Annales Univ. Sci. Budapest., Sect. Comp. 59 (2026) 105–127

LINEAR CANONICAL STURM–LIOUVILLE HAUSDORFF AND CONVOLUTION TYPE OPERATORS

Fethi Soltani (Tunis, Tunisia)

Communicated by László Szili

(Received 12 March 2026; accepted 6 July 2026)

Abstract. In the present paper, we introduce the canonical Sturm–Liouville operator

$$\Delta^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is a positive function satisfying certain conditions. We prove the boundedness of the canonical Sturm–Liouville Hausdorff operators on the space $L^p(\mathbb{R}_+, A(x)\mathrm{d}x)$, $p \in [1, \infty)$. We investigate canonical Sturm–Liouville convolution operator, and obtain some useful results. The relation between the canonical Sturm–Liouville convolution and Hausdorff type operators is also established. The properties of the adjoint canonical Sturm–Liouville Hausdorff operators are further discussed. The harmonic analysis associated with the operator Δ^M plays an important role in establishing the results of this paper.

1. Introduction

The study of Hausdorff operators, which originated from some classical summation methods, has a long history in real and complex analysis. In the one-dimensional setting, Hausdorff operators on the real line were introduced in [11] and studied on the Hardy space in [15]. The natural generalization in several dimensions was introduced and studied in [4, 6, 13]. Particularly, Hausdorff operator is an interesting operator in harmonic analysis [16]. It contains some important operators, such as Hardy operator, adjoint Hardy operator

Key words and phrases: Canonical Sturm–Liouville transform, canonical Sturm–Liouville Hausdorff operators, canonical Sturm–Liouville convolution operators.
2010 Mathematics Subject Classification: 44A05, 44A20, 47G10.
https://doi.org/10.71352/ac.59.060726

[7, 12], and the Riemann–Liouville fractional integral operator [1, 2]. The modern study of general Hausdorff operators on $L^1(\mathbb{R})$ and the real Hardy space $H^1(\mathbb{R})$ over the real line was pioneered by Liflyand and Móricz in [15]. Many research papers have addressed the boundedness of the Hausdorff operator on Hardy spaces. For instance, Liflyand and his collaborators in [13, 14] proved, by more effective ways, that the Hausdorff operator has the same behavior on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$. Recently, Daher and Saadi in [8, 9] investigated the Dunkl Hausdorff operator on the Lebesgue space $L^1_\alpha(\mathbb{R})$ and on the Hardy space $H^1_\alpha(\mathbb{R})$. Subsequently, Mondal and Poria [18] studied Hausdorff operators associated with the Opdam–Cherednik operator. Furthermore, Tyr [30] studied the boundedness of q-Hausdorff operators on q-Hardy spaces. Another fundamental tool in harmonic analysis is the canonical Sturm–Liouville Hausdorff operators, which is the main object of study in this paper.

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2,\mathbb{R})$ such that $b > 0$. We define the canonical Sturm–Liouville operator Δ^M on $(0,\infty)$ by

$$\Delta^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is a positive function satisfying certain conditions.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator Δ^M is reduced to the Sturm–Liouville operator Δ:

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

The classical Sturm–Liouville operator Δ plays an important role in analysis [34, 3]. In particular, the two references [5, 28] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator Δ.

Using the Sturm–Liouville harmonic analysis [5, 28], for all $\lambda \in \mathbb{C}$, the system

$$\begin{cases} \Delta^M u = -(\frac{\lambda^2}{b^2} + \rho^2)u, \\ u(0) = e^{i\frac{d}{2b}\lambda^2}, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ^M_λ and given by

$$\varphi^M_\lambda(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+,$$

where $\varphi_\lambda(x)$ is the Sturm–Liouville kernel [23, 25].

In this paper, we introduce the canonical Sturm–Liouville transform $\mathscr{F}^M$:

$$\mathscr{F}^M(f)(\lambda) := \int_0^\infty \varphi^M_\lambda(x) f(x) A(x)\mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

The canonical Sturm–Liouville transform $\mathscr{F}^M$ can be regarded as a generalization of the Sturm–Liouville transform $\mathscr{F}$ (see [17, 21, 22, 23, 25, 26, 27]):

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) A(x) \mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

Let $\phi \in L^1([1,\infty))$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm–Liouville operator Δ^M for $f \in L^1(\mathbb{R}_+, A(x)\mathrm{d}x)$ by

$$H_\phi f(x) := \int_1^\infty f_t(x)\phi(t)\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

where f_t, $t \geq 1$ is the dilation of f given by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \quad x \in \mathbb{R}_+.$$

The main purpose of this paper is to extend some results of the classical Hausdorff operator given in [33] to the framework of canonical Sturm–Liouville theory, and to investigate the canonical Sturm–Liouville wavelet transform. We prove the boundedness of canonical Sturm–Liouville Hausdorff operator in space $L^p(\mathbb{R}_+, A(x)\mathrm{d}x)$, $p \in [1,\infty)$. The relation between the canonical Sturm–Liouville convolution and Hausdorff type operators is also established. Next, we introduce the adjoint operator H_ϕ^* on $L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$ by

$$H_\phi^* f(x) := \int_1^\infty f(tx)\phi(t)\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

We present the properties of the adjoint operator H_ϕ^*, including its boundedness on $L^p(\mathbb{R}_+, A(x)\mathrm{d}x)$, $p \in [1,\infty)$. We also establish a relation between the canonical Sturm–Liouville convolution operator and the adjoint operator H_ϕ^*.

This paper is organized as follows. In Section 2, we recall some results about the Sturm–Liouville transform $\mathscr{F}$, the Sturm–Liouville translation τ_y and the Sturm–Liouville convolution $*$. In Section 3, we introduce the canonical Sturm–Liouville operator Δ^M, and we investigate the properties of the canonical Sturm–Liouville transform $\mathscr{F}^M$, the canonical Sturm–Liouville translation τ_y^M and the canonical Sturm–Liouville convolution $*^M$ associated with this operator. In Section 4, we introduce the canonical Sturm–Liouville Hausdorff operators H_ϕ and we establish their properties. In the last section, we investigate the canonical Sturm–Liouville convolution operator and derive its relation with the operators H_ϕ and H_ϕ^*.

