \Delta \pmod{P}$. Since $2Ax + B \equiv u \pmod{P}$ is solvable, the $f(x) \equiv 0 \pmod{P}$ congruence has a solution, which proves that $P \in \mathbb{F}_f$.

This completes the proof of Lemma 2.1. ■

Remark 1. The assumption $(p, \Delta) = 1$ cannot be omitted. Let $f(x) = x^2 + x - 1$, $\Delta = 5$, $p = 5$, $n_0 = 2$, $f(2) = 5 \equiv 0 \pmod{5}$. Here, $f(x) \equiv 0 \pmod{P}$ solvable $\iff \left(\frac{5}{P}\right) = 1 \iff P \equiv \pm 1 \pmod{5}$. But $P \rightarrow 5$ implies that $5|f(P)$ and $P \equiv 2 \pmod{5}$. This means that $P \notin \mathbb{F}_f$.

3. The proof of Theorem 1.1

3.1. Case $f \in \mathcal{L}$ and $T = |AC\Delta| \neq 0$

Suppose, for contradiction, that $\sup_{p \in \mathcal{P}} \ell_f(p) < \infty$ and put

$$s = \sup_{p \in \mathcal{P}} \ell_f(p) \in \mathbb{N}.$$

By Schur's theorem ([9]), every non-constant polynomial in $\mathbb{Z}[x]$ has infinitely many prime divisors. Thus, there exists a prime $\pi > 2T$ such that $\pi \in \mathbb{F}_f$. In particular, $(\pi, 2T) = 1$, so Lemma 2.1 applies.

Let

$$\pi = \pi_1 \rightarrow \pi_2 \rightarrow \cdots \rightarrow \pi_t \quad (t \in \mathbb{N})$$

be a simple directed path starting at π . By Lemma 2.1, there exist infinitely many primes P such that

$$P \in \mathbb{F}_f, \quad P \rightarrow \pi.$$

Choose such a prime P with

$$P > \max\{\pi = \pi_1, \dots, \pi_t\}.$$

Then

$$P \rightarrow \pi = \pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_t$$

is a simple directed path, because P is distinct from all vertices in the path beginning at π .

Now repeat the same argument inductively. Starting from $P = P_1$, we can construct a prime P_2 with

$$P_2 \rightarrow P_1, \quad P_2 > P_1 = P > \max\{\pi = \pi_1, \dots, \pi_t\}.$$

and so on. After s steps we obtain primes

$$P_s \rightarrow P_{s-1} \rightarrow \dots \rightarrow P_1 \rightarrow \pi = \pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_t$$

with

$$P_s > P_{s-1} > \dots > P_1 = P$$

This yields a simple directed path of length at least $s + t$, contradicting the definition of s . Therefore $\sup_{p \in \mathcal{P}} \ell_f(p) = \infty$.

This completes the proof of Theorem 1.1 for the case $T \neq 0$.

3.2. The case $f \in \mathcal{L}$ and $T = |AC\Delta| = 0$

We consider the following three cases:

- (a) $A = 0$,
- (b) $A \neq 0, C = 0$,
- (c) $AC \neq 0, \Delta = 0$.

(a) We assume that $A = 0$, therefore $f(x) = Bx + C, BC \neq 0$. We shall prove that $\sup_{p \in \mathcal{P}} \ell_f(p) = \infty$.

Suppose, for contradiction, that every simple directed path in $\mathcal{G}_f$ is bounded in length. Let

$$s := \sup_{p \in \mathcal{P}} \ell_f(p) \quad \text{and} \quad p = p_1 \rightarrow p_2 \rightarrow \dots \rightarrow p_s, \quad (p_i \in \mathcal{P}).$$

Since $p_1 \neq p_2$, $p_1 \rightarrow p_2$, we have $f(p_1) = Bp_1 + C \equiv 0 \pmod{p_2}$. Hence, by Dirichlet's theorem, there exists $t_1 \in \mathbb{N}$ such that

$$Q_1 = p_2 t_1 + p_1 \in \mathcal{P}, Q_1 > \max(|BC|, p_1, \dots, p_s)$$

and

$$f(Q_1) = BQ_1 + C = p_2 \left(Bt_1 + \frac{Bp_1 + C}{p_2} \right) \equiv 0 \pmod{p_2}.$$

