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ON THE EQUATION $F(n^2+1) = bF(n^2-1) + c$

Imre Kátai, Bui Minh Mai Khanh and Bui Minh Phong

(Budapest, Hungary)

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Abstract. Under a suitable conjecture concerning prime numbers, we provide a complete solution to the equation

$$F(n^2+1) = bF(n^2-1) + c \quad \text{for every} \quad n \in \mathbb{N}, \quad n > 1,$$

where $F : \mathbb{N} \to \mathbb{C}$ is a completely multiplicative function and $b, c \in \mathbb{C}, c \neq 0$.

1. Introduction

Let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of primes, positive integers, integers and complex numbers, respectively. Let $\mathbb{N}_1 := \mathbb{N} \setminus \{1\}$. We denote by $\mathcal{M}$ $(\mathcal{M}^*)$ the set of all complex-valued multiplicative (completely multiplicative) functions, respectively. For each $a \in \mathbb{Z}$ and $p \in \mathcal{P}$ let $(\frac{a}{p})$ be the Legendre symbol. Let $P(m)$ be the largest prime divisor of $m \in \mathbb{N}$ and let p_k denote the k-th prime number. For each $k \in \mathbb{N}$ we denote by $\mathcal{F}_k$ the set of all arithmetical functions $F : \mathbb{N} \to \mathbb{C}$ such that $F(n^2+1) = k$ for every $n \in \mathbb{N}$.

Let $\mathbb{E}(n) = 1$ and $\mathbb{I}(n) = n$ for every $n \in \mathbb{N}$.

The problem of characterising the identity function as a multiplicative arithmetical function satisfying certain equations has been studied by several authors. C. Spiro [4] proved that if $f \in \mathcal{M}$ satisfies the relations

$$f(p+q) = f(p) + f(q) \quad \text{for every} \quad p, q \in \mathcal{P}$$

and $f(p_0) \neq 0$ for some $p_0 \in \mathcal{P}$, then f is the identity function.

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In [1], we gave all solutions of the equation

$$F(n^2+m^2+k) = H(n) + H(m) + K \quad \text{for every} \quad n \in \mathbb{N},$$

where $k \in \mathbb{N}$ is the sum of two fixed squares, $K \in \mathbb{C}$ and F, H are completely multiplicative functions.

The equation

$$G(n) = F(n^2-1) + D$$

was completely solved in [2], where $G, F \in \mathcal{M}^*$ and $D \in \mathbb{C}$.

Now we turn to the equation

$$F(Q(n)) = bF(Q(n)-\ell) + c,$$

where $\ell \in \mathbb{N},\ b, c \in \mathbb{C}$ and $Q(x) \in \mathbb{Z}[x]$. The special case

$$(Q(n), \ell, b, c) = \{n^3, 1, 1, D\}$$

was solved in [3].

In this note we consider the case $Q(n) = n^2+1, \ell = 1$ and $b, c \in \mathbb{C}, c \neq 0$. Unfortunately, we can solve the equation only under the following conjecture:

Conjecture 1. *For every prime $p > p_{93} = 487$, there exists a positive integer m such that $p \mid m^2-1$ and*

$$P\left(\frac{(m^2-1)(m^2+1)}{p}\right) < p.$$

We note that, with the help of Maple, we have verified that Conjecture 1 holds for every p_k with $93 < k \leq 10^6$. In the range $k \in \{1, \dots, 93\}$, we could not verify Conjecture 1 for the primes p_k, where

$$k \in \mathcal{F} :=$$
$$:= \{1, 2, 3, 4, 5, 6, 11, 13, 14, 15, 16, 18, 24, 25, 28, 29, 30, 33, 39, 56, 67, 74, 93\}.$$

Define the sets

$$\mathcal{K} := \left\{ p \in \mathcal{P} \;\middle|\; \left(\frac{-1}{p}\right) = \left(\frac{-7}{p}\right) = -1 \right\} \text{ and } \mathcal{H} = \mathcal{P} \setminus \mathcal{K}.$$

It is easy to show that

$$\mathcal{K} = \left\{ p \in \mathcal{P} \;\middle|\; p \equiv t \pmod{28}, \text{ where } t \in \{3, 19, 27\} \right\}.$$

In fact, for the proof of our main result, we require the following weaker conjecture.

