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REMARKS ON A RESULT OF HARDY

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Dedicated to the memory of Professor Zoltán Daróczy

Abstract. We apply results on *equivalent power series* (see K.-H. Indlekofer [5]) to functions with given modulus of continuity and to functions holomorphic in $\{z : |z| \leq 1\} \setminus \{1\}$, and derive Hardy's theorem on the existence of power series which converge uniformly on the unit circle but not absolutely.

1. Introduction

There are series which on the unit circle converge uniformly but not absolutely.

This was proved in 1913 by G. Hardy in [3]. In 1916 E. Landau gave a simplified version of Hardy's proof (see E. Landau - D.Gaier [7]), which can be summarized as follows (see [7], §14).

Let $D := \{z \in \mathbb{C} : \ |z| < 1\}$ and $\overline{D} := \{z \in \mathbb{C} : \ |z| \leq 1\}$ be the open and closed unit disc of $\mathbb{C}$, repectively, and $H(D) := \{f : D \to \mathbb{C},\ f \textit{ holomorphic}\}$ be the algebra of functions holomorphic in D. Let $(i = \sqrt{-1})$

$$g(z) = (1-z)^{-i} = \sum_{n=0}^{\infty} (-1)^n \binom{i}{n} z^n = \sum_{n=0}^{\infty} b_n z^n.$$

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Then $g \in H(D)$, bounded in D and

$$|nb_n| = n\Big|\frac{i(i+1)\cdot\ldots\cdot(i+n-1)}{1\cdot\ldots\cdot(n-1)}\cdot\frac{1}{n}\Big| =$$
$$= \sqrt{(1+\frac{1}{1^2})\cdot\ldots\cdot(1+\frac{1}{(n-1)^2})} \to \sqrt{\prod_{r=1}^{\infty}(1+\frac{1}{r^2})}.$$

i.e.

$$b_n \asymp \frac{1}{n}. \tag{1.1}$$

Further, putting

$$B_m(\varphi) := \sum_{n=0}^{m} b_n e^{in\varphi} \quad \textit{for} \ \ m \geq 1, \quad \varphi \in \mathbb{R},$$

we conclude for $-1 < r < 1$

$$\sum_{n=0}^{m} b_n e^{in\varphi} - g(re^{i\varphi}) = \sum_{n=0}^{m} b_n e^{in\varphi}(1-r^n) - \sum_{n=m+1}^{\infty} b_n r^n e^{in\varphi}$$

and, by (1.1)

$$\Big|\sum_{n=0}^{m} b_n e^{in\varphi} - g(re^{i\varphi})\Big| \leq (1-r)\sum_{n=1}^{m} n|b_n| + \sum_{n=m+1}^{\infty}|b_n|r^n \leq \frac{1}{m}\sum_{n=1}^{m} c + \frac{c}{m}\sum_{n=m+1}^{\infty} r^n$$

if $r = 1 - \frac{1}{m}$. Since $g((1-\frac{1}{m})e^{i\varphi})$ is bounded we have

$$|B_m(\varphi)| \leq c \ \ \textit{independent of } m \textit{ and } \varphi. \tag{1.2}$$

Now, if

$$a_n := \frac{b_n}{\log n}, \quad n \geq 2,$$

then

$$f(z) = \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

is holomorphic for $|z| < 1$,

$$\sum_{n=2}^{\infty} |a_n| = \infty \tag{1.4}$$

and

$$\sum_{n=2}^{\infty} a_n e^{in\varphi} \ \ \textit{is uniformly convergent} \tag{1.5}$$

(which implies that $f(z)$ is continuous in $\overline{D}$).

For the proof of (1.5) we estimate, if $v \geq u \geq 2$ are integers,

$$
\begin{aligned}
|\sum_{n=u}^{v} a_n e^{in\varphi}| &= \Big|\sum_{n=u}^{v} \frac{B_n(\varphi)-B_{n-1}(\varphi)}{\log n}\Big| = \\
(1.6) \qquad &= \Big|\sum_{n=u}^{v} B_n(\varphi)(\frac{1}{\log n} - \frac{1}{\log(n+1)}) - \frac{B_{n-1}(\varphi)}{\log u} + \frac{B_v(\varphi)}{\log(v+1)}\Big| < \\
&< c\sum_{n=u}^{v}(\frac{1}{\log n} - \frac{1}{\log(n+1)} + \frac{c}{\log u} + \frac{c}{\log(v+1)}) = \\
&= \frac{2c}{\log u}.
\end{aligned}
$$

This ends the construction of Landau.

In this paper we shall prove the existence of such functions f

(i) with a given modulus of continuity

and

(ii) which are holomorphic in $\mathbb{C} \setminus K$,

where $1 \in K$ and $K \setminus \{1\} \subset \{z : |z-1| < \varepsilon\} \cap (\mathbb{C} \setminus \overline{D})$.