Then

$$Q_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_s.$$

Since $Q_1 \in \mathcal{P}$, $Q_1 > |B|$, we have $(Q_1, B) = 1$. Thus, there is $n_2 \in \mathbb{N}$ such that $f(n_2) = Bn_2 + C \equiv 0 \pmod{Q_1}$. Since $Q_1 > |BC| \geq |C|$, it is easy to check that $(Q_1, n_2) = 1$, consequently, we infer from Dirichlet's theorem that $Q_2 = Q_1 t_2 + n_2 \in \mathcal{P}$, $Q_2 > \max(Q_1, p_1, \dots, p_s)$ for some $t_2 \in \mathbb{N}$. Hence we have

$$f(Q_2) = B(Q_1 t_2 + n_2) + C = Q_1 \left(Bt_2 + \frac{Bn_2 + C}{Q_1} \right) \equiv 0 \pmod{Q_1}$$

and so

$$Q_2 \rightarrow Q_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_s.$$

Hence, $\ell_f(Q_2) \geq s + 1$, which is impossible because $s = \sup_{p \in \mathcal{P}} \ell_f(p)$.

(b) We assume that $f(x) \in \mathcal{L}$ and $C = 0$, therefore $f(x) = Ax^2 + Bx$, $AB \neq 0$. We note that in this case $p \rightarrow q$ means that $p \neq q$, $q|Ap^2 + Bp = p(Ap + B)$, which implies $q|Ap + B$. In the proof of case (a), replacing B by A and C by B, we see that the proof of (b) is completely analogous to that of case (a). Therefore, we omit the details.

(c) Now assume that $f \in \mathcal{L}$, $AC \neq 0$, $\Delta = B^2 - 4AC = 0$. Then $B^2 = 4AC \neq 0$, i.e. $B \neq 0$. It follows from $4Af(x) = (2Ax + B)^2$ that $p \rightarrow q$ implies $p \neq q$, $q|4Af(p) = (2Ap + B)^2$. Here $p \rightarrow q$ means that $q|2Ap + B$, where $2AB \neq 0$. In the proof of case (a), replacing B by 2A and C by B, the proof of (c) follows exactly the same argument as in case (a), so we do not repeat it here.

Theorem 1.1 is therefore proved. ■

This leads naturally to the following conjectures.

Conjecture 3.1. *There exists a $M_f \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq M_f$, there exists $p \in \mathcal{P}$ such that $\ell_f(p) = n$.*

Conjecture 3.2. *If $p \in \mathcal{P}$ with $\ell_f(p) = n$, then there exists $P \in \mathcal{P}$ such that $p \neq P$, $p|f(P)$ and $\ell_f(\pi) \leq n$ for every prime divisor $\pi|f(P)$.*

Theorem 3.3. *Conjecture 3.2 implies Conjecture 3.1.*

Proof. Let $p \in \mathcal{P}$ with $\ell_f(p) = n$. Assuming Conjecture 3.2, there exists $P \in \mathcal{P}, P \neq p$ such that

$$\begin{cases} p \mid f(P) \\ \pi \mid f(P), \pi \in \mathcal{P} \implies \ell_f(\pi) \leq n. \end{cases}$$

Since $P \neq p$, we have $P \rightarrow p$, and hence

$$\ell_f(P) \geq n + 1.$$

On the other hand, every vertex π adjacent from P satisfies $\pi \mid f(P), \pi \neq P$, and therefore $\ell_f(\pi) \leq n$. Thus every simple directed path starting at P has at most $n + 1$ vertices. Consequently, $\ell_f(P) = n + 1$. Iterating this argument proves Conjecture 3.1.

The proof of Theorem 3.3 therefore is completed. ■

Problem 3.4. *For which polynomials $f(x) \in \mathbb{Z}[x]$ and primes $p \in \mathcal{P}$ is the graph $\mathbb{E}_f(p)$ finite?*

Problem 3.5. *For which polynomials $f(x) \in \mathbb{Z}[x]$ and primes $p \in \mathcal{P}$ is $\ell_f(p)$ unbounded?*

4. Linear polynomials $f(x) = Bx + C$

Let $f(x) = Bx + C \in \mathbb{Z}[x]$ with $B \geq 1$, $2B + C \geq 1$ and $(B, C) = 1$. We recall the directed graph $\mathcal{G}_f$ by $V(\mathcal{G}_f) = \mathcal{P}$ and

$$p \rightarrow q \iff q \mid Bp + C, p \neq q.$$

For $p \in \mathcal{P}$, let $\mathbb{E}_f(p)$ be the set of primes q reachable from p by a directed path in $\mathcal{G}_f$, and let $\mathcal{K}_f$ be the set of primes lying on a directed cycle. Let $b^*(B)$ be the smallest prime not dividing B ($b^*(1) = 2$).