Conjecture 2. *For every prime $p > p_{39} = 167$ with $p \in \mathcal{K}$, there exists a positive integer ℓ such that $p \mid \ell^2 - 1$ and*

$$P\left(\frac{(\ell^2-1)(\ell^2+1)}{p}\right) < p.$$

We have verified Conjecture 2 for every $p = p_k$ with $39 < k \le 10^6$, and we have checked that in the range $k \in \{1, \dots, 39\}$, Conjecture 2 holds for all $p = p_k$ except:

$$p \in \{p_2 = 3,\ p_{11} = 31,\ p_{15} = 47,\ p_{39} = 167\}.$$

In this paper we prove the following result.

Theorem 1. *Assume that $b, c \in \mathbb{C}$, $c \neq 0$ and the function $F \in \mathcal{M}^*$ satisfies the relation*

$$F(n^2+1) = bF(n^2-1) + c \quad \textit{for every} \quad n \in \mathbb{N}_1. \tag{1.1}$$

Then the following assertions hold:

(a) *If $b = 0$, then $c = 1$ and $F \in \mathcal{F}_1$, furthermore*

$$F(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P}, p \equiv 1 \pmod 4.$$

(b) *If $bc \neq 0$ and Conjecture 2 holds, then*

$$(F,\ b,\ c) \in \{(\mathbb{E},\ b,\ -b+1), (\mathbb{I},\ 1,\ 2)\}.$$

2. Lemmas

We first observe from the assumption $F \in \mathcal{M}^*$ and (1.1) that

$$E_n := F(n^2+1) - bF(n-1)F(n+1) - c = 0 \quad \text{for every} \quad n \in \mathbb{N}_1. \tag{2.1}$$

Since

$$(n^2+n+1)^2 + 1 = (n^2+1)\big((n+1)^2+1\big) \quad \text{for every} \quad n \in \mathbb{N}, \tag{2.2}$$

it follows from (2.1) and (2.2) that

$$bF(n^2+n)F(n^2+n+2) + c = \Big(bF(n-1)F(n+1) + c\Big)\Big(bF(n)F(n+2) + c\Big).$$

Therefore, we have the identity

$$\begin{aligned} L_n :&= bF(n)F(n+1)F(n^2+n+2) - \\ &\quad - b^2F(n-1)F(n)F(n+1)F(n+2) - \\ &\quad - bcF(n)F(n+2) - bcF(n-1)F(n+1) + c - c^2 = 0, \end{aligned} \tag{2.3}$$

which holds for every $n \in \mathbb{N}_1$.

From now on, let $F(2) := x$ and $F(3) := y$.

In the proof of our results, we will use the relation (2.1) for the values

$$n \in \{2, 3, 4, 5, 6, 7, 8, 17, 21, 31, 41, 57\}.$$

This yields the following system of equations:

$$(2.4)\qquad \begin{cases} E_2 = F(5) - by - c = 0, \\ E_3 = xF(5) - bx^3 - c = 0, \\ E_4 = F(17) - bxF(5) - c = 0, \\ E_5 = xF(13) - byF(5) - c = 0, \\ E_6 = F(37) - bF(5)F(7) - c = 0, \\ E_7 = xF(5)^2 - bx^4y - c = 0, \\ E_8 = F(5)F(13) - by^2F(7) - c = 0, \\ E_{17} = xF(13)F(17) - -bx^3F(5)F(11) - c = 0, \\ E_{21} = xF(5)F(29) - bx^5y^2 - c = 0, \\ E_{31} = xF(13)F(37) - bx^6yF(5) - c = 0, \\ E_{41} = xF(29)^2 - bx^4yF(5)F(7) - c = 0, \\ E_{57} = xF(5)^3F(13) - bx^4F(7)F(29) - c = 0. \end{cases}$$

Lemma 1. *If $F \in \mathcal{M}^*$ and $c \in \mathbb{C}$ satisfy*

$$(2.5)\qquad F(n^2+1) = c \quad \textit{for every} \quad n \in \mathbb{N},$$

then $c \in \{0, 1\}$.

If $c = 0$, then $F \in \mathcal{F}_0$.

If $c = 1$, then $F(2) = 1$ and

$$(2.6)\qquad F(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P},\ p \equiv 1 \pmod 4.$$

Proof. From (2.2) and (2.5), we have

$$c = F\Big((n^2+n+1)^2+1\Big) = F\Big((n^2+1)((n+1)^2+1)\Big) = \\ = F(n^2+1)F\big((n+1)^2+1\big) = c^2,$$

which implies $c \in \{0, 1\}$.