2. Sturm–Liouville harmonic analysis

In this section we recall some results about the harmonic analysis associated with the Sturm–Liouville operator (Sturm–Liouville transform, Sturm–Liouville translation and Sturm–Liouville convolution).

We consider the second-order differential operator Δ defined on $(0,\infty)$ by

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0)=1$. Moreover we assume that A satisfies the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$,

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho > 0$,

(iii) there exists a constant $\delta > 0$, such that

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x}D(x), \tag{2.1}$$

where D is an infinitely differentiable function on $(0,\infty)$, bounded and with bounded derivatives on all intervals $[x_0,\infty)$, for $x_0 > 0$.
This operator was studied in [5, 28], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -(\lambda^2+\rho^2)u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases} \tag{2.2}$$

admits a unique solution, denoted by φ_λ, with the following properties:

- for $x \in \mathbb{R}_+$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$.
- For $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.

(II) For nonzero $\lambda \in \mathbb{C}$, the equation

$$\Delta u = -(\lambda^2+\rho^2)u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}}V(x,\lambda),$$

with

$$\lim_{x\to\infty} V(x,\lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \in \mathbb{R}_+,$$

for nonzero $\lambda \in \mathbb{C}$.
Moreover there exist positive constants k_1, k_2, k, such that

$$k_1|\lambda|^{2\alpha+1} \le |c(\lambda)|^{-2} \le k_2|\lambda|^{2\alpha+1},$$

for all λ such that $\mathrm{Im}\lambda \le 0$ and $|\lambda| \ge k$.

(III) From (2.2) the Sturm–Liouville kernel $\varphi_\lambda(t)$ satisfies

$$\varphi'_\lambda(t)\varphi''_\lambda(t) + \frac{A'(t)}{A(t)}(\varphi'_\lambda(t))^2 = -(\lambda^2+\rho^2)\varphi'_\lambda(t)\varphi_\lambda(t).$$

Integrating this equation between 0 and x we obtain

$$\frac{(\varphi'_\lambda(x))^2}{2} + \int_0^x \frac{A'(t)}{A(t)}(\varphi'_\lambda(t))^2 \mathrm{d}t = -(\lambda^2+\rho^2)\frac{(\varphi_\lambda(x))^2-1}{2}.$$

From the property (ii) of the function A we have

$$\int_0^x \frac{A'(t)}{A(t)}(\varphi'_\lambda(t))^2 \mathrm{d}t \ge 0.$$

Therefore, $(\varphi_\lambda(t))^2 - 1 \le 0$. Hence

$$|\varphi_\lambda(x)| \le 1, \quad \lambda, x \in \mathbb{R}_+.$$

We denote by

- μ the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\mu(x) := A(x)\mathrm{d}x,$$

and by $L^p(\mu)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_0^\infty |f(x)|^p \mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad p \in [1,\infty),$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty.$$

- ν the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu(\lambda) := \frac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2},$$

and by $L^p(\nu)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm–Liouville transform is the Fourier transform associated with the operator Δ and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) \mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+. \tag{2.3}$$

Some of the properties of the Sturm–Liouville transform $\mathscr{F}$ are collected below.

Theorem 2.1. (See [3, 5, 28, 34]).

(i) *Plancherel theorem for $\mathscr{F}$. The Sturm–Liouville transform $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(ii) *Inversion theorem for $\mathscr{F}$. Let $f \in L^1(\mu)$, such that $\mathscr{F}(f) \in L^1(\nu)$. Then*

$$f(x) = \int_0^\infty \varphi_\lambda(x) \mathscr{F}(f)(\lambda) d\nu(\lambda), \quad a.e. \quad x \in \mathbb{R}_+.$$

The Sturm–Liouville kernel φ_λ satisfies the product formula [5, 28]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z) w(x, y, z) \mathrm{d}\mu(z) \quad \text{for} \quad x, y \in \mathbb{R}_+; \tag{2.4}$$

where $w(x, y, z)$ is a measurable positive function on $\mathbb{R}_+$, with support in $[|x - y|, x + y]$, satisfying

$$\int_0^\infty w(x, y, z) \mathrm{d}\mu(z) = 1,$$

$$w(x, y, z) = w(y, x, z) \quad \text{for} \quad z \in \mathbb{R}_+, \tag{2.5}$$

$$w(x, y, z) = w(x, z, y) \quad \text{for} \quad z > 0. \tag{2.6}$$

We now define the generalized translation operator induced by (2.4). For $f \in L^1(\mu)$, the linear operator

$$\tau_y f(x) := \int_0^\infty f(z)w(x,y,z)\mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+, \tag{2.7}$$

will be called Sturm–Liouville translation [5, 28].
As a first remark, we note that the relation (2.5) means that

$$\tau_y f(x) = \tau_x f(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 2.2. (See [19, 23, 25]).

(i) *For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1,\infty]$, we have*

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) *For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have*

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y)\mathscr{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\mu)$. The Sturm–Liouville convolution $f * g$ of f and g is defined by

$$f * g(x) := \int_0^\infty \tau_x f(y)g(y)\mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{2.8}$$

The convolution $*$ is commutative, associative and satisfies the Young inequality (see [19]). Let $p, q, r \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f * g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}.$$

Theorem 2.3. (See [19, 29]).

(i) *For $f, g \in L^2(\mu)$, the function $f * g$ belongs to $L^\infty(\mu)$, and*

$$f * g(x) = \int_0^\infty \varphi_\lambda(x)\mathscr{F}(f)(\lambda)\mathscr{F}(g)(\lambda)\,d\nu(\lambda), \quad x \in \mathbb{R}_+.$$

(ii) *Let $f, g \in L^2(\mu)$. Then*

$$\int_0^\infty |f * g(x)|^2\,d\mu(x) = \int_0^\infty |\mathscr{F}^M(f)(\lambda)|^2|\mathscr{F}^M(g)(\lambda)|^2\,d\nu(\lambda),$$

where both sides are finite or infinite.

