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THE REGULARITY INDEX OF SEVEN ALMOST EQUIMULTIPLE POINTS IN PROJECTIVE SPACE $\mathbb{P}^4$

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Abstract. In this paper, we compute the regularity index of the set of seven almost equimultiple points and give an example for the set of equimultiple points in projective space $\mathbb{P}^3$ which does not attain Segre's upper bound.

1. Introduction

In this paper, we always denote by $\mathbb{P}^n{:=}\mathbb{P}^n_K$ the n-dimensional projective space over an algebraically closed K of arbitrary characteristic, and denote by $R := K[x_0, \ldots, x_n]$ the polynomial ring in variables $x_0, \ldots, x_n$ over K. If $P_1, \ldots, P_s$ are distinct points in $\mathbb{P}^n$, we denote by $\wp_1, \ldots, \wp_s$ the defining homogeneous prime ideals of $P_1, \ldots, P_s$ in R, respectively.

Let $P_1, \ldots, P_s$ be distinct points in $\mathbb{P}^n$ and $m_1, \ldots, m_s$ be positive integer. Then the set of all homogeneous polynomials that vanish at P_i to order, for $i = 1, \ldots, s$ is homogeneous ideal $I := \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. We call the zero-scheme defined by I fat points (or a set of fat points) in $\mathbb{P}^n$ and we denote it by

$$Z := m_1 P_1 + \cdots + m_s P_s.$$

If $m_1 = \cdots = m_s = 2$, then fat points $Z = 2P_1 + \cdots + 2P_s$ is called double points. If $m_1 = \cdots = m_s = m$, then $mP_1 + \cdots + mP_s$ is called equimultiple.

If $Z = m_1 P_1 + \cdots + m_s P_s$, with $m_i \in \{m, m+1\}$, then we call the points P_i the points of support of Z.

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If the support of Z span $\mathbb{P}^n$ then $P_1, \dots, P_s$ is called non-degenerate in $\mathbb{P}^n$ and we also call Z a set of non-degenerate fat points in $\mathbb{P}^n$.

The homogeneous coordinate ring R/I of Z is a grade ring. $R/I = \bigoplus\limits_{t\geq 0} R/I$. We call the function

$$H_{R/I}(t) := \dim_K (R/I)_t$$

the Hilbert function of Z in $\mathbb{P}^n$. The Hilbert function $H_{R/I}(t)$ strictly increases until it reaches multipliable number $e(R/I) = \sum\limits_{i=1}^{s} \left(\begin{array}{c} m_i + n - 1 \\ n \end{array} \right)$, at which it stabilizes. We call the least integer such that $H_{R/I}(t) = e(R/I)$ the regularity index of Z in $\mathbb{P}^n$, and we denote it by $\operatorname{reg}(Z)$. It is well know that $\operatorname{reg}(Z) = = \operatorname{reg}(R/I)$, the Castelnuovo–Mumford regularity index of the ring R/I.

From now on, if a is a rational number, we denote by $[a]$ integer part. For generic fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^2$ with $m_1 \geq \cdots \geq m_s$, Segre [7] showed that

$$\operatorname{reg}(Z) \leq \max\left\{ m_1 + m_2 - 1, \left[\frac{m_1 + \cdots + m_s}{2} \right] \right\}.$$

It was conjectured by Trung (see [8]) and, independently, by Fatabbi and Lorenzini (see [5]) that

$$\operatorname{reg}(R/I) \leq \max\left\{ T_j(Z) |\, j = 1, \dots, n \right\},$$

where

$$T_j(Z) := \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \right\},$$

if $P_{i_1}, \dots, P_{i_q}$on a j-dimensional linear space.

The number

$$T(Z) = \max\left\{ T_j(Z) \mid j = 1, \dots, n \right\}$$

is called the Segre's bound because it generalizes Segre' upper bound. There were many different results of proving the Segre's upper bound. In 2016, Ballico et al. (see [1]) successfully proved the Segre's bound for $n + 3$ non-degenerate fat points in $\mathbb{P}^n$. Recently, Nagel and Trok (see [6]) have successfully proved the Segre's bound for arbitrary fat points Z in $\mathbb{P}^n$.

