EXPONENTIAL SUMS AND ARITHMETIC FUNCTIONS RUNNING ON SUBINTERVALS

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Dedicated to Professor Zsolt Páles on the occasion of his 70^{th} *anniversary*

Abstract. Let $\mathcal{M}_1$ stand for the set of all complex valued multiplicative functions satisfying $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Some fifty years ago, Hedi Daboussi proved that, given any irrational number α, we have $\sup_{f\in\mathcal{M}_1} \sum_{n\leq x} f(n)e^{2\pi i\alpha n} = o(x)$ as $x \to \infty$. We show that the same result holds if f satisfies the additional condition $|f(n)| = 1$ for all $n \in \mathbb{N}$ and if we replace the exponential expression $e^{2\pi i\alpha n}$ by $h(u(n))$, where h is the characteristic function of a set made of a collection of subintervals of $[0, 1)$ and $u(n)$ is an arithmetic function with specific properties.

1. Introduction

Let $\mathcal{M}_1$ stand for the set of all complex valued multiplicative functions such that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Some fifty years ago, Daboussi (see Daboussi and Delange [1]) proved that given any irrational number α,

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \frac{1}{x} \sum_{n\leq x} f(n)e(\alpha n) = 0, \tag{1.1}$$

where $e(r) := \exp\{2\pi ir\}$.

While the classic proof of (1.1) is based on the large sieve inequality, the second author [3] gave a simple proof using essentially the Turán-Kubilius inequality. In [2], he also proved the following.

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Theorem A. *Let $u : \mathbb{N} \to \mathbb{R}$ be such that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e\big(u(p_1 n) - u(p_2 n)\big) = 0 \tag{1.2}$$

holds for every pair of primes $p_1 \ne p_2$. Then,

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \frac{1}{x} \left| \sum_{n\le x} f(n) e\big(u(n)\big) \right| = 0.$$

From here on, we let $P(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n$ be a polynomial with real coefficients with the property that at least one among $a_k, \dots, a_1$ is an irrational number.

Remark 1. Setting $u(n) = P(n)$, the conditions of Theorem A are satisfied and therefore the conclusion of Theorem A holds for this particular choice of $u(n)$.

Here, letting $\mathcal{M}_1^*$ be the set of those functions $f \in \mathcal{M}_1$ satisfying the additional condition $|f(n)| = 1$ for all $n \in \mathbb{N}$, we show that the Daboussi estimate (1.1) holds if $f \in \mathcal{M}_1^*$ and if we replace the exponential expression $e^{2\pi i \alpha n}$ by $h(u(n))$, where h is the characteristic function of a set made of a collection of subintervals of $[0,1)$ and $u(n)$ is an arithmetic function with specific properties.

2. Setting the table

Given an integer $k \ge 2$, consider the family of real intervals $I_j := [\alpha_j, \beta_j]$, $j = 1, \dots, k$, where $0 \le \alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1} < 1$ for each integer $j \in \{1, \dots, k-1\}$ and then consider the set

$$E := \bigcup_{j=1}^{k} I_j \subseteq [0,1)$$

and the function

$$h(x) := \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \in [0,1) \setminus E, \end{cases}$$

which we extend to the set of all non-negative real numbers by setting $h(x+n) := h(x)$ for $x \in E$ and $n \in \mathbb{N}$.

Given an arbitrarily small but fixed $\delta > 0$, we introduce the two functions

$$h_1(x) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} h(x+u)\,du,$$

$$h_2(x) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} h_1(x+u)\,du.$$

We then have

$$h_2(x) = \sum_{j=-\infty}^{\infty} c_j\, e\big(jx\big)$$

for some real numbers c_j with $\sum_{j=-\infty}^{\infty} |c_j| < \infty$. With the interval $R(u) :=$ $:= (u-\delta, u+\delta)$, set

$$S := \left(\bigcup_{j=1}^{k} R(\alpha_j)\right) \bigcup \left(\bigcup_{j=1}^{k} R(\beta_j)\right).$$

Let $K \in \mathbb{N}$ be such that

$$\sum_{|j|\geq K} |c_j| < \delta. \tag{2.1}$$

Since

$$c_0 = \int_0^1 h_2(u)\,du,$$

and letting $\lambda(A)$ stand for the measure of the set A, one can easily see that

$$|c_0 - \lambda(E)| \leq 2\,\delta. \tag{2.2}$$

Further define the function

$$H_\delta(x) := \lambda(E) + \sum_{1\leq |j| \leq K} c_j\, e\big(jx\big). \tag{2.3}$$

