WALSH–FEJÉR MEANS AND WEIGHTED HARDY SPACES

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Abstract. In this paper we consider the A_p weights and weighted dyadic Hardy spaces. We investigate the convergence of Fejér means of Walsh–Fourier series. We prove that, under some conditions, the maximal operator of the Fejér means is bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$. This implies almost everywhere convergence of the Fejér means.

1. Introduction

We investigate the convergence of the Fejér means of Walsh–Fourier series. It was proved by Fine [3] that the Fejér means $\sigma_n(f)$ converge to f almost everywhere if $f \in L_1$. Schipp [11] obtained the same result by proving the weak type inequality of the maximal operator σ_* of the Fejér means. Next Fujii [4] showed that σ_* is bounded from the dyadic Hardy space H_1 to L_1 (see also Schipp and Simon [12]). Later the author [18] proved that σ_* is bounded from H_p to L_p for $1/2 < p < \infty$.

In this paper, we generalize these results to weighted spaces. We introduce the concept of A_p weights. With the help of the weighted $L_p(w)$-norm of the dyadic maximal function, we introduce the weighted dyadic Hardy spaces and

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give their atomic decomposition. Using the atomic decomposition, we verify a sufficient condition for an operator to be bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$.

In the next section, we prove that the maximal Fejér operator σ_* is bounded from $H_p(w)$ to $L_p(w)$ if w satisfies some conditions. Finally, we obtain the almost everywhere and norm convergence of the Fejér means to the function.

Throughout this paper, let C denote a positive constant which may vary from line to line, but is independent of the main parameters. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \le Cg$. For any subset E of $[0,1)$, we use $\mathbf{1}_E$ to denote its characteristic function. For $1 \le p \le \infty$, the dual index p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Walsh system

In this paper let w be a strictly positive integrable weight function. The weighted space $L_p(w)$ is equipped with the norm (or quasinorm)

$$\|f\|_{L_p(w)} := \left(\int_0^1 |f|^p w \, d\lambda \right)^{1/p} \qquad (0 < p < \infty)$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. For a set I, we use the notation

$$w(I) := \int_I w \, d\lambda. \tag{2.1}$$

The Rademacher functions are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (0 \le n_k < 2).$$

By a dyadic interval, we mean one of the form $[k2^{-n},(k+1)2^{-n})$ for some $k,n\in\mathbb{N}$ and $0\le k<2^n$. For any given $n\in\mathbb{N}$ and $x\in[0,1)$, denote by $I_n(x)$ the dyadic interval of length 2^{-n} which contains x. For any $n\in\mathbb{N}$, the σ-algebra generated by the dyadic intervals $\{I_n(x):x\in[0,1)\}$ is denoted by $\mathcal{F}_n$. It is clear that, for any $n\in\mathbb{N}$, r_n is $\mathcal{F}_{n+1}$ measurable.

Let us denote by $\mathbb{E}_n$ the conditional expectation operator with respect to $\mathcal{F}_n$ and the Lebesgue measure. A sequence $(f_n, n\in\mathbb{N})$ is called a dyadic martingale if f_n is $\mathcal{F}_n$ measurable and $\mathbb{E}_{n-1}(f_n)=f_{n-1}$ $(n\in\mathbb{N})$. If $f\in L^1[0,1)$, for any $n\in\mathbb{N}$, the number

$$\widehat{f}(n):=\int_0^1 f w_n\,d\lambda$$

is called the nth Walsh–Fourier coefficient of f. We can extend this definition to martingales as follows. If $f:=(f_k)_{k\in\mathbb{N}}$ is a martingale, then

$$\widehat{f}(n):=\lim_{k\to\infty}\int_0^1 f_k w_n\,d\lambda \qquad (n\in\mathbb{N}).$$

3. Weighted dyadic Hardy spaces

The definition of A_p weights for martingales was introduced by Izumisawa and Kazamaki in [8] as follows.

Definition 3.1. Let $q\in[1,\infty)$. A weight w is said to satisfy the A_q condition (briefly $w\in A_q$) if there exists a positive constant K such that, when $q\in(1,\infty)$,

$$\sup_{n\in\mathbb{N}}\mathbb{E}_n(w)\left[\mathbb{E}_n\left(w^{-\frac{1}{q-1}}\right)\right]^{q-1}\le K$$

and, when $q=1$,

$$\sup_{n\in\mathbb{N}}\frac{\mathbb{E}_n(w)}{w}\le K.$$

w is said to satisfy A_∞ (briefly $w\in A_\infty$) if $w\in A_q$ for some $q\in[1,\infty)$.

Now, we give the following definition of the reverse Hölder inequality.