The problem to exactly determine $\operatorname{reg}(R/I)$ is more fair difficult. So far, there have been only a few results of computing $\operatorname{reg}(R/I)$.

For arbitrary fat points $Z = m_1P_1 + \cdots + m_sP_S$ in $\mathbb{P}^n$, Davis and Geramita (see [4, Corollary 2.3]) proved that

$$\operatorname{reg}(R/I) = m_1 + m_2 + \cdots + m_s - 1$$

if and only if the points $P_1, \dots, P_s$ lie on a line in $\mathbb{P}^n$, the points $P_1, \dots, P_s$ in $\mathbb{P}^n$ is said to be in linearly general position. If no $j + 2$ of the points $P_1, \dots, P_s$

are on a j-dimensional linear subspace for $j < n$, we also call the fat points $Z = m_1P_1 + \cdots + m_sP_s$ in general position in $\mathbb{P}^n$. A rational normal curve in $\mathbb{P}^n$ is a curve of degree n with parametric equations

$$x_0 = t^n, x_1 = t^{n-1}u, \ldots, x_{n-1} = tu^{n-1}, u^n.$$

For fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ with $m_1 \geq \cdots \geq m_s$, Catalisano et al. [3] showed formulas to compute $\operatorname{reg}(Z)$ in the following two cases:

If $s \geq 2$ and $P_1, \ldots, P_s$ are on a rational normal curve in $\mathbb{P}^n$ [3, Theorem 7],

$$\operatorname{reg}(R/I) = \max\left\{m_1 + m_2 - 1, \left[(\sum_{i-1}^{s} m_i + n - 2)/n\right]\right\}$$

If $n \geq 3$, $2 \leq s \leq n+2$, $2 \leq m_1 \geq \cdots \geq m_s$ and $P_1, \ldots, P_s$ are in general position in $\mathbb{P}^n$, then

$$\operatorname{reg}(R/I) = m_1 + m_2 - 1.$$

For fat points $Z = m_1P_1 + \cdots + m_{s+2}P_{s+2}$ in $\mathbb{P}^n$ with $P_1, \ldots, P_{s+2}$ not in a linear $(s-1)$-space in $\mathbb{P}^n, s \leq n$, Thien [9, Theorem 3.4] pointed out:

$$\operatorname{reg}(R/I) = T(Z).$$

For equimultiple fat points $Z = mP_1 + \cdots + mP_s$ in $\mathbb{P}^n$ with $m \neq 2$; $P_1, \ldots, P_s$ not in a linear $(r-1)$-space and $s \leq r+3$, Thien and Sinh [11, Theorem 4.5] showed:

$$\operatorname{reg}(R/I) = T(Z).$$

In this paper, we compute regularity index of a set of seven almost equitulple points in the projective space $\mathbb{P}^4$ (Theorem 3.3) and provide an example for the set of equitulple points projective space $\mathbb{P}^3$ which does not attain Segre' upper bound (Example 4.1).

2. Preliminaries

We use the following lemmas which have been proved yet. The first lemma allows us to compute the regularity index where the set of fat points is on a line.

Lemma 2.1 ([4], Corollary 2.3). *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of arbitrary fat points in $\mathbb{P}^n$. Then,*

$$\operatorname{reg}(Z) = m_1 + \cdots + m_s - 1$$

if and only if the points $P_1, \ldots, P_s$ lie on a line.

Let $\{i_1, \ldots, i_r\}$ be the subset of the set of indices $\{1, \ldots, s\}$. We call $Y = m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r}$ the subset of fat points of $Z = m_1P_1 + \cdots + m_sP_s$. The following lemma helps us to compare the regularity index of a subset of a given set of fat points.