We recall that for $f(x) = x + C$ this is the classical Kátai graph [3, 5]. The finiteness of $\mathcal{K}_f$ was proved in [3, 5]. In [2] we conjectured:

Conjecture 4.1. *Let $f(x) = Bx + C$, $B \geq 2$, $2B + C \geq 1$, $(B, C) = 1$. Then*

- (1) $\mathbb{E}_f(p)$ is finite for every $p \in \mathcal{P}$,
- (2) $\mathcal{K}_f$ is finite.

For $p \in \mathcal{P}$ denote by $\mathbb{K}_f(p)$ the set of increasing directed paths

$$L : p = p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_s, \quad |C| < p_1 < \cdots < p_s$$

in $\mathcal{G}_f$ and put $k_f(p) = \sup\{|L| : L \in \mathbb{K}_f(p)\}$ with $\sup \emptyset = 0$.

Lemma 4.1. *Let $p_1 \in \mathcal{P}$, $p_1 > \max(B, |C|)$ and*

$$p_1 < p_2 < \cdots < p_s, \quad Bp_i + C = p_{i+1}v_i, \quad v_i \in \mathbb{N}.$$

Then $1 \leq v_i \leq B$ and $(v_i, B) = 1$. In particular, if $v_i < b^(B)$ then $v_i = 1$. Moreover, if $v_i = 1$ for all i , i.e. $p_{i+1} = Bp_i + C$ for all $i \in \{1, \dots, s-1\}$ and $(p_1, B) = 1$, then $s \leq p_1$.*

Proof. Since $p_{i+1} > p_i > |C|$,

$$v_i = \frac{Bp_i + C}{p_{i+1}} < \frac{Bp_i + |C|}{p_i} < B + 1,$$

consequently $v_i \leq B$. Since $(B, C) = 1$, $(B, Bp_i + C) = 1$, hence $(v_i, B) = 1$. Let $\mathcal{A}_B = \{1 \leq v \leq B : (v, B) = 1\}$. Then $1 \in \mathcal{A}_B$ and $v_i \in \mathcal{A}_B$ for every $i = 1, \dots, s$.

If $v_i < b^*(B)$ and $v_i \neq 1$, then there exists a prime factor $\pi \mid v_i$, $\pi < b^*(B)$. It follows from the definition of $b^*(B)$ that $\pi \mid B$, so $(v_i, B) \geq \pi > 1$, which is contradiction to $v_i \in \mathcal{A}_B$. Thus, the condition $v_i < b^*(B)$ implies $v_i = 1$.

Assume now $v_i = 1$, i.e. $p_{i+1} = Bp_i + C$ for every $i \in \{1, \dots, s\}$. By induction,

$$(4.1) \quad p_m = B^{m-1}p_1 + C(1 + B + \cdots + B^{m-2}), \quad m \geq 2.$$

Suppose $s > p_1$. Then p_{p_1}, p_{p_1+1} exist.

If $B \equiv 1 \pmod{p_1}$, (4.1) gives $p_{p_1+1} \equiv p_1 + Cp_1 \equiv 0 \pmod{p_1}$, hence $p_{p_1+1} = p_1$, contradiction to strict increase.

If $B \not\equiv 1 \pmod{p_1}$, $B - 1$ is invertible mod p_1 and by Fermat $B^{p_1-1} \equiv 1 \pmod{p_1}$. Hence

$$\frac{B^{p_1-1} - 1}{B - 1} = 1 + \cdots + B^{p_1-2} \equiv 0 \pmod{p_1}$$

and from (4.1), $p_{p_1} \equiv 0 \pmod{p_1}$, i.e. $p_{p_1} = p_1$, again contradiction. Thus $s \leq p_1$. ■

Theorem 4.2. *Let $f(x) = Bx + C$ as above and $U = \prod_{q \leq B} q$ ($U = 1$ if $B = 1$). Let*

$$\max(B, |C|) < p_1 < p_2 < \cdots < p_s$$

be an increasing directed path in $\mathcal{G}_f$. Then $v_i \in \mathcal{A}_B$ and in particular $v_i \leq B$. If moreover $(Bp_i + C, U) = 1$ for $1 \leq i < s$, then $v_i = 1$, $p_{i+1} = Bp_i + C$ and $s \leq p_1$.