Definition 3.2. Let $q\in(1,\infty]$. A weight w is said to satisfy the R_q condition (briefly $w\in R_q$) if there exists a positive constant K such that, when $q\in(1,\infty)$,

$$\sup_{n\in\mathbb{N}}\left[\mathbb{E}_n\left(w^q\right)\right]^{\frac{1}{q}}\left[\mathbb{E}_n(w)\right]^{-1}\le K$$

and, when $q=\infty$,

$$\sup_{n\in\mathbb{N}} w\,[\mathbb{E}_n(w)]^{-1} \leq K.$$

w is said to satisfy R_1 (briefly $w \in R_1$) if $w \in R_q$ for some $q \in (1,\infty]$.

For any $p,q \in (1,\infty)$ with $p \leq q$, we have $A_1 \subset A_p \subset A_q \subset A_\infty$ and $R_\infty \subset R_q \subset R_p \subset R_1$. So we can introduce the indices $q(w)$ and $r(w)$ as follows

$$q(w) := \inf\{q \in [1,\infty) : \; w \in A_q\}$$

and

$$r(w) := \sup\{q \in (1,\infty] : \; w \in R_q\}.$$

Let $d\widehat{\lambda} := wd\lambda$. By Long [9, p. 229], we find that, for any measurable set A,

$$\lambda(A) = 0 \Longleftrightarrow \widehat{\lambda}(A) = 0.$$

The following lemmas can be found in Doléans-Dade and Meyer [2] and Long [9] (see also Xie et al. [22]).

Lemma 3.1. *Let $J \subset I$ two dyadic intervals. If $1 \leq q < \infty$ and $w \in A_q$, then*

$$\left(\frac{\lambda(J)}{\lambda(I)}\right)^q \lesssim \frac{w(J)}{w(I)}.$$

If $1 < q \leq \infty$ and $w \in R_q$, then

$$\left(\frac{w(J)}{w(I)}\right)^{q'} \lesssim \frac{\lambda(J)}{\lambda(I)}.$$

Note that $w(I)$ is defined in (2.1). If $w \in A_q$, then w satisfies the reverse Hölder's inequality and $w \in A_{q-\varepsilon}$ for some $\varepsilon > 0$.

Lemma 3.2.

(i) *If $1 \leq q < \infty$ and $w \in A_q$, then there exists $\epsilon > 0$ such that $w \in R_{1+\epsilon}$.*

(ii) *If $1 < q < \infty$ and $w \in A_q$, then there exists $\varepsilon > 0$ such that $w \in A_{q-\varepsilon}$.*

We define the maximal operator of the dyadic martingale $f = (f_n, n \in \mathbb{N})$ by

$$f^* := \sup_{n\in\mathbb{N}} |f_n|.$$

The weighted dyadic Hardy space $H_p(w)$ $(0 < p < \infty)$ consists of all dyadic martingales $f = (f_n, n \in \mathbb{N})$ for which

$$\|f\|_{H_p(w)} := \|f^*\|_{L_p(w)} < \infty.$$

For $w = 1$, we write again H_p. It is known (see e.g. Weisz [19]) that

$$H_p \sim L_p \qquad (1 < p \leq \infty),$$

where $\sim$ denotes the equivalence of spaces and norms. For weighted Hardy spaces, we can formulate this theorem as follows (see Long [9], Xie et al. [22], Weisz [21]).

Theorem 3.1.

(i) *If* $w \in A_\infty$ *and* $q(w) < p < \infty$, *then*

$$H_p(w) \sim L_p(w).$$

(ii) *If* $w \in A_1$ *and* $f \in L_1(w)$, *then*

$$\sup_{\rho>0} \rho\, w(f^* > \rho) \leq C \|f\|_{L_1(w)} .$$

Here $w(f^* > \rho)$ denotes $w(\{x \in [0,1) : f^*(x) > \rho\})$ (see (2.1)). The atomic decomposition provides a useful characterization of Hardy spaces. First we introduce the concept of atoms.

Definition 3.3. A bounded function a is a weighted dyadic p-atom if there exists a dyadic interval $I \subset [0,1)$ such that

(a) $\operatorname{supp} a \subset I$,

(b) $\|a\|_\infty \leq w(I)^{-1/p}$,

(c) $\int_I a(x)\, dx = 0$.

The atomic decomposition of the Hardy space $H_p(w)$ means that every function (more exactly, martingale) from the Hardy space can be decomposed into the sum of atoms. The following theorem was shown in Weisz [21].