Lemma 2.2 ([9], Lemma 3.3). *Let* $X = \{P_1, \ldots, P_s\}$ *be a set of distinct points in* $\mathbb{P}^n$ *and* $m_1, \ldots, m_s$ *be positive integers. Put* $I = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \ldots \cap \wp_s^{m_s}$. *If* $Y = \{P_{i_1}, \ldots, P_{i_r}\}$ *is a subset of* X *and* $J = \wp_{i_1}^{m_{i_1}} \cap \wp_{i_2}^{m_{i_2}} \cap \ldots \cap \wp_{i_r}^{m_{i_r}}$,

$$\operatorname{reg}(R/J) \leq \operatorname{reg}(R/I).$$

This implies that if $Z = m_1P_1 + \cdots + m_sP_s$ is the set of fat points defining by I, and $Y = m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r}$ is the set of fat points defining by J,

$$\operatorname{reg}(Y) \leq \operatorname{reg}(Z).$$

The next two lemmas allow us to compute the regularity index for a given set of points.

Lemma 2.3 ([9], Theorem 3.4). *Let* $P_1, \ldots, P_{s+2}$ *be distinct points not in a linear* $(s-1)$*-space in* $\mathbb{P}^n$, $s \leq n$ *and* $m_1, \ldots, m_s$ *are positive integers. Put* $I = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \ldots \cap \wp_{s+2}^{m_{s+2}}$, $A = R/I$. *Then,*

$$\operatorname{reg}(A) = \max\{T_j | j = 1, \ldots, n\},$$

where

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

if $P_{i_1}, \ldots, P_{i_q}$ *are on a* j*-dimensional linear subspace.*

Lemma 2.4 ([11], Theorem 3.1). *Let* $X = \{P_1, \ldots, P_{s+3}\}$ *be a set of distinct points in a general position on a linear* s*-space, not on a* $(s-1)$*-dimensional linear subspace in* $\mathbb{P}^n$, $s \leq n$, *and* m_i *are positive integers. Let* $Z = m_1P_1 + + \cdots + m_{s+3}P_{s+3}$ *be the set of fat points. Then,*

$$\operatorname{reg}(A) = \max\{T_j | j = 1, \ldots, n\},$$

where

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

$P_{i_1}, \ldots, P_{i_q}$ *are on aj-dimensional linear subspace.*

The following lemma gives the upper bound for the set $n+3$ non-degenerate fat points in $\mathbb{P}^n$.

Lemma 2.5 ([1], Theorem 2.1). *Let* $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ *be a set of* $n+3$ *non-degenerate fat points in* $\mathbb{P}^n$. *Then,*

$$\operatorname{reg}(Z) \leq \max\{T_j | j = 1, \ldots, n\},$$

where

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

$P_{i_1}, \ldots, P_{i_q}$ *are on a* j*-dimensional linear subspace.*

3. The regularity index of a set of seven almost equimultiple points in $\mathbb{P}^4$

In this section, we compute the regularity index of a set of seven almost equimultiple points in $\mathbb{P}^4$.

Lemma 3.1. *Let $X = \{P_1, \dots, P_7\}$ be seven distinct non-degenerate points in $\mathbb{P}^4$ such that there are not $s+3$ points of X on an s-dimensional linear subspace, $s < 4$. Let $3 \leq m_1 \geq \dots \geq m_7$ be positive integers and*

$$Z = m_1P_1 + \cdots + m_7P_7.$$

Put

$$T_j = \max\left\{\left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j}\right] | P_{i_1}, \dots, P_{i_q} \textit{ on a } j\textit{-dimensional linear space}\right\},$$

and

$$T = \max\left\{T_j | j = 1, 2, 3, 4\right\},$$

when if $T = T_1$ or $T = T_2$ then $\operatorname{reg}(Z) = T$.

Proof. Consider the two following cases:

- $T = T_1$: Calling ℓ a line passes through $P_{l_1}, \dots, P_{l_r}, r \leq 3$ such that

$$T_1 = m_{l_1} + \cdots + m_{l_r} - 1.$$

Consider the set of fat points

$$Z_\ell = m_{l_1}P_{l_1} + \cdots + m_{l_r}P_{l_r}.$$

Since Lemma 2.1 and Lemma 2.2 we have $\operatorname{reg}(Z) \geq T = T_1$, since Lemma 2.5 we have $\operatorname{reg}(Z) \leq T = T_1$. Therefore