Here the modulus of continuity $\omega(f;h)$ for functions $f \in H(D)$, which are continuous in $\overline{D}$, is defined by

$$
(1.7) \qquad \omega(f;h) := \sup_{|\varphi|\leq\pi} \sup_{|t|\leq h} |f(e^{i(\varphi+t)}) - f(e^{i\varphi})|.
$$

If f is given by (1.3) then (see (1.6))

$$
f(e^{it}) - f(e^{i(t+h)}) = \sum_{n\leq u} \frac{b_n}{\log n} e^{itn}(1-e^{ihn}) + O(\frac{1}{\log u})
$$

and

$$
\Big|\sum_{n\leq u} \frac{b_n}{\log n} e^{itn}(1-e^{ihn})\Big| \ll \sum_{n\leq u} \frac{|b_n|}{\log n} hn = O\left(h\sum_{n\leq u}\frac{1}{\log n}\right) = O\left(h\frac{u}{\log u}\right).
$$

Choosing $u = \frac{1}{h}$ gives $\omega(f;h) = O\left(\frac{1}{\log 1/h}\right)$.

2. Functions with a given modulus of continuity

If $\zeta_0 \in D, \zeta_0 \neq 0$ then

$$\Phi_{\zeta_0}(z) := \frac{1-\overline{\zeta_0}}{1-\zeta_0} \cdot \frac{z-\zeta_0}{1-\overline{\zeta_0}z}$$

defines a conformal and bijective mapping Φ_{ζ_0} of D onto itself. In 1973 [4] we introduced the space

$$\begin{aligned}\mathcal{A}_{\alpha,c} := \Big\{ f \in H(D) : f(w) = \sum_{n=0}^{\infty} a_n w^n, \sum_n |a_n| < \infty, \omega(f;h) = \\ = O_f\left(\frac{h^{1/2}}{\log \frac{1}{h} \alpha(c \log \frac{1}{h})}\right) \quad \mathit{for}\ h \to 0^+ \Big\},\end{aligned} \tag{2.1}$$

where $\alpha : [x_0, \infty] \to (0, \infty)$ and $c > 0$.

We showed ([4], Satz 1)

Proposition 1. *Let c_1, c_1', x_0 be positive numbers and let $\alpha : [x_0, \infty) \to (0, \infty)$ be monotonically increasing such that*

$$\sum_{n \geq x_0} \frac{1}{n\alpha(n)} = \infty \tag{2.2}$$

and

$$\alpha(t(n+1)) \leq c'\alpha(tn) \quad \textit{for all real } t > 1, n \in \mathbb{N}, n \geq x_0.$$

Then there exist $f \in \mathcal{A}_{\alpha,c}$ such that $f_1(z) = f(\Phi_{\zeta_0}(z)) = \sum b_n z^n$ diverges absolutely for $|z| = 1$.

Since $\Phi_{\zeta_0}(z)$ is holomorphic on the boundary of D, the modulus of continuity of $f_1 = f \circ \Phi_{\zeta_0}$ can be estimated by

$$\omega(f_1;h) = O(\omega(f;h)).$$

L. Alpár [1] showed that if $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is absolutely convergent for $|z| = 1$, then $f_1(z) = f(\Phi_{\zeta_0}(z)) = \sum_{n=0}^{\infty} b_n z^n$ is uniformly convergent for $z = e^{i\varphi}$, $\varphi \in [0, 2\pi]$.

Thus we have

Theorem 1. *There exist functions $f \in \mathcal{A}_{\alpha,c}$ satisfying (2.2) such that $f_1(z) = = f(\Phi_{\varphi_0}(z)) = \sum_{n=0}^{\infty} b_n z^n$ converges uniformly but not absolutely for $|z| = 1$.*

Corollary 1. *There exist functions $f_1 \in H(D), f_1(z) = \sum_{n=0}^{\infty} b_n z^n$, continuous on $\overline{D}$ such that*

(i) $\omega(f_1;h) = O\left(\frac{h^{1/2}}{\log\frac{1}{h}\log\log\frac{1}{h}\log\log\log\frac{1}{h}}\right)$ *for* $h \to 0^+$,

(ii) $\sum_{n=0}^{\infty} |b_n| = \infty$,

(iii) $\sum_{n=0}^{\infty} b_n e^{in\varphi}$ *is uniformly convergent for* $\varphi \in [0, 2\pi]$.

Remark 1. The result is best possible in the following sense (cf. [4] Satz 2): If $f \in \mathcal{A}_{\alpha,c}$ such that

$$\sum_{n \geq x_0} \frac{1}{n\alpha(n)} < \infty$$

then $f_1(z) = f(\Phi_{\zeta_0}(z)) = \sum_{n=0}^{\infty} b_n z^n$ is absolutely convergent ($|z| = 1$) for every $\zeta_0 \in D$.