Theorem 3.2. *Let* $0 < p \leq 1$ *and* $w \in A_\infty$. *A martingale* f *is in* $H_p(w)$ *if and only if there exist a sequence* (μ_k) *of real numbers and a sequence* (a^k) *of weighted dyadic p-atoms such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k a_n^k \quad \text{a.e.}, \quad n \in \mathbb{N}.$$

Moreover,

$$\|f\|_{H_p(w)} \sim \inf \left(\sum_{k\in\mathbb{Z}} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f *as above.*

The next result gives a sufficient condition for an operator to be bounded from $H_p(w)$ to $L_p(w)$ (see Weisz [21]). For the unweighted case see Weisz [19] and, for $p_0 = 1$, Schipp, Wade, Simon and Pál [13] and Móricz, Schipp and Wade [10].

Let X be a martingale space and Y a measurable function space. An operator $T: \ X \to Y$ is said to be a σ-sublinear operator if, for any complex number α,

$$\left| T\left(\sum_{k=1}^{\infty} f_k\right)\right| \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha||T(f)|.$$

Theorem 3.3. *Let $w \in A_\infty$. Suppose that V is a σ-sublinear operator and*

$$\int_{[0,1)\setminus I} |Va|^{p_0} w \, d\lambda \leq C_{p_0} \tag{3.1}$$

for all weighted dyadic p_0-atoms a and for some fixed $0 < p_0 \leq 1$, where the interval I is the support of the atom. If V is bounded from $L_{p_1}(w)$ to $L_{p_1}(w)$ for some $1 < p_1 \leq \infty$, then

$$\|Vf\|_{L_p(w)} \leq C_p \|f\|_{H_p(w)} \qquad (f \in H_p(w))$$

for all $p_0 \leq p \leq p_1$. Moreover, if $w \in A_1$ and $p_0 < 1$, then the operator V is of weak type $(1,1)$, i.e., if $f \in L_1(w)$, then

$$\sup_{\rho>0} \rho \, w(|Vf| > \rho) \leq C \|f\|_{L_1(w)}.$$

4. Fejér summability of Walsh–Fourier series

Denote by $s_n f$ the nth partial sum of the Walsh–Fourier series of a martingale f, namely,

$$s_n(f) := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \qquad (n \in \mathbb{N}).$$

If $f \in L_1$, then

$$s_n(f)(x) = \int_0^1 f(t) D_n(x \dot{+} t) \, dt \qquad (n \in \mathbb{N}),$$

where $\dot{+}$ denotes the dyadic addition and

$$D_n(u) := \sum_{k=0}^{n-1} w_k(u)$$

is the nth Walsh-Dirichlet kernel (see, for example, Schipp, Wade, Simon and Pál [13]).

It is easy to see that, for any martingale $f = (f_n)$,

$$s_{2^n}(f) = f_n \qquad (n \in \mathbb{N}).$$

Hence, by the martingale convergence theorem, we know that, for $1 \leq p < \infty$ and $f \in L_p$,

$$\lim_{n\to\infty} s_{2^n}(f) = f \qquad \text{in the } L_p\text{-norm}.$$

This result was generalized (see e.g. Schipp et al. [13, Theorem 4.1]). More precisely, it was proved that, for any $1 < p < \infty$ and $f \in L_p$,

$$\lim_{n\to\infty} s_n(f) = f \qquad \text{in the } L_p\text{-norm}.$$

The generalization of this theorem to the $L_p(w)$ spaces can be found in [21, 22].

Theorem 4.1. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n\in\mathbb{N}} \|s_n(f)\|_{L_p(w)} \leq C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Corollary 4.1. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n\to\infty} s_n(f) = f \qquad \textit{in the } L_p(w)\textit{-norm.}$$

The inequality of Theorem 4.1 and the convergence in Corollary 4.1 do not hold for $p = 1$ or $p = \infty$. With the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [16], Butzer and Nessel [1], Trigub and Belinsky [17], Grafakos [7] and Weisz [19, 20] and the references therein.

In this paper we consider the Fejér means defined by

$$\sigma_n(f)(x) := \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) = \sum_{j=0}^{n}\left(1-\frac{j}{n}\right)\widehat{f}(j)w_j(x) = \int_0^1 f(x-u)K_n(u)\,du,$$

where the Fejér kernels are given by

$$K_n(u) := \sum_{j=0}^{n}\left(1-\frac{j}{n}\right)w_j(u) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(u).$$

It is known (see [13, Theorem 1.16]) that

$$D_{2^k}(x) = \begin{cases} 2^k & \text{when } x \in [0, 2^{-k}), \\ 0 & \text{when } x \in [2^{-k}, 1) \end{cases}$$

and

$$|K_n(x)| \le \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} \left[D_{2^i}(x) + D_{2^i}\left(x \dot{+} 2^{-j-1}\right) \right], \tag{4.1}$$

where $x \in [0,1)$, $n \in \mathbb{N}$ and N is a positive integer such that $2^{N-1} \le n < 2^N$.