Proof. Write $Bp_i + C = p_{i+1}v_i$. By Lemma 4.1, $v_i \in \mathcal{A}_B$. If $(Bp_i + C, U) = 1$ then $(v_i, U) = 1$. If $v_i \neq 1$, its least prime factor $> B$, so $v_i \geq B + 1$, contradicting $v_i \leq B$. Hence $v_i = 1$. Lemma 4.1 gives $s \leq p_1$. ■

Theorem 4.3. *Let $B \in \{1, 2, 3\}$, $2B + C \geq 1$, $(B, C) = 1$ and C even if $B = 3$. Then for every $p > \max(B, |C|)$*

$$k_f(p) \leq p$$

and $\mathbb{E}_f(p)$ is finite.

Proof. If $q > \max(B, |C|)$ and $q < q', q \rightarrow q'$ then writing $Bq + C = q'v$ we have $v \leq B$, $(v, B) = 1$. Put $\mathcal{A}_B = \{1 \leq v \leq B : (v, B) = 1\}$, $b^* = b^*(B)$. Then $\mathcal{A}_B \cap [1, b^* - 1] = \{1\}$. For $B = 1, 2, 3$ with C even if $B = 3$, $(B, C) = 1$ gives

$$(4.2) \quad q < q' \implies q' = Bq + C \text{ prime.}$$

Hence an increasing path satisfies $p_{i+1} = Bp_i + C$ and by Lemma 4.1

$$(4.3) \quad s \leq p_1,$$

so $k_f(p) \leq p$.

Put $M := b^* + (b^* - 1)B + |C|$. If $q > M$ and $Bq + C$ is composite, $Bq + C = q'v$, $v \geq b^*$, so

$$(4.4) \quad q' \leq \frac{Bq + |C|}{b^*} < q,$$

because $(b^* - B)q > |C|$ for $B = 1, 2$ and $q > |C|$ for $B = 3$. Thus

$$(4.5) \quad q > M, Bq + C \text{ composite} \implies q' < q.$$

Let $L : p = p_0 \rightarrow \dots$ and let q_0 be the first vertex $> M$ (if none, $L \subset [2, M]$). Then $r \leq M$, $r \rightarrow q_0$ must be increasing, so by (4.2) $q_0 = Br + C$ and

$$(4.6) \quad q_0 \leq BM + |C|.$$

Let $q_0 < q_1 < \dots < q_t$ be the maximal increasing block from q_0 . By (4.2) and Lemma 4.1,

$$(4.7) \quad t \leq q_0,$$

$$(4.8) \quad q_t = B^t q_0 + C \frac{B^t - 1}{B - 1} \leq B^{q_0} q_0 + |C| B^{q_0}.$$

If $Bq_t + C$ were prime, the block would extend to $t + 1 > q_0$, contradicting (4.7). Hence $Bq_t + C$ is composite and $q_t \rightarrow q_{t+1}$ is decreasing, $q_{t+1} < q_t$ by (4.4).

Thus every excursion above M starts from $\leq BM + |C|$, goes up at most q_0 steps inside $[2, B^{q_0} q_0 + |C| B^{q_0}]$ and goes down. $\mathbb{E}_f(p)$ lies in a finite interval depending only on p, B, C , hence finite. ■

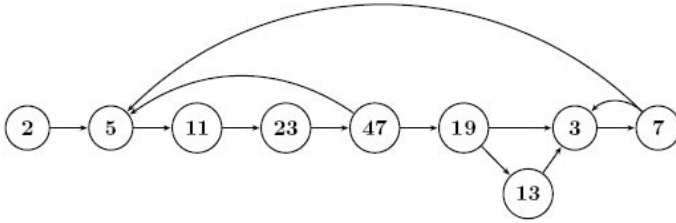
4.1. The cases $f(x) = 2x \pm 1$

a. The case $f(x) = 2x + 1$.

In this case, we have

$$\mathbb{B}_f(p) = \{q \in \mathcal{P} : q \mid 2p + 1\}$$

and the directed graph $\mathcal{G}_f(2)$ is the following



We note that $\mathbb{B}_f(47) = \{5, 19\}$, $\mathbb{B}_f(19) = \{3, 13\}$ and $\mathbb{B}_f(7) = \{3, 5\}$. It follows from the above $\mathcal{G}_f(2)$ graph that

$$2 \rightarrow 5 \rightarrow 11 \rightarrow 23 \rightarrow 47 \rightarrow 19 \rightarrow 13 \rightarrow 3 \rightarrow 7$$

is a longest simple directed path. Hence $\ell_f(2) = 9$.