It is easily seen that

$$\lambda(S) \leq 8\,\delta\,k. \tag{2.4}$$

Letting $P(n)$ be as in Section 1 and letting E and h be as in Section 2, we define the set $\mathcal{T}$ as the set of those positive integers n for which $h(n) = 1$, that is, those n for which $\{P(n)\} \in E$, where here $\{y\}$ stands for the fractional part of y. Given $f \in \mathcal{M}_1^*$, we introduce the two summation functions

$$S^{(f)}(x) := \sum_{n \le x} f(n) \quad \text{and} \quad S_{\mathcal{T}}^{(f)}(x) := \sum_{n \le x} f(n) h(P(n)).$$

Finally, let

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

stand for the normal distribution function.

3. Main results

With the notation introduced in Section 2, we have the following results.

Theorem 1. *We have*

$$\lim_{x \to \infty} \sup_{f \in \mathcal{M}_1^*} \frac{1}{x} \left| S_{\mathcal{T}}^{(f)}(x) - \lambda(E) S^{(f)}(x) \right| = 0.$$

Theorem 2. *Let $u : \mathbb{N} \to \mathbb{R}$ be such that*

(a) *the sequence $u(n)$ is uniformly distributed modulo 1,*

(b) *for every pair of primes $p_1 \ne p_2$ and each $\ell \in \mathbb{N}$, we have*

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} e\Big(\ell\big(u(p_1 n) - u(p_2 n)\big) \Big) = 0.$$

Let E and h be as in Section 2 and let $\mathcal{T}_1$ be the set of those positive integers n for which $h(u(n)) = 1$. Moreover, given $f \in \mathcal{M}_1^*$, set

$$S_{\mathcal{T}_1}^{(f)}(x) := \sum_{n \le x} f(n) h(u(n))$$

and

$$c_0 := \lim_{x \to \infty} \frac{1}{x} \#\{n \le x : \{u(n)\} \in E\}.$$

Then,

$$\lim_{x \to \infty} \sup_{f \in \mathcal{M}_1^*} \frac{1}{x} \left| S_{\mathcal{T}_1}^{(f)}(x) - \lambda(E) S^{(f)}(x) \right| = 0.$$

As a consequence of these results, we have the following applications, the second of which echoes the well-known Erdős–Kac theorem.

Corollary 1. *Let g be a strongly additive function. Then g has a limiting distribution on $\mathbb{N}$ if it has a limiting distribution on the set $\mathcal{T} = \{n \in \mathbb{N} : : \{P(n)\} \in E\}$, and the two limiting distributions are equal.*

Corollary 2. *Let $g : \mathbb{N} \to \mathbb{R}$ be a strongly additive function such that $g(p) = O(1)$ and set*

$$A(x) := \sum_{p\le x} \frac{g(p)}{p} \qquad \text{and} \qquad B(x) := \sqrt{\sum_{p\le x} \frac{g^2(p)}{p}}.$$

Then,

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : g(n) \le A(x) + zB(x)\} = \Phi(z) \tag{3.1}$$

holds for every real number z if and only if

$$\lim_{x\to\infty} \frac{1}{\lambda(E)x} \#\{n \le x : n \in \mathcal{T} \text{ and } g(n) \le A(x) + zB(x)\} = \Phi(z) \tag{3.2}$$

for every real number z.

4. Proof of Theorems 1 and 2

We shall only provide the proof of Theorem 2, since the conditions and conclusion of Theorem 1 follow from those of Theorem 2.

We have

$$S^{(f)}_{\mathcal{T}_1}(x) = \sum_{n\le x} f(n)h(u(n)).$$

Given an arbitrarily small but fixed $\delta > 0$, let K be such that the sequence $\{c_j\}$ (defined in Section 2) satisfies condition (2.1). Then,

$$\left| S^{(f)}_{\mathcal{T}_1}(x) - \sum_{n\le x} f(n)H_\delta(u(n)) \right| \le \delta\, x + \#\{n \le x : \{u(n)\} \in S\}. \tag{4.1}$$

Now, observe that, with H_δ as in (2.3), we have

$$\sum_{n\le x} f(n)H_\delta(u(n)) = \lambda(E)\, S^{(f)}(x) + \sum_{1\le|j|\le K} c_j \sum_{n\le x} f(n)e\big(ju(n)\big). \tag{4.2}$$