If $c = 0$, then $F(n^2+1) = 0$ for every $n \in \mathbb{N}$ and so $F \in \mathcal{F}_0$.

If $c = 1$, then $F(n^2+1) = 1$ for every $n \in \mathbb{N}$ and so $F \in \mathcal{F}_1$.

We now prove (2.6). It is easy to check that $F(2) = 1, F(5) = F(13) = = F(17) = 1$.

Assume that $F(q) = 1$ for every $q \in \mathcal{P}, q \equiv 1 \pmod 4), q < P$ and let $P \in \mathcal{P}, P \equiv 1 \pmod 4), P > 17$. Since $P \equiv 1 \pmod 4)$, there is $m, Q \in \mathbb{N}$ with $m < P$, such that

$$m^2 + 1 = PQ.$$

Clearly, $Q < P$ since $PQ = m^2+1 \le (P-1)^2+1$. For any $q \in \mathcal{P}, q|Q$, we have $m^2+1 \equiv 0 \pmod q$ and

$$(-1)^{\frac{q-1}{2}} = \left(\frac{-1}{q}\right) = \left(\frac{m^2}{q}\right) = 1,$$

so $q \equiv 1 \pmod 4$, and by our assumption $F(q) = 1$. Therefore $F(Q) = 1$, and thus

$$1 = F(m^2+1) = F(PQ) = F(P)F(Q) = F(P).$$

This proves (2.6) and completes the proof of Lemma 1. ∎

Lemma 2. *If $F \in \mathcal{M}^*$ satisfies (1.1) and $bc \neq 0$, then*

$$F(2)F(3)F(5) \neq 0.$$

Proof. (i) The proof of $x = F(2) \neq 0$.

From (2.4), we have

$$E_3 = xF(5) - bx^3 - c = 0,$$

which, together with $c \neq 0$, implies $x = F(2) \neq 0$.

(ii) The proof of $y = F(3) \neq 0$.

Assume that $y = 0$. Then from E_2 and E_4 in (2.4), we obtain:

$$F(5) = F(17) = c. \tag{2.7}$$

Consequently,

$$\begin{aligned} c^3 &= F(5)F(17)^2 = F(1445) = F(38^2+1) = bF(38^2-1)+c = \\ &= bF(3)F(13)F(37)+c = c, \end{aligned}$$

which gives

$$c^2 = 1. \tag{2.8}$$

Thus, we have

$$x = xc^2 = F(2)F(5)^2 = F(50) = bF(48)+c = bx^4y+c = c. \tag{2.9}$$

On the other hand, from (1.1) and (2.8)–(2.9), we have

$$c^2 = F(2)F(5) = F(10) = F(3^2+1) = bF(3^2-1)+c = bF(2)^3+c = bc^3+c.$$

This with (2.8) implies $c = b+1$, and hence

$$1 = c^2 = (b+1)^2 = b(b+2)+1 \Rightarrow b(b+2) = 0.$$

Since $b \neq 0$, it follows that

$$b = -2 \quad \text{and} \quad c = b + 1 = -1. \tag{2.10}$$

Now, from (2.7) and (2.10), we have

$$x = F(2) = -1, F(5) = -1 \quad \text{and} \quad F(17) = -1. \tag{2.11}$$

Now from (2.4), we compute:

$$\begin{cases} 0 = E_5 = xF(13) - bx^3y - c = -F(13) + 1, \\ 0 = E_{17} = xF(5)F(29) - bx^5y^2 - c = F(29) + 1, \\ 0 = E_{57} = xF(5)^2F(13) - bx^4F(7)F(29) - c = \\ \qquad = -F(13) + 2F(7)F(29) + 1. \end{cases}$$

From this, we get

$$F(13) = 1, \; F(29) = -1 \quad \text{and} \quad F(7) = 0.$$

Now consider:

$$\begin{cases} 0 = E_6 = F(37) - bF(5)F(7) - c = F(37) + 1 \Rightarrow F(37) = -1, \\ 0 = E_{31} = xF(13)F(37) - bx^6yF(5) - c = -F(37) + 1 \Rightarrow F(37) = 1, \end{cases}$$

which are impossible. Therefore, our assumption $y = 0$ leads to a contradiction, so $y \neq 0$.

(iii) The proof of $F(5) \neq 0$.