All computations were performed using Maple. An exhaustive computation for all primes ($p < 10^6$) gives

$$\ell_f(p) \leq 27 \text{ for every } p \in \mathcal{P}, p < 10^6$$

with equality for $p = 779111$.

We also verified computationally that for each integer n , $8 \leq n \leq 27$, the least prime π_n satisfying $\ell_f(\pi_n) = n$ is given in Table 1.

n	8	9	10	11	12	13	14	15	16	17	18	19
π_n	3	2	59	29	41	61	53	79	179	89	311	1223

n	20	21	22	23	24	25	26	27
π_n	1889	2833	3821	23371	18149	63521	222323	779111

Table 1. The smallest prime π_n such that $\ell_f(\pi_n) = n$.

Consequently, Conjecture 3.1 has been verified computationally for $n \in \mathbb{N}$, $8 \leq n \leq 27$.

b. The case $f(x) = 2x - 1$.

$$\mathbb{B}_f(p) = \{q \in \mathcal{P} : q \mid 2p - 1\}.$$

If $|\mathbb{B}_f(p)| = 1$ and $2p - 1$ is prime, then

$$q = 2p - 1 > p$$

and the edge is increasing. Otherwise the edge goes to a proper prime divisor

$$q \leq (2p - 1)/3 < p \quad \text{for } p > 3.$$

Hence every directed walk consists of blocks of increasing prime steps (Cunningham chains) separated by decreasing steps.

Recall that a Cunningham chain of the first kind is a sequence

$$p_{i+1} = 2p_i + 1 \quad \text{prime,}$$

of the second kind

$$p_{i+1} = 2p_i - 1 \quad \text{prime.}$$

The length of an increasing directed path starting at p is precisely the length of a Cunningham chain of the second kind starting at p .

5. The case $f(x) = x^2 + 1$

In the case $f(x) = x^2 + 1$, the graph $\mathcal{G}_f(p)$ is rather complicated. For example, we verified computationally that $\ell_f(2) \geq 130$. It is unknown whether it is finite or infinite.

This leads to the following natural questions:

Problem 5.1. *Is $\ell_f(2)$ finite or infinite?*

Problem 5.2. *Does there exist a prime $p \in \mathcal{P}$ such that $\ell_f(p)$ is infinite?*

The following sequence of 130 distinct primes satisfying $p_{k+1} \mid p_k^2 + 1$:

2, 5, 13, 17, 29, 421, 401, 37, 137, 1877, 661, 218461, 89, 233, 61, 1861, 1229, 773, 59753, 733, 4133, 1708169, 112224666637, 4241, 8993041, 84069424561, 19777, 39112973, 154981573, 1406849, 3877, 1503113, 225934869077, 142601, 10167522601, 2293, 8089, 32715961, 102677, 9281, 113, 1277, 313, 97, 941, 34057, 449, 100801, 769, 17393, 853, 193, 149, 653, 42641, 909127441, 2477, 21157, 373, 13913, 19357157, 99709, 1697, 4721, 11143921, 2801, 109, 457, 4177, 1744733, 3637, 45613, 241, 257, 1321, 1637, 1549, 1009, 39157, 39877, 159017513, 4817, 4289, 9197761, 21661, 2635949, 69389, 1093, 23893, 11417509, 660375841, 1889, 1069, 571381, 433, 18749, 175762501, 202493, 5821, 997, 99401, 4940279401, 71597, 512613041, 54001, 1458054001, 177217, 43021733, 209736673, 294093097, 69233, 337, 277, 7673, 203017, 881, 388081, 2068037, 627041, 11564129873, 869461, 58976629, 19895297, 10695024121, 7717, 11173, 734329, 19993, 275669, 173.

For every k , $p_{k+1} \mid p_k^2 + 1$, for example in the last row above:

$$\begin{aligned} 19895297 &\mid 58976629^2 + 1 \\ 10695024121 &\mid 19895297^2 + 1 \\ 7717 &\mid 10695024121^2 + 1. \end{aligned}$$

To extend the particular path beyond 130 vertices, one needs a prime divisor of $p_k^2 + 1$ that has not previously occurred in the path. The construction may require prime divisors of extremely large numbers, making further extension computationally difficult.

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