3. Functions holomorphic in $\overline{D} \setminus \{1\}$

Define $u : \mathbb{C} \to \mathbb{C}$ by

$$u(z) = z^2 - (\frac{z-1}{2})^4. \tag{3.1}$$

Then $u(1) = 1$ and $|u(z)| < 1$ for $z \in \overline{D} \setminus \{1\}$(see Figure 1). Especially we have

$$|u(e^{i\varphi})| = 1 - \sin^4\left(\frac{\varphi}{2}\right) \quad (\varphi \in [0, 2\pi]).$$

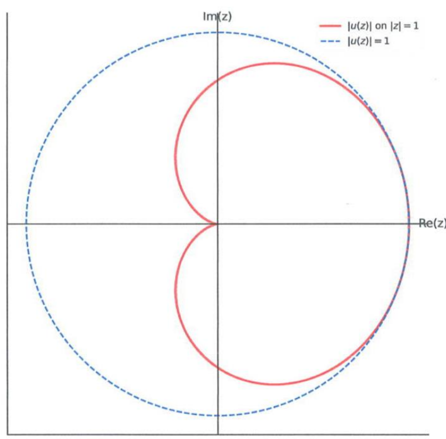

Figure 1. Modulus of $u(z) = z^2 - ((z-1)/2)^4$ on $|z| = 1$

Then K.-H. Indlekofer and R. Trautner [6] proved

Proposition 2.

(i) *If* $g \in H(D), g(z) = \sum_{n=0}^{\infty} a_n z^n$ *is absolutely convergent for* $|z| = 1$, *then* $g(u(z)) = \sum_{n=0}^{\infty} b_n z^n$ *is absolutely convergent for* $|z| = 1$.

(ii) *There exist* $g \in H(D)$ *such that* $g(z) = \sum_{n=0}^{\infty} a_n z^n$ *is absolutely convergent for* $|z| = 1$ *and* $g(u(\Phi_{\varphi_0}(z))) = \sum_{n=0}^{\infty} b_n z^n$ *is absolutely divergent for* $|z| = 1$.

The function $g(u(z))$ in (i) and (ii) is holomorphic in $\overline{D} \setminus \{1\}$ (see Figure 2). Choose g as in (ii) and put $f = g \circ u \circ \Phi_{\zeta_0}$. Then $f(z) = \sum_{n=0}^{\infty} b_n z^n$ is uniformly convergent and $\sum_{n=0}^{\infty} |b_n| = \infty$.

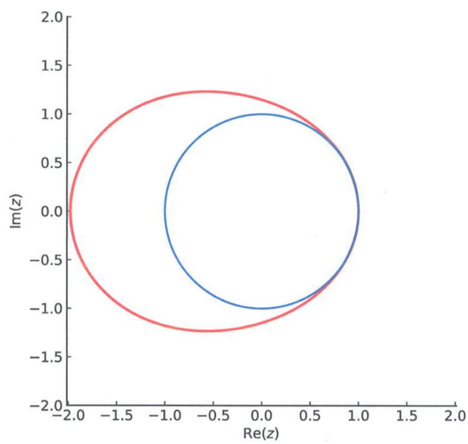

Figure 2. $|u(z)| = 1$ for $u(z) = z^2 - ((z-1)/2)^4$

Let G be the maximal region where f is holomorphic, and put $K = \mathbb{C} \setminus G$. Define

$$K_1 := K \cap \{z \in \mathbb{C} : |1 - z| \leq \varepsilon\} \tag{3.2}$$

where $0 < \varepsilon < \min\{1, \frac{1}{2\zeta_0}\}$. Let $\{z_1, z_2\} = \{z : |u(z)| = 1\} \cap \{z : |z - 1| = \varepsilon\}$. Then $|z_1| = |z_2| = 1 + \varepsilon'$ with $0 < \varepsilon' < \varepsilon$. Define

$$K_2 := K \cap \{z : |z| \geq 1 + \varepsilon'\}, \tag{3.3}$$

so that by (3.2) and (3.3) $K = K_1 \cup K_2$. Then by Théorème A of N. Aronszajn in [2]

$$f = f_1 + f_2$$

where f_1, f_2 are holomorphic in $G_1 = \mathbb{C} \setminus K_1$ and $G_2 = \mathbb{C} \setminus K_2$, respectively. Since the function $f_2(z)$ is holomorphic for $|z| < 1 + \varepsilon'$ it converges absolutely for $|z| = 1$, and for f_1 the following holds.

Theorem 2. *There exists $f_1(z)$ holomorphic for $z \in \mathbb{C} \setminus K_1$ such that $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ converges uniformly but not absolutely for $|z| = 1$.*

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