Assume that $xy = F(2)F(3) \neq 0$, but $F(5) = 0$. Then from (2.4), we get:

$$\begin{cases} E_2 = -by - c = 0, \\ E_3 = -bx^3 - c = 0, \\ E_7 = -bx^4y - c = 0, \\ E_8 = -by^2F(7) - c = 0, \\ E_{41} = xF(29)^2 - c = 0, \\ E_{57} = -bx^4F(7)F(29) - c = 0. \end{cases} \tag{2.12}$$

From E_3 and E_2, we get:

$$c = -bx^3, \; y = -\frac{c}{b} = x^3.$$

Using E_7:

$$E_7 = -bx^3(x^4 - 1) = 0 \implies x^4 = 1.$$

Also

$$E_8 = -by^2F(7) - c = -bx^3(x^3F(7) - 1) = 0 \implies F(7) = \frac{1}{x^3} = x.$$

Thus, we have

$$E_{57} = -bx^4F(7)F(29) - c = -bx^3(x^2F(29)-1) = 0,$$

which implies

$$F(29) = x^2$$

and

$$E_{41} = xF(29)^2 - c = x^3(x^2+b) = 0.$$

Hence

$$b = -x^2.$$

Therefore, we have

$$x^4 = 1,\ b = -x^2,\ c = x^5,\ y = x^3,\ F(7) = x,\ F(29) = x^2.$$

Now compute:

$$\begin{aligned} L_2 &= bx^6yF(5) - b^2x^3yF(5)F(7) - bcF(5)F(7) - bcx^3y+ \\ &+ c - c^2 = x^5(x^8 - x^5 + 1) = x(-x+2) = 0, \end{aligned}$$

which leads to $x = F(2) = 2$. This shows that $1 = x^4 = F(2)^4 = 2^4 = 16$, which is impossible.

Thus, Lemma 2 is proved. ■

Lemma 3. *If $F \in \mathcal{M}^*$ satisfies (1.1) and $bc \neq 0$, then*

$$\begin{aligned} &\Big(b, c, x, y, F(5), F(7), F(11), F(13), F(17)\Big) \in \\ &\in \Big\{(b, -b+1, 1, 1, 1, 1, 1, 1, 1), (1, 2, 2, 3, 5, 7, 11, 13, 17)\Big\}. \end{aligned}$$

Proof. Assume that $bc \neq 0$. Then Lemma 2 implies that $bcF(2)F(3)F(5) \neq$ $\neq 0$.

By applying (2.4) concerning $E(2)$, $E(4)$, $E(5)$, $E(8)$, $E(17)$, $E(21)$, $E(6)$, we obtain the following expressions:

$$(2.13)\qquad \begin{cases} F(5) &= bF(3)+c = by+c,\\ F(17) &= bF(3)F(5)+c = b^2y^2+bcy+c,\\ F(13) &= \dfrac{bF(2)^3F(3)+c}{F(2)} = \dfrac{bx^3y+c}{x},\\ F(7) &= \dfrac{F(5)F(13)-c}{bF(3)^2} = \dfrac{b^2x^3y^2+bcx^3y+bcy+c^2-cx}{bxy^2},\\ F(29) &= \dfrac{bF(2)^5F(3)^2+c}{F(2)F(5)} = \dfrac{bx^5y^2+c}{x(by+c)},\\ F(11) &= \dfrac{F(2)F(13)F(17)-c}{bF(2)^3F(5)} =\\ &= \dfrac{b^3x^3y^3+b^2cx^3y^2+bcx^3y+b^2cy^2+bc^2y+c^2-c}{bx^3(by+c)},\\ F(37) &= bF(5)F(7)+c =\\ &= \frac{b^3x^3y^3+2b^2cx^3y^2+bc^2x^3y+b^2cy^2+2bc^2y-bcxy+cxy^2+c^3-c^2x}{xy^2}. \end{cases}$$

From (1.1), we also have

$$(2.14)\qquad E(n) = F(n^2+1) - bF(n^2-1) - c = 0 \quad \text{for every} \quad n\in\mathbb{N}_1.$$