$$\operatorname{reg}(Z) = T = T_1.$$

- $T > T_1$: when $T = T_2$, since the condition of this Lemma, there are not three points on a line and five points of X on a 2-dimensional linear subspace. Calling γ a 2-dimensional linear subspace passes through $P_{i_1}, \dots, P_{i_q}$, $q \leq 4$, such that

$$T_2 = \left[\frac{m_{i_1} + \cdots + m_{i_q}}{2}\right].$$

Consider the set of fat points

$$Z\gamma = m_{i_1}P_{i_1} + \cdots + m_{i_q}P_{i_q},$$

since Lemma 2.3 we have $\operatorname{reg}(Z_\gamma) = T_2$, by Lemma 2.2 we have $\operatorname{reg}(Z) \geq$ $\geq \operatorname{reg}(Z_\gamma) = T_2 = T$. By Lemma 2.5 we have $\operatorname{reg}(Z) \leq T$. Therefore, we have

$$\operatorname{reg}(Z) = T. \qquad \blacksquare$$

Proposition 3.1. *Let $X = \{P_1, \dots, P_7\}$ be seven distinct non-degenerate points and not in linearly general position in $\mathbb{P}^4$ such that there are not six points of X on a 3-dimensional linear subspace. Let $m \geq 3$ be a positive integer, we consider the set of seven almost equimultiple fat points:*

$$Z = m_1P_1 + m_2P_2 + \cdots + m_7P_7,$$

with $m_1 = m_2 = m_3 = m$, $m_4 = \cdots = m_7 = m + 1$. Put

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

$P_{i_1}, \dots, P_{i_q}$ on a j-dimensional linear subspace and

$$T = \max \Big\{ T_j | j = 1, 2, 3, 4 \Big\},$$

then

$$\operatorname{reg}(Z) = T.$$

Proof. Since there are not six points of X on a 3-dimensional linear subspace, there are not four points of X on a line and five of X on a 2-dimensional linear subspace. We consider the two following cases:

Case 1. There are three points of X on a line. When

$$T_4 = \left[\frac{7m+6}{4}\right], \quad T_3 \leq \left[\frac{5m+4}{3}\right], \quad T_2 \leq 2m + 2, \quad T_1 \geq 3m - 1$$

we have

- $3m - 1 - \dfrac{7m+6}{4} = \dfrac{5m-10}{4} \geq 0$. This implies $3m - 1 \geq \left[\dfrac{7m+6}{4}\right]$.

Therefore

$$T_1 \geq T_4.$$

- $3m - 1 - \dfrac{5m+4}{3} = \dfrac{4m-7}{3} \geq 0$. This implies $3m - 1 \geq \left[\dfrac{5m+4}{3}\right]$.

Therefore

$$T_1 \geq T_3.$$

So,

$$T = \max \Big\{ T_1, T_2 \Big\}.$$

By Lemma 2.1 we have

$$\operatorname{reg}(Z) = T.$$

Case 2. There are not three points of X on a line. When

$$T_4 = \left[\frac{7m+6}{4}\right], \quad T_1 = 2m + 1.$$

Since $2m+1-\dfrac{7m+6}{4}=\dfrac{m-2}{2}\geq 0$, this implies $2m+1\geq\left[\dfrac{7m+6}{4}\right]$ or $T_1\geq T_4$.

SUBCASE 2.1. There are four points of X on a 2-dimensional linear subspace α.

a) If 2-dimensional linear subspace α passes through $Y=\{P_4,P_5,P_6,P_7\}$. We have

$$T_3=\left[\frac{5m+5}{3}\right],\quad T_2=2m+2,$$

therefore,

$$T=T_2.$$

By Lemma 2.1 we have

$$\operatorname{reg}(Z)=T_2=2m+2.$$

b) If 2-dimensional linear subspace α passes through the most three points of $Y=\{P_4,P_5,P_6,P_7\}$. When

$$T_3=\left[\frac{5m+5}{3}\right],\quad T_2\leq\left[\frac{4m+3}{2}\right]=2m+1=T_1$$

we have

- since $2m+1-\dfrac{7m+6}{4}=\dfrac{8m+4-7m-6}{4}=\dfrac{m-2}{4}\geq 0$, apply $2m+1\geq$ $\geq\left[\dfrac{7m+6}{4}\right]$. Therefore,