Example 2.4. (See [10, 20]). Note that if $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha > \beta \geq -1/2$ and $\rho = \alpha+\beta+1$, the Sturm–Liouville operator Δ is reduced to the Jacobi operator denoted by $\Delta_{(\alpha,\beta)}$:

$$\Delta_{(\alpha,\beta)} := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + [(2\alpha+1)\coth(x) + (2\beta+1)\tanh(x)]\frac{\mathrm{d}}{\mathrm{d}x}.$$

In this case $\varphi_\lambda(x)$ is the Jacobi function denoted by $\varphi_\lambda^{(\alpha,\beta)}(x)$:

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\Big(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho+i\lambda), \alpha+1, -\sinh^2(x)\Big),$$

where ${}_2F_1(a,b,c,z)$ is the hypergeometric function. We denote by $\mu_{\alpha,\beta}$ and $\nu_{\alpha,\beta}$ the two measures defined respectively by

$$\mathrm{d}\mu_{\alpha,\beta}(x) := \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)\mathrm{d}x \quad \text{and} \quad \mathrm{d}\nu_{\alpha,\beta}(\lambda) := \frac{\mathrm{d}\lambda}{2\pi|c_{\alpha,\beta}(\lambda)|^2},$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

The Jacobi transform denoted by $\mathscr{F}_{(\alpha,\beta)}$ is defined for $f \in L^1(\mu_{\alpha,\beta})$ by

$$\mathscr{F}_{(\alpha,\beta)}(f)(\lambda) := \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(x) f(x)\mathrm{d}\mu_{\alpha,\beta}(x), \quad \lambda \in \mathbb{R}_+.$$

The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ satisfies the product formula:

$$\varphi_\lambda^{(\alpha,\beta)}(x)\varphi_\lambda^{(\alpha,\beta)}(y) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(z) w_{\alpha,\beta}(x,y,z)\mathrm{d}\mu_{\alpha,\beta}(z), \quad \lambda, x, y \in \mathbb{R}_+,$$

where $w_{\alpha,\beta}(x,y,.)$ is the kernel given by

$$\begin{aligned} w_{\alpha,\beta}(x,y,z) = a_\alpha &\frac{[\cosh(x)\cosh(y)\cosh(z)]^{-(\alpha+\beta+1}}{[\sinh(x)\sinh(y)\sinh(z)]^{2\alpha}}(1-Q^2(x,y,z))^{\alpha-\frac{1}{2}} \\ &\times {}_2F_1\Big(\alpha+\beta, \alpha-\beta, \alpha+\frac{1}{2}, \frac{1}{2}(1-Q(x,y,z))\Big)\chi_{(|x-y|,x+y)}(z), \end{aligned}$$

and

$$Q(x,y,z) = \frac{\cosh^2(x)+\cosh^2(y)+\cosh^2(z)-1}{2\cosh(x)\cosh(y)\cosh(z)},$$

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}.$$

Here $\chi_{(|x-y|,x+y)}$ is the characteristic function of the interval $(|x-y|, x+y)$.

The Jacobi translation operator is given by

$$\tau_y^{(\alpha,\beta)} f(x) := \int_0^\infty f(z) w_{\alpha,\beta}(x,y,z) \mathrm{d}\mu_{\alpha,\beta}(z), \quad x, y \in \mathbb{R}_+.$$

Let $f, g \in L^2(\mu_{\alpha,\beta})$. The Jacoby convolution $f *_{(\alpha,\beta)} g$ of f and g is defined by

$$f *_{(\alpha,\beta)} g(x) := \int_0^\infty \tau_x^{(\alpha,\beta)} f(y) g(y) \mathrm{d}\mu_{\alpha,\beta}(y), \quad x \in \mathbb{R}_+.$$

3. Canonical Sturm–Liouville operator

Throughout this paper, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2,\mathbb{R})$ such that $b > 0$. We define the canonical Sturm–Liouville operator Δ^M on $(0,\infty)$ by

$$\Delta^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x \right) \frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b} \right),$$

where A is the positive function given in Section 2.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator Δ^M is reduced to the Sturm–Liouville operator Δ:

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)} \frac{\mathrm{d}}{\mathrm{d}x}.$$

For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta^M u = -(\frac{\lambda^2}{b^2} + \rho^2) u, \\ u(0) = e^{i\frac{d}{2b}\lambda^2}, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+.$$

For $f \in L^1(\mu)$, we define the canonical Sturm–Liouville transform $\mathscr{F}^M(f)$ by

$$\mathscr{F}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^M(x) f(x) \mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+.$$

This transform can be written as

$$\mathscr{F}^M(f)(\lambda) = e^{i\frac{d}{2b}\lambda^2}\mathscr{F}(e^{i\frac{a}{2b}x^2}f)(\frac{\lambda}{b}), \quad f \in L^1(\mu), \tag{3.1}$$

where $\mathscr{F}$ is the Sturm–Liouville transform given by (2.3).

We denote by ν_b, $b > 0$, the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu_b(\lambda) := \frac{\mathrm{d}\lambda}{2\pi b|c(\frac{\lambda}{b})|^2},$$

and by $L^p(\nu_b)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu_b)} < \infty$.

Theorem 3.1. (i) *Let $f \in L^1(\mu)$, such that $\mathscr{F}^M(f) \in L^1(\nu_b)$. Then*

$$f(x) = \int_0^\infty \varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)\,d\nu_b(\lambda), \quad a.e. \quad x \in \mathbb{R}_+,$$

where N is the matrix given by $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

(ii) *For $f \in L^2(\mu)$ we have*

$$\|\mathscr{F}^M(f)\|_{L^2(\nu_b)} = \|f\|_{L^2(\mu)}.$$

Proof. (i) follows from Theorem 2.1 (ii) and relation (3.1).

(ii) follows from Theorem 2.1 (i) and relation (3.1). ■

For $f \in L^1(\mu)$, we define the canonical Sturm–Liouville translation operators by

$$\tau_y^N f(x) := e^{-i\frac{a}{2b}(x^2+y^2)}\int_0^\infty f(z)e^{i\frac{a}{2b}z^2}w(x,y,z)\mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+.$$

It is easy to prove the following results.