Combining (4.1) and (4.2), we may write that

$$\begin{aligned} \left| \frac{\lambda(E)S^{(f)}(x)}{x} - \frac{S^{(f)}_{\mathcal{T}_1}(x)}{x} \right| \quad \le \quad & \sum_{1\le|j|\le K} \frac{|c_j|}{x} \left| \sum_{n\le x} f(n)e\big(ju(n)\big) \right| + \\ & + \frac{1}{x}\#\{n \le x : \{u(n)\} \in S\} + 2\delta. \end{aligned} \tag{4.3}$$

Since by hypothesis (a) of Theorem 2, the sequence $\{u(n)\}$ is uniformly distributed modulo 1, we have that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \{u(n)\} \in S\} = \lambda(S). \tag{4.4}$$

Using (4.4), it follows from (4.3) that, as $x \to \infty$,

$$\begin{aligned} \left| S^{(f)}_{\mathcal{T}_1}(x) - \lambda(E) S^{(f)}(x) \right| &\le 2\delta x + \\ &+ (\lambda(S) + o(1))\, x + \sum_{1\le |j| \le K} |c_j| \left| \sum_{n\le x} f(n) e\big(ju(n)\big) \right|. \end{aligned} \tag{4.5}$$

Recalling condition (2.1) and inequality (2.2), we have from (4.5) that, as $x \to \infty$,

$$\begin{aligned} \sup_{f\in\mathcal{M}_1^*} \left| \frac{S^{(f)}_{\mathcal{T}_1}(x)}{x} - \lambda(E) \frac{S^{(f)}(x)}{x} \right| &\le 2\delta + \lambda(S) + o(1) + \\ &+ \sum_{1\le |j| \le K} |c_j| \sup_{f\in\mathcal{M}_1^*} \frac{1}{x} \left| \sum_{n\le x} f(n) e\big(ju(n)\big) \right|. \end{aligned} \tag{4.6}$$

Then, in light of the generalised form of the Daboussi theorem (see Remark 1) and using inequality (2.4), we may replace inequality (4.6) by

$$\sup_{f\in\mathcal{M}_1^*} \left| \frac{S^{(f)}_{\mathcal{T}_1}(x)}{x} - \lambda(E) \frac{S^{(f)}(x)}{x} \right| \le 8k\delta + 2\delta.$$

Since δ can be chosen arbitrarily small, the conclusion of Theorem 2 follows.

■

5. Proof of Corollaries 1 and 2

Corollary 1 is an immediate consequence of Theorem 1. So, we only prove Corollary 2. For any real number τ, we set

$$f_\tau(n) := e^{i\tau g(n)}.$$

From Theorem 1, we have that

$$\sup_{\tau\in\mathbb{R}} \left| \frac{\lambda(E)}{x} \sum_{n\le x} f_\tau(n) - \sum_{\substack{n\le x\\ n\in\mathcal{T}}} f_\tau(n) \right| \to 0 \qquad (x\to\infty).$$

Letting

$$h_{x,\tau}(n) := e^{i\tau\left(\frac{g(n)-A(x)}{B(x)}\right)} = f_{\tau/B(x)}(n)\cdot e^{-\tau A(x)/B(x)}.$$

Hence, from Theorem 1, we have that

$$\lim_{x\to\infty}\sup_{\tau\in\mathbb{R}}\left|\frac{1}{x}\sum_{n\le x}h_{x,\tau}(n)-\frac{1}{\lambda(E)x}\sum_{\substack{n\le x\\ n\in\mathcal{T}}}h_{x,\tau}(n)\right|=0.$$

Hence, letting $\varphi(\tau) := e^{-\tau^2/2}$, assertion (3.1) holds if and only if

$$\frac{1}{x}\sum_{n\le x}h_{x,\tau}(n)\to\varphi(\tau)\qquad (x\to\infty),$$

which itself holds if and only if

$$\frac{1}{\lambda(E)x}\sum_{n\le x}h_{x,\tau}(n)\to\varphi(\tau)\qquad (x\to\infty),$$

which is equivalent to assertion (3.2), thus concluding the proof of Corollary 2. ■

6. Final remark

It would be interesting to characterize all those functions $u(n)$ for which condition (1.2) holds. We believe that, setting $u(n) := An^{\xi}$, where $A\in\mathbb{R}\setminus\{0\}$ and $\xi > 1$ is not an integer, condition (1.2) can be verified, implying that Theorem 2 would hold in that case. However, we could not prove that.

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