$$T_1\geq T_4.$$

- since $2m+1-\dfrac{5m+5}{3}=\dfrac{6m+3-5m-5}{3}=\dfrac{m-2}{3}\geq 0$ apply $2m+1\geq$ $\geq\left[\dfrac{5m+5}{3}\right]$. Therefore,

$$T_1\geq T_3.$$

So,

$$T=\max\left\{T_1,T_2,T_3,T_4\right\}=2m+1.$$

By Lemma 2.1 we have

$$\operatorname{reg}(Z)=T=T_1=2m+1.$$

SUBCASE 2.2. There are not four points of X on a 2-dimensional linear subspace β. When

$$T_3\leq\left[\frac{5m+5}{3}\right],\quad T_2=\left[\frac{3m+3}{2}\right]$$

from he proof of Subcase 2.1. b) we have

$$T_3 \leq 2m + 1 = T_1, \quad T_2 = \left[\frac{3m+3}{2} \right] \leq \left[\frac{4m+3}{2} \right] \leq 2m + 1 = T_1.$$

Therefore

$$T = T_1.$$

By Lemma 2.1 we have

$$\operatorname{reg}(Z) = T = T_1. \qquad \blacksquare$$

Proposition 3.2. *Let $X = \{P_1, \ldots, P_7\}$ be seven distinct non-degenerate points and not in linearly general position in $\mathbb{P}^4$. Let $m \geq 3$ be a positive integer, we consider the set of seven almost equimultiple fat points:*

$$Z = m_1 P_1 + m_2 P_2 + \cdots + m_7 P_7,$$

with $m_1 = m_2 = m_3 = m$, $m_4 = \cdots = m_7 = m + 1$. Put

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

if $P_{i_1}, \ldots, P_{i_q}$ on a j-dimensional linear subspace, and

$$T = \max \Big\{ T_j | j = 1, 2, 3, 4 \Big\}.$$

Then

$$\operatorname{reg}(Z) = T.$$

Proof. Since X is not non-degenerate points in $\mathbb{P}^4$ then there are not five points of X on a line. We consider the two following cases:

Case 1. There are four points of X on a line. When

$$T_4 = \left[\frac{7m+6}{4} \right], \quad T_3 \leq \left[\frac{6m+5}{3} \right], \quad T_2 \leq \left[\frac{5m+4}{2} \right], \quad T_1 \geq 4m.$$

So

- since $4m - \dfrac{7m+6}{4} = \dfrac{16m - 7m - 61}{4} = \dfrac{9m-6}{4} \geq 0$, we get $4m \geq$ $\geq \left[\dfrac{7m+6}{4} \right]$, therefore

$$T_1 \geq T_4;$$

- since $4m - \dfrac{6m+5}{3} = \dfrac{6m-5}{3} \geq 0$, we get $4m \geq \left[\dfrac{6m+5}{3} \right]$, therefore

$$T_1 \geq T_3;$$

- since $4m - \dfrac{5m+4}{2} = \dfrac{3m-4}{2} \geq 0$, we get $4m \geq \left[\dfrac{5m+4}{2}\right]$, therefore
$$T_1 \geq T_2.$$

So,
$$T = \max\Big\{T_1, T_2, T_3, T_4\Big\} = T_1.$$

By Lemma 2.1 we have
$$\operatorname{reg}(Z) = T = T_1.$$

Case 2. There are not four points of X on a line. We consider the two following subcases:

Subcase 2.1. There are three points of X on a line ℓ. When
$$T_4 = \left[\frac{7m+6}{4}\right], \quad T_3 \leq \left[\frac{6m+5}{3}\right], \quad T_2 \leq \left[\frac{5m+4}{2}\right], \quad T_1 \geq 3m-1$$

we have

- since $3m-1-\dfrac{7m+6}{4} = \dfrac{5m-10}{4} \geq 0$, so $3m-1 \geq \left[\dfrac{7m+6}{4}\right]$, therefore
$$T_1 \geq T_4;$$
- since $3m-1-\dfrac{6m+5}{3} = \dfrac{3m-8}{3}$ and $m \geq 3$, we get $\dfrac{3m-8}{3} \geq 0$ imply $3m-1 \geq \left[\dfrac{6m+5}{3}\right]$, therefore
$$T_1 \geq T_3,$$

so
$$T = \max\big\{T_1, T_2\big\}.$$

Calling α 2-dimensional linear subspace passes through $P_{i_1}, \dots, P_{i_k}$ of X such that
$$T_2 = \left[\frac{m_{i_1} + \cdots + m_{i_k}}{2}\right].$$

To rate T_2, we consider the two following cases:

(i) α passes through four points of X. When
$$T_2 \leq \left[\frac{4m+4}{2}\right]$$

since $3m-1-\dfrac{4m+4}{2} = \dfrac{2m-6}{2}$ and $m \geq 3$, we get $\dfrac{2m-6}{2} \geq 0$. This implies $3m-1 \geq \left[\dfrac{4m+4}{2}\right]$. Therefore,
$$T_1 \geq T_2.$$

(ii) If α passes through five points of X, α has to contain ℓ, (if not there are seven points on a 3-dimensional linear subspace, contradiction).

(ii$_1$) If ℓ passes through P_1, P_2, P_3,

$$T_1 = 3m - 1, \quad T_2 = \left[\frac{5m+2}{2}\right].$$

Sine $3m - 1 - \dfrac{5m+2}{2} = \dfrac{m-4}{2}$,

- if $m \geq 4$, $3m - 1 \geq \left[\dfrac{5m+2}{2}\right]$, therefore $T_1 \geq T_2$.
- if $m = 3$, $T_1 = 3.3 - 1 = 8$ and $T_2 = \left[\dfrac{5.3+2}{2}\right] = 8$, therefore $T_1 = T_2$.

So

$$T_1 \geq T_2.$$

(ii$_2$) If ℓ passes through two points of $\{P_1, P_2, P_3\}$,

$$T_1 = 3m, \quad T_2 \leq \left[\frac{5m+3}{2}\right].$$

Since $3m - \dfrac{5m+3}{2} = \dfrac{m-3}{2}$ and $m \geq 3$, $\dfrac{m-3}{2} \geq 0$. This implies $3m \geq$ $\geq \left[\dfrac{5m+3}{2}\right]$. Therefore

$$T_1 \geq T_2.$$

(ii$_3$) If ℓ passes through at most one points of $\{P_1, P_2, P_3\}$,

$$T_1 \geq 3m + 1, \quad T_2 \leq \left[\frac{5m+4}{2}\right].$$

Since $3m + 1 - \dfrac{5m+4}{2} = \dfrac{m-2}{2}$ and $m \geq 3$, we get $\dfrac{m-2}{2} \geq 0$. This implies $3m + 1 \geq \left[\dfrac{5m+4}{2}\right]$. Therefore,

$$T_1 \geq T_2.$$

So, we have $T = T_1$. Calling $P_{l_1}, P_{l_2}, P_{l_3}$ three points on a line ℓ, the corresponding multiples are $m_{l_1}, m_{l_2}, m_{l_3}$ such that

$$T_1 = m_{l_1} + m_{l_2} + m_{l_3} - 1.$$

We consider the set of almost equimultiple points

$$Z_\ell = m_{l_1}P_{l_1} + m_{l_2}P_{l_2} + m_{l_3}P_{l_3}.$$

By Lemma 2.1 we have $\operatorname{reg}(Z_\ell) = T_1$, by Lemma 2.2 we have $\operatorname{reg}(Z_\ell) \le \operatorname{reg}(Z)$ and by Lemma 2.5 we have $\operatorname{reg}(Z) \le T = T_1$. So

$$\operatorname{reg}(Z) = T = T_1.$$

Subcase 2.2. There are not three points of X on a line ℓ. In this case there are not six points of X on a 3-linear space. Since Proposition 3.1 we have

$$\operatorname{reg}(Z) = T.$$

So, we consider for in case, there are six points of X on a 3-linear space.