Theorem 3.2. *The operators τ_y^N, $y \in \mathbb{R}_+$, satisfy:*

(i) $\tau_y^N f(x) = \tau_x^N f(y)$, $x, y \in \mathbb{R}_+$.

(ii) $\tau_y^N f(x) = e^{-i\frac{a}{2b}(x^2+y^2)}\tau_y\left(f(z)e^{i\frac{a}{2b}z^2}\right)(x)$,
where τ_y is the Sturm–Liouville translation given by (2.7).

(iii) $\tau_y^M\varphi_\lambda^M(x) = e^{-i\frac{d}{2b}\lambda^2}\varphi_\lambda^M(x)\varphi_\lambda^M(y)$.

Theorem 3.3. (i) *For all* $y \in \mathbb{R}_+$ *and* $f \in L^p(\mu)$, $p \in [1,\infty]$, *we have*

$$\|\tau_y^N f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}.$$

(ii) *For* $f \in L^2(\mu)$ *and* $y \in \mathbb{R}_+$, *we have*

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{i\frac{d}{2b}\lambda^2}\varphi_\lambda^N(y)\mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+,$$

where $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

Proof. (i) follows from Theorem 2.2 (i) and Theorem 3.2 (ii).

(ii) Let $f \in L^1(\mu) \cap L^2(\mu)$. Then

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = \int_0^\infty \tau_y^N f(x)\varphi_\lambda^M(x)\mathrm{d}\mu(x) =$$

$$= \int_0^\infty \left[e^{-i\frac{a}{2b}(x^2+y^2)} \int_0^\infty f(z)e^{i\frac{a}{2b}z^2} w(x,y,z)\mathrm{d}\mu(z)\right] \varphi_\lambda^M(x)\mathrm{d}\mu(x).$$

By using Fubini's theorem, (2.5) and (2.6) we obtain

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{-i\frac{a}{2b}y^2}\int_0^\infty f(z)e^{i\frac{a}{2b}z^2}\left[\int_0^\infty \varphi_\lambda^M(x)e^{-i\frac{a}{2b}x^2} w(z,y,x)\mathrm{d}\mu(x)\right]\mathrm{d}\mu(z).$$

And by Theorem 3.2 (iii) we deduce that

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{i\frac{d}{2b}\lambda^2}\varphi_\lambda^N(y)\mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+. \tag{3.2}$$

Since $L^1(\mu) \cap L^2(\mu)$ is a dense in $L^2(\mu)$, the formula (3.2) remains valid for $f \in L^2(\mu)$. ∎

Let $f, g \in L^2(\mu)$. The canonical Sturm–Liouville convolution $f *^N g$ of f and g is defined by

$$f *^N g(x) := \int_0^\infty \tau_x^N f(y)\left[e^{i\frac{a}{b}y^2} g(y)\right]\mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{3.3}$$

Then we can write

$$f *^N g(x) = e^{-i\frac{a}{2b}x^2}\left(e^{i\frac{a}{2b}z^2} f\right) * \left(e^{i\frac{a}{2b}z^2} g\right)(x), \quad x \in \mathbb{R}_+, \tag{3.4}$$

where $*$ is the Sturm–Liouville convolution given by (2.8).

The canonical Sturm–Liouville convolution $*^N$ is commutative, associative and satisfies the Young inequality. Let $p, q, r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f *^N g\|_{L^r(\mu)} \le \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}.$$

Theorem 3.4. (i) *For $f,g\in L^2(\mu)$, the function $f*^N g$ belongs to $L^\infty(\mu)$, and*

$$f*^N g(x)=\int_0^\infty e^{-i\frac{d}{2b}\lambda^2}\varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)\mathscr{F}^M(g)(\lambda)d\nu_b(\lambda),\quad x\in\mathbb{R}_+.$$

(ii) *Let $f,g\in L^2(\mu)$. Then*

$$\int_0^\infty |f*^N g(x)|^2 d\mu(x)=\int_0^\infty |\mathscr{F}^M(f)(\lambda)|^2|\mathscr{F}^M(g)(\lambda)|^2 d\nu_b(\lambda),$$

where both sides are finite or infinite.

Proof. (i) follows from (3.4), Theorem 2.3 (i) and (3.1).

(ii) follows from (3.4), Theorem 2.3 (ii) and (3.1). ■

Example 3.5. (See [24]). Note that if $A(x)=\sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha>\beta\geq -1/2$, the operator Δ^M is reduced to the canonical Jacobi operator $\Delta^M_{(\alpha,\beta)}$:

$$\Delta^M_{(\alpha,\beta)}:=\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\left(h_{\alpha,\beta}(x)-2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x}-\left(\frac{a^2}{b^2}x^2+i\frac{a}{b}xh_{\alpha,\beta}(x)+i\frac{a}{b}\right),$$

where

$$h_{\alpha,\beta}(x)=(2\alpha+1)\coth(x)+(2\beta+1)\tanh(x).$$

In this case

$$\varphi_\lambda^M(x)=\varphi_\lambda^{(\alpha,\beta),M}(x)=e^{\frac{i}{2}(\frac{d}{b}\lambda^2+\frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}^{(\alpha,\beta)}(x).$$

The canonical Jacobi transform $\mathscr{F}^M_{(\alpha,\beta)}$ is defined for $f\in L^1(\mu_{\alpha,\beta})$ by

$$\mathscr{F}^M_{(\alpha,\beta)}(f)(\lambda):=\int_0^\infty \varphi_\lambda^{(\alpha,\beta),M}(x)f(x)\mathrm{d}\mu_{\alpha,\beta}(x),\quad \lambda\in\mathbb{R}_+.$$

The canonical Jacobi translation operators are given by

$$\tau_y^{(\alpha,\beta),M}f(x):=e^{i\frac{a}{2b}(x^2+y^2)}\int_0^\infty f(z)e^{-i\frac{a}{2b}z^2}w_{\alpha,\beta}(x,y,z)\mathrm{d}\mu_{\alpha,\beta}(z),\quad x,y\in\mathbb{R}_+.$$

Let $f,g\in L^2(\mu_{\alpha,\beta})$. The canonical Jacobi convolution $f*^N_{(\alpha,\beta)}g$ of f and g is defined by

$$f*^N_{(\alpha,\beta)}g(x):=\int_0^\infty \tau_x^{(\alpha,\beta),N}f(y)\left[e^{i\frac{a}{b}y^2}g(y)\right]\mathrm{d}\mu_{\alpha,\beta}(y),\quad x\in\mathbb{R}_+.$$

4. Canonical Sturm–Liouville Hausdorff operator

In this section we define and study the Hausdorff operator associated with the canonical Sturm–Liouville operator Δ^M.

Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and $t \geq 1$. We define the dilation function f_t by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \tag{4.1}$$

and satisfies

$$\|f_t\|_{L^p(\mu)} \leq \left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} \|f\|_{L^p(\mu)}, \tag{4.2}$$

where

$$k(t) = \sup_{x \in \mathbb{R}_+} \left(\frac{A(x)}{A(tx)}\right).$$

From (2.1), there exist two constants $C_1, C_2 > 0$, such that

$$C_1 e^{2\rho x} \leq A(x) \leq C_2 e^{2\rho x}, \quad x > 0.$$

Therefore,

$$\frac{1}{C} e^{2\rho(1-t)x} \leq \frac{A(x)}{A(tx)} \leq C e^{2\rho(1-t)x}, \quad x > 0,$$

where $C = \frac{C_2}{C_1}$.

Let $\phi \in L^1([1, \infty))$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm–Liouville operator Δ^M for $f \in L^1(\mu)$ by

$$H_\phi f(x) := \int_1^\infty f_t(x)\phi(t)\mathrm{d}t. \tag{4.3}$$

Theorem 4.1. *Let ϕ be a measurable function on $[1, \infty)$ such that*

$$C_{\phi,p} := \int_1^\infty \left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} |\phi(t)| dt < \infty. \tag{4.4}$$

Then the Hausdorff operator H_ϕ is bounded on $L^p(\mu)$, $p \in [1, \infty)$ with

$$\|H_\phi f\|_{L^p(\mu)} \leq C_{\phi,p} \|f\|_{L^p(\mu)}.$$

Proof. By using Minkowski's inequality for integrals, we have

$$\|H_\phi f\|_{L^p(\mu)} = \left[\int_0^\infty \left|\int_1^\infty f_t(x)\phi(t)\mathrm{d}t\right|^p \mathrm{d}\mu(x)\right]^{1/p} \leq$$

$$\leq \left[\int_0^\infty \left(\int_1^\infty |f_t(x)||\phi(t)|\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p} \leq$$

$$\leq \int_1^\infty \left(\int_0^\infty |f_t(x)|^p|\phi(t)|^p\mathrm{d}\mu(x)\right)^{1/p} \mathrm{d}t =$$

$$= \int_1^\infty \|f_t\|_{L^p(\mu)}|\phi(t)|\mathrm{d}t.$$

Then by (4.2) we obtain

$$\|H_\phi f\|_{L^p(\mu)} \leq C_{\phi,p}\|f\|_{L^p(\mu)}.$$

Going back to the definition of

$$\left[\int_0^\infty \left(\int_1^\infty |f_t(x)||\phi(t)|\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p},$$

we deduce that the integral

$$H_\phi f(x) = \int_1^\infty f_t(x)\phi(t)\mathrm{d}t$$

is absolutely convergent for almost all $x \in \mathbb{R}_+$, and defines a function $H_\phi f \in$ $\in L^p(\mathbb{R}_+)$. ■

Theorem 4.2. *Let $\phi \in L^1([1,\infty))$. Then for $f \in L^1(\mu)$, we have*

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_1^\infty \mathscr{F}^M(f_t)(\lambda)\phi(t)\,dt, \quad \lambda \in \mathbb{R}_+.$$

Proof. Let $\phi \in L^1(\mathbb{R}_+)$, and let $f \in L^1(\mu)$. From Theorem 4.1, $H_\phi f \in L^1(\mu)$.

Then by (4.3) we have

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_0^\infty H_\phi f(x)\varphi_\lambda^M(x)\mathrm{d}\mu(x) = \\ = \int_0^\infty \left[\int_1^\infty f_t(x)\phi(t)\mathrm{d}t\right]\varphi_\lambda^M(x)\mathrm{d}\mu(x).$$

Since

$$\int_0^\infty\int_1^\infty |f_t(x)||\phi(t)||\varphi_\lambda^M(x)|\mathrm{d}t\mathrm{d}\mu(x) \le \|\phi\|_{L^1([1,\infty))}\|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_1^\infty \left[\int_0^\infty f_t(x)\varphi_\lambda^M(x)\mathrm{d}\mu(x)\right]\phi(t)\mathrm{d}t = \\ = \int_1^\infty \mathscr{F}^M(f_t)(\lambda)\phi(t)\mathrm{d}t.$$

The theorem is proved. ■

Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $[1,\infty)$ satisfying the condition

$$C_{\phi,2} := \int_1^\infty \left(\frac{k(t)}{t}\right)^{\frac{1}{2}} |\phi(t)|\mathrm{d}t < \infty. \tag{4.5}$$

We define the adjoint operator H_ϕ^* by the relation

$$\int_0^\infty H_\phi^* f(x)g(x)\mathrm{d}\mu(x) = \int_0^\infty f(x)H_\phi g(x)\mathrm{d}\mu(x).$$

Theorem 4.3. *Let $f \in L^2(\mu)$, and let ϕ be a measurable function on $[1,\infty)$ satisfying the condition (4.5). Then*

$$H_\phi^* f(x) = \int_1^\infty f(tx)\phi(t)\,dt. \tag{4.6}$$

Proof. Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $[1, \infty)$ satisfying the condition (4.5). From (4.3) and Fubini's theorem we have

$$\int_0^\infty f(x) H_\phi g(x) \mathrm{d}\mu(x) = \int_0^\infty f(x) \left[\int_1^\infty g_t(x)\phi(t)\mathrm{d}t \right] \mathrm{d}\mu(x) =$$
$$= \int_1^\infty \left[\int_0^\infty f(x) g_t(x) \mathrm{d}\mu(x) \right] \phi(t)\mathrm{d}t =$$
$$= \int_1^\infty \left[\int_0^\infty f(tx) g(x) \mathrm{d}\mu(x) \right] \phi(t)\mathrm{d}t.$$

