CORRECTION OF OUR PAPER "ON THE ITERATES OF SOME MULTIPLICATIVE FUNCTIONS"

Imre Kátai and Bui Minh Phong

(Budapest, Hungary)

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Abstract. In this paper, we correct our article mentioned in the title.

The above mentioned paper was published in this journal **56** (2024) 263–270 (see [1]). The Section 6 of this paper will be changed.

Section 6. New results

Theorem 1. *Let $H \in \mathcal{M}^*$, $|H(n)| = 1$ for every $n \in \mathbb{N}$. Let $H(p) = \kappa$ for every $p \in \mathcal{P}$. Let $r \in \mathbb{N}$, $r \geq 2$. Assume that $H(r) = \kappa^{\Omega(r)} \neq 1$, and choose an arbitrary real number $Y = Y(x) \in \left[x^{\frac{7}{12}+\varepsilon},\ x^{\frac{2}{3}}\right]$. Then*

$$\lim_{x\to\infty} \frac{1}{Y} \sum_{x\leq n\leq x+Y} H\left(\tau^{(r)}(n)\right) = 0.$$

Theorem 2. *Let $H \in \mathcal{M}^*$, $|H(n)| = 1$ for every $n \in \mathbb{N}$. Let $H(p) = \kappa$ for every $p \in \mathcal{P}$. Assume that $\kappa \neq 1$. Then*

$$\lim_{x\to\infty} \frac{1}{Y} \sum_{x\leq n\leq x+Y} H(n) = 0.$$

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Proof of Theorem 2. If $|\kappa|=1$, $\kappa\neq 1$, then either

(a) $\kappa^\ell=1$ for some $\ell\in\mathbb{N}$, $\ell\geq 2$, and then

$$\sum_{j=0}^{\ell-1}\kappa^j=0$$

or

(b) $\kappa=e^{2\pi i\alpha}$, where α is an irrational number, and then

$$\left|\sum_{\nu=0}^{L-1}\kappa^\nu\right|=\left|\frac{\kappa^L-1}{\kappa-1}\right|\leq\frac{2}{|\kappa-1|}=:C.$$

Thus

$$\liminf_{L\to\infty}\left|\sum_{\nu=0}^{L-1}\kappa^\nu\right|\leq C$$

in the cases (a) and (b).

Thus

$$\sum_{x\leq n\leq x+Y}H(n)=\sum_{K\in\mathcal{K}}H(K)\sum_{x\leq Km\leq x+Y}\kappa^{\omega(m)}|\mu(m)|=\sum_{K\in\mathcal{K}}H(K)\sum\nolimits_K.$$

Let $T>0$. We have

$$\sum_{\substack{K\in\mathcal{K}\\K>T}}H(K)|\Sigma_K|\leq\sum_{T<K<Y}\frac{Y}{K}+\sum_{\substack{K\in\mathcal{K}\\Y\leq K}}1 \tag{6.1}$$

Since

$$\sum_{\substack{K\in\mathcal{K}\\K>u}}1\leq c\sqrt{u},$$

the right-hand side of (6.1) is $o_T(1)\,Y(x)$.

Let $\delta>0$. Let $T=T_\delta$ be such a number for which the right-hand side of (6.1) is less than $\delta Y(x)$.

We shall estimate

$$\Sigma_K=\sum_{\frac{x}{K}\leq m\leq\frac{x+Y(x)}{K}}\kappa^{\omega(m)}\,|\mu(m)|$$

for $K\leq T_\delta$. Let $x_K=\frac{x}{K}$, $\qquad y_K=\frac{Y}{K}$. Then

$$x_k^{\frac{7}{2}+\frac{\varepsilon}{2}}\leq y_K\leq x_K^{\frac{2}{3}}$$

holds if x is large enough.

We shall prove that

$$\lim_{x\to\infty}\max_{K<T_x}\left|\frac{1}{y_K}\sum_{\substack{(m,K)=1\\ x_K\le m\le x_K+y_K}}\kappa^{\omega(m)}|\mu(m)|\right|=0.$$

Let

$$U_x=\left[(1-\sigma_x)x_2,\ x_2+\rho_x\sqrt{x_2}\right],$$

where $x_2=\log\log x$, $\rho_x\to\infty$ and $\sigma_x\to 0$ slowly. From Theorem A we obtain that

$$\max_{K\le T_\delta}\frac{1}{y_K}\sum_{\substack{(m,K)=1\\ \omega(m)\notin U_x}}|\mu(m)|\longrightarrow 0 \qquad \text{as } x\to\infty.$$

By using Theorem A we obtain that

$$\Sigma_K=y_K\sum_{k\in U_x}\kappa^h\rho_h(x)+o(y_K).$$

Since

$$\max_{\ell\le L}\left|\frac{\rho_{h+\ell}(x)}{\rho_h(x)}-1\right|\longrightarrow 0 \qquad \text{as } x\to\infty,$$

it follows that

$$\rho_h(x)=\frac{1}{L}\sum_{\ell=1}^{L}\rho_{h+\ell}(x)+o_x(1)$$

and so

$$\left|\sum_{k\in U_x}\kappa^h\rho_h(x_K)\right|\le\frac{1}{L}\sum_{t\in U_x}\rho_t(x_K)\left|\sum_{j=1}^{t}\kappa^{t-j}\right|\le\delta+o_x(1).$$

Thus

$$\frac{1}{y_K}|\Sigma_K|\longrightarrow 0 \qquad \text{for every } K\in\mathcal{K},\ K\le T_\delta.$$

We keep the notations $\mathcal{K},\rho_h$ and Theorem A from [1, Section 2].

The proof of Theorem 2 is therefore complete. ■

Proof of Theorem 1. Let

$$n=Km, \qquad \mu(m)\ne 0,$$

then

$$\tau^{(r)}(n)=\tau^{(r)}(K)\tau^{(r)}(m)=r^{\omega(K)}r^{\omega(m)},$$

Let $H_1(n):=H\big(\tau^{(r)}(n)\big)$. Then $H_1\in\mathcal{M}^*$, $|H_1(n)|=1$ $(n\in\mathbb{N})$, and $H_1(p)=H(r)$. Since $|H(r)|=1$, $H(r)\ne 1$.

Thus, Theorem 1 is a special case of Theorem 2. ■

References

[1] **Kátai, I. and B.M. Phong,** On the iterates of some multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **56** (2024), 263–270. https://doi.org/10.71352/ac.56.263

Imre Kátai
https://orcid.org/0000-0002-5209-7591
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary
katai@inf.elte.hu

Bui Minh Phong
https://orcid.org/0000-0001-8371-5279
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary
bui@inf.elte.hu