Theorem 4.1 and Corollary 4.1 imply the next results.

Corollary 4.2. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n\in\mathbb{N}} \|\sigma_n(f)\|_{L_p(w)} \le C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Corollary 4.3. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n\to\infty} \sigma_n(f) = f \qquad \text{in the } L_p(w)\text{-norm.}$$

Now let us investigate maximal operator of the Fejér means:

$$\sigma_*(f) := \sup_{n\in\mathbb{N}} |\sigma_n(f)|.$$

Applying Theorem 3.3, we can generalize the previous results to the $L_p(w)$ spaces and to the maximal operator.

Theorem 4.2. *Suppose that $w \in A_\infty$ is a weight and*

$$\max\left\{ \frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'} \right] \right\} < 1.$$

If

$$\max\left\{ \frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'} \right] \right\} < p < \infty, \tag{4.2}$$

then, for any $f \in H_p(w)$,

$$\|\sigma_*(f)\|_{L_p(w)} \le C \|f\|_{H_p(w)}.$$

Moreover, if $w \in A_1$ and $f \in L_1(w)$, then

$$\sup_{\rho>0} \rho\, w(\sigma_*(f) > \rho) \le C \|f\|_{L_1(w)}.$$

Proof. For any $f \in L_\infty(w)$,

$$\|\sigma_*(f)\|_{L_\infty(w)} \le \|f\|_{L_\infty(w)} \sup_{n\in\mathbb{N}} \|K_n\|_1 = \|f\|_{L_\infty(w)} . \tag{4.3}$$

Note that $\|f\|_{L_\infty(w)} = \|f\|_{L_\infty}$.

In addition to the conditions of the theorem, suppose that $p \le 1$. We have to prove (3.1) for $\sigma_*(a)$, for $p_0 = p$ and for all weighted dyadic p-atoms a.

Denote the support of a by I and let $\lambda(I) = 2^{-K}$ $(K \in \mathbb{N})$. Without loss of generality we can suppose that $I = [0, 2^{-K})$. It is easy to see that $\widehat{a}(n) = 0$ if $n < 2^K$, so, in this case, $\sigma_n(a) = 0$. Therefore we assume that $n \ge 2^K$.

Let $x \in I^c := [0,1) \setminus I$. If $j \ge K$, then $x \dotplus 2^{-j-1} \notin I$. Hence, for $x \notin I$ and $i \ge j \ge K$, we have

$$a(t) D_{2^i}(x \dotplus t) = a(t) D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) = 0.$$

If $2^N > n \ge 2^{N-1}$, then $N-1 \ge K$. Henceforth, for $x \notin I$,

$$\begin{aligned}
|\sigma_n(a)(x)| &\le \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} \int_I |a(t)| \left(D_{2^i}(x \dotplus t) + D_{2^i}(x \dotplus t \dotplus 2^{-j-1})\right) dt \le \\
&\le C w(I)^{-1/p} 2^{-K} \sum_{j=0}^{K-1} 2^j \sum_{i=j}^{K-1} \int_I \left(D_{2^i}(x \dotplus t) + D_{2^i}(x \dotplus t \dotplus 2^{-j-1})\right) dt + \\
&\quad + C w(I)^{-1/p} \sum_{j=0}^{K-1} 2^j \sum_{i=K}^{\infty} 2^{-i} \int_I \left(D_{2^i}(x \dotplus t) + D_{2^i}(x \dotplus t \dotplus 2^{-j-1})\right) dt =: \\
&=: A_1(x) + A_2(x).
\end{aligned}$$

Note that the right hand side is independent of n. Using (4.1), we can verify that for $x \notin I$,

$$\int_I D_{2^i}(x \dotplus t \dotplus 2^{-j-1})\, dt = 2^{i-K} \mathbf{1}_{[2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})}(x)$$

if $j \le i \le K-1$,

$$\int_I D_{2^i}(x \dotplus t)\, dt = 2^{i-K} \mathbf{1}_{[2^{-K}, 2^{-i})}(x)$$

if $i \in \mathbb{N}$ and

$$\int_I D_{2^i}(x \dotplus t \dotplus 2^{-j-1})\, dt = \mathbf{1}_{[2^{-j-1}, 2^{-j-1} \dotplus 2^{-K})}(x)$$

if $i \geq K$. Therefore, for $x \notin I$,

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim$$

$$\lesssim 2^{-2Kp} w(I)^{-1} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=j}^{K-1} 2^{ip} \Big[w([2^{-K}, 2^{-i})) + w([2^{-j-1}, 2^{-j-1} + 2^{-i})) \Big].$$