Using (4.2), this calculation is justified by the fact that

$$\int_1^\infty \int_0^\infty |f(x)||g_t(x)| \mathrm{d}\mu(x) |\phi(t)| \mathrm{d}t \leq C_{\phi,2} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} < \infty.$$

Then according to Fubini's theorem we obtain

$$\int_0^\infty f(x) H_\phi g(x) \mathrm{d}\mu(x) = \int_0^\infty \left[\int_1^\infty f(tx)\phi(t)\mathrm{d}t \right] g(x) \mathrm{d}\mu(x) =$$
$$= \int_0^\infty H_\phi^* f(x) g(x) \mathrm{d}\mu(x),$$

where

$$H_\phi^* f(x) = \int_1^\infty f(tx)\phi(t)\mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_1^\infty \int_0^\infty |f(tx)||g(x)| \mathrm{d}\mu(x) |\phi(t)| \mathrm{d}t \leq C_{\phi,2} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} < \infty.$$

This completes the proof of the theorem. ∎

Remark 4.1. From Theorem 4.1, the operator H_ϕ^* is bounded on $L^p(\mu)$, $p \in [1, \infty)$ with

$$\|H_\phi^* f\|_{L^p(\mu)} \leq C_{\phi, \frac{p}{p-1}} \|f\|_{L^p(\mu)},$$

where $C_{\phi,p}$ is the constant given by (4.4).

As in the same of Theorem 4.2, we obtain the following result.

Theorem 4.4. *Let ϕ be a measurable function on $[1,\infty)$ satisfying the condition*

$$C_{\phi,\infty} := \int_1^\infty \frac{k(t)}{t}|\phi(t)|\,dt < \infty. \tag{4.7}$$

Then for $f \in L^1(\mu)$, we have

$$\mathscr{F}^M(H_\phi^* f)(\lambda) = \int_1^\infty \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\,dt, \quad \lambda \in \mathbb{R}_+,$$

where $f_t^*(x) = f(tx)$.

Proof. Let ϕ be a measurable function on $[1,\infty)$ satisfying the condition (4.7), and let $f \in L^1(\mu)$. Then by (4.6) we have

$$\begin{aligned}\mathscr{F}^M(H_\phi^* f)(\lambda) &= \int_0^\infty H_\phi^* f(x)\varphi_\lambda^M(x)\mathrm{d}\mu(x) = \\ &= \int_0^\infty \left[\int_1^\infty f(tx)\phi(t)\mathrm{d}t\right]\varphi_\lambda^M(x)\mathrm{d}\mu(x).\end{aligned}$$

Since

$$\int_0^\infty\int_1^\infty |f(tx)||\phi(t)||\varphi_\lambda^M(x)|\mathrm{d}t\mathrm{d}\mu(x) \le C_{\phi,\infty}\|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\begin{aligned}\mathscr{F}^M(H_\phi^* f)(\lambda) &= \int_1^\infty \left[\int_0^\infty f(tx)\varphi_\lambda^M(x)\mathrm{d}\mu(x)\right]\phi(t)\mathrm{d}t = \\ &= \int_1^\infty \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\mathrm{d}t.\end{aligned}$$

The theorem is proved. ■

Example 4.5. If we choose $\phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, we obtain the canonical Sturm–Liouville Hardy operator denoted by $\mathcal{H}$ and given by

$$\mathcal{H}f(x) := \int_1^\infty f_t(x)\frac{\mathrm{d}t}{t}.$$

It is well known that Hardy operators are important operators in harmonic analysis, for instance, see [7, 12].

Example 4.6. If we choose $\phi(t) = \frac{1}{\Gamma(\eta)} \frac{(1-\frac{1}{t})^{\eta-1}}{t} \chi_{(1,\infty)}(t)$, $\eta > 0$ we obtain the canonical Sturm–Liouville Riemann–Liouville fractional integral operator denoted by $\mathcal{I}_\eta$ and given by

$$\mathcal{I}_\eta f(x) := \frac{1}{\Gamma(\eta)} \int_1^\infty f_t(x)(1 - \frac{1}{t})^{\eta-1} \frac{\mathrm{d}t}{t}.$$

The study of Riemann–Liouville fractional integral operators can be found in [1].

5. Canonical Sturm–Liouville convolution operators

In this section, we first recall some fundamental results on the canonical Sturm–Liouville convolution operators. In the following we establish a relation between the canonical Sturm–Liouville convolution operator and the canonical Sturm–Liouville Hausdorff operator.

Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and let $g \in L^q(\mu)$, with $\frac{1}{p} + \frac{1}{q} = 1$. We define the canonical Sturm–Liouville convolution operator by

$$\mathscr{C}_g^N(f)(s) := f *^N g(s) = \int_0^\infty e^{i\frac{a}{b}x^2} f(x) \tau_s^N g(x) \mathrm{d}\mu(x),$$

where $*^N$ is the generalized convolution product given by (3.3).

From Hölder's inequality we have

$$\|\mathscr{C}_g^N(f)\|_{L^\infty(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

We obtain a relation between the canonical Sturm–Liouville convolution operator and the canonical Sturm–Liouville Hausdorff operator.

Theorem 5.1. *Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and let $g \in L^q(\mu)$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then for ϕ a measurable function on $[1, \infty)$ satisfying the condition (4.4) we have*

$$\mathscr{C}_g^N(H_\phi f)(s) = \int_1^\infty \mathscr{C}_g^N(f_t)(s) \phi(t) \, dt,$$

where f_t is the dilation of f given by (4.1).