(+) There are four points of X on a 2-linear space. Then $T_3 \le \left[\dfrac{6m+5}{3}\right] \le T_2 \le \left[\dfrac{4m+1}{2}\right]$. So

$$T = \max\{T_1, T_2\}.$$

From Lemma 3.1 we get

$$\operatorname{reg}(Z) = T.$$

(+) If there are not three points on a 2-linear space. Since $T_4 \le T_1$ thus

$$T = \max\{T_1, T_2, T_3\}.$$

We consider the set of six almost equimultiple points

$$Z_\gamma = m_{i_1} P_{i_1} + \cdots + m_{i_6} P_{i_6}.$$

Then, the set of $P_{i_1}, \ldots, P_{i_6}$ in a general position. By Lemma 2.4 we have

$$\operatorname{reg}(Z_\gamma) = T.$$

By Lemma 2.2 we have $\operatorname{reg}(Z_\gamma) \le \operatorname{reg}(Z)$ and by Lemma 2.5 we have $\operatorname{reg}(Z) \le$ $\le T$, so

$$\operatorname{reg}(Z) = T. \qquad \blacksquare$$

Theorem 3.3. *Let* $X = \{P_1, \ldots, P_7\}$ *be seven distinct non-degenerate points in* $\mathbb{P}^4$. *Let* $m \ge 3$ *be a positive integer. We consider the set of seven almost equimultiple fat points:*

$$Z = m_1 P_1 + m_2 P_2 + \cdots + m_7 P_7$$

with $m_1 = m_2 = m_3 = m$, $m_4 = \cdots = m_7 = m + 1$. *Put*

$$T_j = \max \left[\frac{\sum\limits_{l=1}^{q} m_{i_l} + j - 2}{j} \right]$$

$P_{i_1}, \ldots, P_{i_q}$ *on a* j*-dimensional linear subspace and*

$$T = \max\Big\{T_j | j = 1, 2, 3, 4\Big\}.$$

Then

$$\operatorname{reg}(Z) = T.$$

Proof. We consider the following cases:

Case 1. If X is in general position in $\mathbb{P}^4$, by Lemma 2.4 we have

$$\operatorname{reg}(Z) = T.$$

Case 2. If X is not in general position in $\mathbb{P}^4$, then we consider the two following subcases:

Subcase 2.1. There are not six points of X on a 3-dimensional linear subspace. Then by Proposition 3.1 we have

$$\operatorname{reg}(Z) = T.$$

Subcase 2.2. There are six points of X on a 3-dimensional linear subspace. Then by Proposition 3.2 we have

$$\operatorname{reg}(Z) = T. \qquad \blacksquare$$

4. A case about the regularity index of the set of fat points such that Segre's bound is not attained

In this section, we give a case about the regularity index of set of fat points such that Segre's bound which is not attained.

Lemma 4.1 ([3], Lemma 1). *Let* $P_1, \ldots, P_r, P$ *be distinct points in* $\mathbb{P}^n$ *and* $\wp$ *be the defining ideals in* R *corresponding to the point* P. *If* $m_1, \ldots, m_r, a$ *are positive integers,* $J := \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ *and* $I = J \cap \wp^a$, *then*

$$\operatorname{reg}(R/I) = \max\{a - 1, \operatorname{reg}(R/J), \operatorname{reg}(R/J + \wp^a)\}.$$

Note that $R/(J + \wp^a)$ *is a zero-dimensional graded ring algebra, the regularity index* $\operatorname{reg}(R/(J + \wp^a))$ *is the least integer such that* $[R/(J + \wp^a)]_t = 0$. *To estimate* $\operatorname{reg}(R/(J + \wp^a))$ *we shall use the following lemma.*

Lemma 4.2 ([3], Lemma 3). *Let* $P_1, \ldots, P_r, P$ *be distinct points in* $\mathbb{P}^n$ *and* $\wp$ *be the defining ideals in* R *corresponding to the point* P. *If* $m_1, \ldots, m_r, a$ *are positive integers,* $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ *and* $\wp = (x_1, \ldots, x_n)$ *then* $\operatorname{reg}(R/(J + +\wp^a)) \le b$ *if and only if* $x_0^{b-i} M \in J + \wp^{i+1}$ *for every monomial* M *of degree* i *in* $x_1, \ldots, x_n$, $i = 0, 1, \ldots, a - 1$.