Let L, J, K three dyadic intervals such that $L, J \subset K$. Moreover, let $\tau > q(w)$ and $\rho < r(w)$. By Lemma 3.1, we conclude that

$$\frac{w(L)}{w(J)} = \frac{w(K)}{w(J)} \frac{w(L)}{w(K)} \lesssim \left[\frac{\lambda(K)}{\lambda(J)}\right]^{\tau} \left[\frac{\lambda(L)}{\lambda(K)}\right]^{\frac{1}{\rho'}}. \tag{4.4}$$

Let $L = [2^{-K+s}, 2^{-i})$ or $L = [2^{-j-1}, 2^{-j-1} + 2^{-i})$, $J = I$ and $K = [0, 2^{-j}]$. From this it follows that

$$\frac{w([2^{-K}, 2^{-i}))}{w(I)} + \frac{w([2^{-j-1}, 2^{-j-1} + 2^{-i}))}{w(I)} \lesssim 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}},$$

where $\tau > q(w)$ and $\rho < r(w)$. Consequently,

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim 2^{-2Kp} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=j}^{K-1} 2^{ip} 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}} \lesssim$$

$$\lesssim 2^{K(\tau-2p)} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \sum_{i=j}^{K-1} 2^{i(p-1/\rho')}.$$

If $p < \frac{1}{r(\varphi_1)'}$, then we can choose ρ near enough to $r(w)$ such that $p < \frac{1}{\rho'} < \frac{1}{r(w)'}$. Since $p > q(w)/2$, we can choose τ such that $p > \tau/2$. Then

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim 2^{K(\tau-2p)} \sum_{j=0}^{K-1} 2^{j(2p-\tau)} \lesssim C.$$

On the other hand, if $p \geq \frac{1}{r(\varphi_1)'}$, then we can choose ρ such that $\frac{1}{\rho'} < p$. Since $p > q(w) - \frac{1}{r(w)'}$, we can choose τ and ρ such that $p > \tau - \frac{1}{\rho'}$. Then

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim 2^{K(\tau-p-1/\rho')} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \lesssim C.$$

Similarly, for $x \notin I$,

$$\int_{I^c} A_2(x)^p w(x)dx \lesssim w(I)^{-1} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} w([2^{-j-1}, 2^{-j-1} + 2^{-K})).$$

Using (4.4) for $L = [2^{-j-1}, 2^{-j-1} + 2^{-K})$, $J = I$ and $K = [0, 2^{-j}]$, we have

$$\frac{w([2^{-j-1}, 2^{-j-1} + 2^{-K}))}{w(I)} \lesssim 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}},$$

where $\tau > q(w)$ and $\rho < r(w)$. Therefore,

$$\int_{I^c} A_2(x)^p w(x)dx \lesssim \sum_{j=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}} \lesssim$$

$$\lesssim 2^{-K(p-\tau+1/\rho')} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \lesssim C.$$

In the last step we have used that we can choose τ and ρ such that $p > \tau - \frac{1}{\rho'}$ because of the condition $p > q(w) - \frac{1}{r(w)'}$.

Combining the above inequalities, we can establish that

$$\int_{I^c} |\sigma_*(a)|^p w \, d\lambda \leq C_p.$$

Now Theorem 3.3 finishes the proof for

$$\max\left\{\frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'}\right]\right\} < p \leq 1.$$

For $1 < p < \infty$, the theorem follows from (4.3) and from interpolation. ■

The first inequality of Theorem 4.2 was proved by Fujii [4] for $w = p = 1$ (see also Schipp and Simon [12]) and for $w = 1$ and $1/2 < p < \infty$ by the author [18, 19]. Unfortunately, the second condition of (4.2) is missing in [21]. Theorem 4.2 implies the following convergence results.

Corollary 4.4. *Besides the conditions of Theorem 4.2, if $f \in H_p(w)$, then $\sigma_n(f)$ converges almost everywhere on $[0, 1)$ as well as in the $L_p(w)$-norm as $n \to \infty$. Moreover, if $w \in A_1$ and $f \in L_1(w)$, then*

$$\lim_{n\to\infty} \sigma_n(f) = f \qquad a.e.$$

Note that $q(w) \geq 1$. If $w = 1$ and $p \leq 1/2$, then σ_* is not bounded anymore (see Simon and Weisz [15], Simon [14] and Gát and Goginava [5, 6]). For integrable functions and for the unweighted case, the convergence was proved by Fine [3] and Schipp [11].

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