Proof. Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and let $g \in L^q(\mu)$, with $\frac{1}{p} + \frac{1}{q} = 1$. From Theorem 4.1 we have $H_\phi f \in L^p(\mu)$, $p \in [1, \infty)$. Then by (4.3) we get

$$\begin{aligned}\mathscr{C}_g^N(H_\phi f)(s) &= \int_0^\infty e^{i\frac{a}{b}x^2} H_\phi f(x)\tau_s^N g(x)\mathrm{d}\mu(x)\\ &= \int_0^\infty e^{i\frac{a}{b}x^2}\left[\int_1^\infty f_t(x)\phi(t)\mathrm{d}t\right]\tau_s^N g(x)\mathrm{d}\mu(x)\\ &= \int_1^\infty \left[\int_0^\infty e^{i\frac{a}{b}x^2} f_t(x)\tau_s^N g(x)\mathrm{d}\mu(x)\right]\phi(t)\mathrm{d}t\\ &= \int_1^\infty \mathscr{C}_g^N(f_t)(s)\phi(t)\mathrm{d}t.\end{aligned}$$

Using (4.2), this calculation is justified by the fact that

$$\int_1^\infty\int_0^\infty |f_t(x)||\tau_s^N g(x)|\mathrm{d}\mu(x)|\phi(t)|\mathrm{d}t \le C_{\phi,p}\|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)} < \infty.$$

This ends the proof of the theorem. ■

As in the same of Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let $f \in L^p(\mu)$, $p \in [1,\infty)$ and let $g \in L^q(\mu)$, with $\frac{1}{p}+\frac{1}{q}=1$. Then for ϕ a measurable function on $[1,\infty)$ satisfying the condition (4.4) we have*

$$\mathscr{C}_g^N(H_\phi^* f)(s) = \int_1^\infty \mathscr{C}_g^N(f_t^*)(s)\phi(t)\,dt,$$

where $f_t^*(x) = f(tx)$.

Proof. Let $f \in L^p(\mu)$, $p \in [1,\infty)$ and let $g \in L^q(\mu)$, with $\frac{1}{p}+\frac{1}{q}=1$. From Remark 4.1 we have $H_\phi^* f \in L^p(\mu)$, $p \in [1,\infty)$. Then by (4.6) we get

$$\begin{aligned}\mathscr{C}_g^N(H_\phi^* f)(s) &= \int_0^\infty e^{i\frac{a}{b}x^2} H_\phi^* f(x)\tau_s^N g(x)\mathrm{d}\mu(x) =\\ &= \int_0^\infty e^{i\frac{a}{b}x^2}\left[\int_1^\infty f_t^*(x)\phi(t)\mathrm{d}t\right]\tau_s^N g(x)\mathrm{d}\mu(x) =\\ &= \int_1^\infty \left[\int_0^\infty e^{i\frac{a}{b}x^2} f_t^*(x)\tau_s^N g(x)\mathrm{d}\mu(x)\right]\phi(t)\mathrm{d}t =\\ &= \int_1^\infty \mathscr{C}_g^N(f_t^*)(s)\phi(t)\mathrm{d}t.\end{aligned}$$

This calculation is justified by the fact that

$$\int_1^\infty \int_0^\infty |f_t^*(x)||\tau_s^N g(x)|\mathrm{d}\mu(x)|\phi(t)|\mathrm{d}t \le C_{\phi,q}\|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)} < \infty.$$

This ends the proof of the theorem. ∎

Conclusion. In this work we have succeeded in generalizing the results of Brown and Móricz for the classical Hausdorff operator [4], Upadhyay et al. for the Hankel Hausdorff operator [31, 32] and Daher et al. for the Dunkl Hausdorff operator [8, 9] to the setting of canonical Sturm–Liouville theory. In this paper, we have studied the canonical Sturm–Liouville Hausdorff operator on the Lebesgue space $L^p(\mu)$, $p \in [1, \infty)$. If $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha > \beta \ge -1/2$, we obtain the results of the canonical Jacobi case. Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain the results of the classical Sturm–Liouville case.

Acknowledgments. The author wishes to thank the Reviewer for the careful reading and editing of the paper.

References

[1] **Andersen, A. and E. Sawyer,** Weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators, *Trans. Amer. Math. Soc.*, **308(2)** (1988), 547–558.
https://doi.org/10.2307/2001091

[2] **Behera, B.,** Hardy and Hardy–Littlewood–Pólya operators and their commutators on local fields, *Period. Math. Hung.*, **89(2)** (2024), 318–334.
https://doi.org/10.1007/s10998-024-00589-y

[3] **Bloom, W.R. and Z. Xu,** Fourier multipliers for L^p on Chébli–Trimèche hypergroups, *Proc. London Math. Soc.*, **80(03)** (2000), 643–664.
https://doi.org/10.1112/S0024611500012326

[4] **Brown, G. and F. Móricz,** Multivariate Hausdorff operators on the spaces $L^p(\mathbb{R}^n)$, *J. Math. Anal. Appl.*, **271(2)** (2002), 443–454.
https://doi.org/10.1016/S0022-247X(02)00128-2

[5] **Chébli, H.,** Théorème de Paley–Wiener associé à un opérateur différentiel singulier sur $(0, \infty)$, *J. Math. Pures Appl.*, **58(1)** (1979), 1–19.
https://doi.org/10.5802/jedp.178

[6] **Chen, J. and X. Zhu,** Boundedness of multidimensional Hausdorff operators on $H^1(\mathbb{R}^n)$, *J. Math. Anal. Appl.*, **409(1)** (2014), 428–434.
https://doi.org/10.1016/j.jmaa.2013.07.042

[7] **Christ, M. and L. Grafakos,** Best constants for two nonconvolution inequalities, *Proc. Amer. Math. Soc.*, **123(6)** (1995), 1687–1693.
https://doi.org/10.2307/2160978

[8] **Daher R. and F. Saadi,** The Dunkl–Hausdorff operator is bounded on the real Hardy space $H^1_\alpha(\mathbb{R})$, *Integral Transform. Spec. Funct.*, **30(11)** (2019), 882–892.
https://doi.org/10.1080/10652469.2019.1636236

[9] **Daher, R. and F. Saadi,** The Dunkl–Hausdorff operators and the Dunkl continuous wavelets transform, *J. Pseudo-Differ. Oper. Appl.*, **11(4)** (2020), 1821–1831.
https://doi.org/10.1007/s11868-020-00351-1

[10] **Flensted–Jensen, M. and T. Koornwinder,** The convolution structure for Jacobi function expansions, *Ark. Mat.*, **11(1-2)** (1973), 245–262.
https://doi.org/10.1007/BF02388521