Example 4.1. Let α, β be two 2-linear space in $\mathbb{P}^3$. Let $P_1, \ldots, P_9$ be nine distinct points such that $P_1, P_2, P_3, P_4 \in \alpha \setminus \beta$ and P_1, P_2, P_3, are on a line ℓ, $P_5, P_6, P_7, P_8 \in \beta \setminus \alpha$, $P_4, P_9 \notin \alpha \cup \beta$. Let m be a positive integers. Then $Z = mP_1 + \cdots + mP_9$ is a set of nine non-degenerate equimultiple fat points in $\mathbb{P}^3$. We have

$$T_1 = 3m - 1, \quad T_2 = \left[\frac{4m}{2}\right] = 2m, \quad T_3 = \left[\frac{9m+1}{3}\right] = 3m.$$

Let $\wp_j$ be the homogeneous prime ideal in $R = K[x_0, x_1, x_2, x_3]$ corresponding to be point P_j $j = 1, \ldots, 9$. Choose $P_9 = (1, 0, 0, 0)$, then $\wp_9 = (x_1, x_2, x_3)$. Put $J = \wp_1^m \cap \cdots \cap P_8^m$, $I = J \cap \wp_9^m$. Then R/I is the coordinate ring of Z, so $\operatorname{reg}(Z) = \operatorname{reg}(R/I)$, the Castelnuovo–Mumford regularity index of the ring is R/I. By Lemma 4.1 we get

$$\operatorname{reg}(R/I) = \max\{m - 1, \operatorname{reg}(R/J), \operatorname{reg}(R/J + \wp_6^m)\}.$$

Since ℓ is a line passing through three points P_1, P_2, P_3, then there are a 2-linear space β passing through four points P_1, P_2, P_3, P_4 we have $\beta^m \in$ $\in \cap_{i=1}^4 \wp_i^m$. Since α is a 2-linear space passing through four points P_4, P_6, P_7, P_8 we have $\alpha^m \in \cap_{j=5}^8 = \wp_j^m$. Hence $\alpha^m \beta^m \in \cap_{l=1}^8 \wp_l^m$. This implies that for every monomial M of degree i in x_1, x_2, x_3, $i = 0, 1, \ldots, m-1$. We have

$$\alpha^m \beta^m M \in J.$$

Since α, β are a 2-dimensional linear subspace avoiding $P_9 = (1, 0, 0, 0)$, we can write $\alpha = x_0 + g_\alpha$, $\beta = x_0 + g_\beta$ for some g_α, $g_\beta \in \wp_9$. Thus we have

$$(x_0 + g_\alpha)^m (x_0 + g_\beta)^m \in J.$$

Moreover, since g_α, $g_\beta \in \wp_9$ and $M \in \wp_9^i$ we get $x_0^{2m} M \in J + \wp_9^{i+1}$ for $i = 0, \ldots, m-1$. This implies that

$$x_0^{3m-1-i} M \in J + \wp_9^{i+1}$$

for $i = 0, \ldots, m-1$. By Lemma 4.2 we get

$$\operatorname{reg}(R/(J + \wp_9^m)) \leq 3m - 1.$$

Let $U = mP_1 + \cdots + mP_8$. We have

$$T_1(U) = 3m - 1, \quad T_2(U) = \left[\frac{4m}{2}\right] = 2m \leq 3m - 1,$$

$$T_3(U) = \left[\frac{8m+1}{3}\right] \leq 3m - 1.$$

By Lemma 2.1 we have $\operatorname{reg}(U) = 3m - 1$. Since R/J is the homogenous coordinate ring of U, thus

$$\operatorname{reg}(R/J) = \operatorname{reg}(U) = 3m - 1.$$

Therefore,

$$\operatorname{reg}(Z) = \operatorname{reg}(R/I) = \max\{m-1, 3m-1, \operatorname{reg}(R/(J+\wp_9^m))\} = 3m-1 < T(Z).$$

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