[11] **Georgakis, C.,** The Hausdorff mean of a Fourier–Stieltjes transform, *Proc. Amer. Math. Soc.*, **116(2)** (1992), 465–471.
https://doi.org/10.2307/2159753

[12] **Kufner, A. and L.-E. Persson,** *Weighted Inequalities of Hardy Type*, Singapore: World Scientific Publishing Co Pte Ltd, 2003.
https://doi.org/10.1142/5129

[13] **Lerner, A.K. and E. Liflyand,** Multidimensional Hausdorff perators on the real Hardy space, *J. Austr. Math. Soc.*, **83(1)** (2007), 79–86.
https://doi.org/10.1017/S1446788700036399

[14] **Liflyand, E.,** Boundedness of multidimensional Hausdorff operators on $H^1(\mathbb{R}^n)$, *Acta Sci. Math.*, **74(3-4)** (2008), 845–851.

[15] **Liflyand, E. and F. Móricz,** The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$, *Proc. Amer. Math. Soc.*, **128(5)** (2000), 1391–1396.
https://doi.org/10.1090/S0002-9939-99-05159-X

[16] **Liflyand, E.,** Hausdorff operators on Hardy spaces, *Eurasian Math. J.*, **4(4)** (2013), 101–141.

[17] **Ma, R.,** Heisenberg uncertainty principle on Chébli–Trimèche hypergroups, *Pacific J. Math.*, **235(2)** (2008), 289–296.
https://doi.org/10.2140/pjm.2008.235.289

[18] **Mondal, S.S. and A. Poria,** Hausdorff operators associated with the Opdam–Cherednik transform in Lebesgue spaces, *J. Pseudo-Differ. Oper. Appl.*, **13(3)** (2022), Art. 31.
https://doi.org/10.1007/s11868-022-00462-x

[19] **Mourou, A.A. and K. Trimèche,** Calderón's reproducing formula for a singular differential operator and inversion of the generalized Abel transform, *J. Fourier Anal. Appl.*, **4(2)** (1998), 229–245.
https://doi.org/10.1007/BF02475991

[20] **Platonov, S.S.,** Fourier–Jacobi harmonic analysis and some problems of approximation of functions on the half-axis in L_2 metric: Nikol'ski–Besov type functions spaces, *Integral Transform. Spec. Funct.*, **31(4)** (2020), 281–298.
https://doi.org/10.1080/10652469.2019.1691548

[21] **Soltani, F.,** Extremal functions on Sturm–Liouville hypergroups, *Complex Anal. Oper. Theory*, **8(1)** (2014), 311–325.
https://doi.org/10.1007/s11785-013-0303-9

[22] **Soltani, F.,** Parseval–Goldstein type theorems for the Sturm–Liouville transform, *Integral Transform. Spec. Funct.*, **36(8)** (2025), 634–646.
https://doi.org/10.1080/10652469.2024.2443949

[23] **Soltani, F. and M. Aloui,** Lipschitz and Dini–Lipschitz functions for the Sturm–Liouville transform, *Integral Transform. Spec. Funct.*, **35(11)** (2024), 612–625.
https://doi.org/10.1080/10652469.2024.2364790

[24] **Soltani, F. and M. Aloui,** Boas–type theorems for the linear canonical Sturm–Liouville transform, *Methods Funct. Anal. Topol.*, **31(1)** (2025), 56–69.
https://doi.org/10.31392/MFAT-npu26_1.2025.06

[25] **Soltani, F. and Y. Zarrougui,** Localization operators and Shapiro's inequality for the Sturm–Liouville–Stockwell transform, *J. Math. Sci.*, **289(1)** (2025), 45–58.
https://doi.org/10.1007/s10958-024-07090-4

[26] **Soltani, F. and Y. Zarrougui,** Reconstruction and best approximate inversion formulas for the Sturm–Liouville–Stockwell transform, *Appl. Math. E-Notes*, **25** (2025), 57–72.

[27] **Soltani, F. and Y. Zarrougui,** Toeplitz operators and spectrogram associated with the Sturm–Liouville–Stockwell transform, *Filomat*, **39(18)** (2025), 6295–6310.
https://doi.org/10.2298/FIL2518295S

[28] **Trimèche, K.,** Transformation intégrale de Weyl et théorème de Paley–Wiener associés à un opérateur différentiel singulier sur $(0,\infty)$, *J. Math. Pures Appl.*, **60(1)** (1981), 51–98.

[29] **Trimèche, K.,** Inversion of Lions transmutation operators using generalized wavelets, *Appl. Comput. Harmonic Anal.*, **4(1)** (1997), 97–112.
https://doi.org/10.1006/acha.1996.0206

[30] **Tyr, O.,** On the boundedness of q-Hausdorff operators on q-Hardy spaces, *Kragujev. J. Math.*, **50(8)** (2025), 1261–1278.

[31] **Upadhyay, S.K., R.N. Yadav and L. Debnath,** Properties of the Hankel–Hausdorff operator on Hardy space $H^1(0,\infty)$, *Anal.*, **32(3)** (2012), 221–230.
https://doi.org/10.1524/anly.2012.1164

[32] **Upadhyay, S.K., R.S. Pandey and R.N. Mohapatra,** H^P-boundedness of Hankel Hausdorff operator involving Hankel transformation, *Dynam. Cont. Dis. Ser. A: Math. Anal.*, **21(2)** (2014), 243–258.

[33] **Weisz, F.,** The boundedness of the Hausdorff operator on multi-dimensional Hardy spaces, *Anal.*, **24(2)** (2004), 183–195.
https://doi.org/10.1524/anly.2004.24.14.183

[34] **Zeuner, H.,** The central limit theorem for Chébli–Trimèche hypergroups, *J. Theor. Probab.*, **2(1)** (1989), 51–63.
https://doi.org/10.1007/BF01048268

Fethi Soltani
https://orcid.org/0000-0001-7519-0994
Université de Carthage
Ecole Nationale d'Ingénieurs de Carthage
Tunis
Tunisia
fethi.soltani@fst.utm.tn