Thus we infer from (2.13) and (2.14) that

$$E(3) = F(10) - bF(2)F(4) - c = -bx^3 + bxy + cx - c = 0,$$

$$\begin{aligned} &E(12) = F(5)F(29) - bF(11)F(13) - c =\\ &= -\frac{1}{x^4(by+c)}\Big(b^4x^6y^4 + b^3cx^6y^3 - b^2x^8y^3 - bcx^8y^2 + b^2cx^6y^2+\\ &+ 2b^3cx^3y^3 + 2b^2c^2x^3y^2 + 2bc^2x^3y + bcx^4y + b^2c^2y^2 - 2bcx^3y + c^2x^4 + bc^3y-\\ &- c^2x^3 + c^3 - c^2\Big) = 0. \end{aligned}$$

By applying the above relations with $n\in\{2,3,8\}$, we infer from (2.3) that

$$\begin{aligned} L(2) &= bF(2)F(3)F(8) - b^2F(1)F(2)F(3)F(4) - bcF(2)F(4) - bcF(3)-\\ &\quad - c^2 + c = bF(2)^4F(3) - b^2F(2)^3F(3) - bcF(2)^3 - bcF(3) - c^2 + c =\\ &= bx^4y - b^2x^3y - bcx^3 - bcy - c^2 + c = 0, \end{aligned}$$

$$\begin{aligned} L(3) &= bF(3)F(4)F(14) - b^2F(2)F(3)F(4)F(5) - bcF(3)F(5) - bcF(2)F(4)-\\ &- c^2 + c = bF(3)F(2)^3F(7) - b^2F(2)^3F(3)F(5) - bcF(3)F(5)-\\ &- bcF(2)^3 - c^2 + c = -\frac{1}{y}\Big(b^3x^3y^3 - b^2x^5y^2 + b^2cx^3y^2 - bcx^5y + b^2cy^3+\\ &+ bcx^3y + bc^2y^2 - bcx^2y - c^2x^2 + cx^3 + c^2y - cy\Big) = 0, \end{aligned}$$

$$\begin{aligned} L(8) &= bF(8)F(9)F(74) - b^2F(7)F(8)F(9)F(10) - bcF(8)F(10)- \\ &- bcF(7)F(9) - c^2 + c = bF(2)^4F(3)^2F(37 - b^2F(7)F(2)^4F(3)^2F(5)- \\ &- bcF(2)^4F(5) - bcF(7)F(3)^2 - c^2 + c = \\ &= \frac{-c}{x}\Big(b^2x^5y - bx^5y^2 + b^2x^3y^2 + bcx^5 + bcx^3y + bcy + c^2 - x\Big) = 0. \end{aligned}$$

Define:

$$\begin{aligned} W_1 &= -E(3), \\ W_2 &= -L(2), \\ W_3 &= -xL(8)/c, \\ W_4 &= -yL(3), \\ W_5 &= -x^4(by+c)E(12). \end{aligned}$$

Then we have the following system of equations

$$W_1 = bx^3 - bxy - cx + c = 0,$$

$$W_2 = b^2x^3y - bx^4y + bcx^3 + bcy + c^2 - c = 0,$$

$$W_3 = b^2x^5y - bx^5y^2 + b^2x^3y^2 + bcx^5 + bcx^3y + bcy + c^2 - x = 0,$$

$$\begin{aligned} W_4 = {} & b^3x^3y^3 - b^2x^5y^2 + b^2cx^3y^2 - bcx^5y + b^2cy^3 + bcx^3y + bc^2y^2 - bcx^2y- \\ & - c^2x^2 + cx^3 + c^2y - cy = 0, \end{aligned}$$

$$\begin{aligned} W_5 = {} & b^4x^6y^4 + b^3cx^6y^3 - b^2x^8y^3 - bcx^8y^2 + b^2cx^6y^2 + 2b^3cx^3y^3 + 2b^2c^2x^3y^2+ \\ & + 2b^2c^2x^3y^2 + 2bc^2x^3y + bcx^4y + b^2c^2y^2 - 2bcx^3y + c^2x^4 + bc^3y- \\ & - c^2x^3 + c^3 - c^2 = 0. \end{aligned}$$

These yield the following system of five polynomial equations in variables b, c, x, y. Using a computer algebra system (Maple), we find that the system has the following solutions:

$$\begin{aligned} (b,c,x,y) \in \{&(b,0,0,y),(0,1,1,y),(b,-b+1,1,1),(1,2,2,3), \\ &(-1,1,1,1),(-2,-1,-1,0)\}. \end{aligned}$$

From these, the only tuples consistent with $bcF(2)F(3)F(5) = bcxy(by+c) \neq 0$ are

$$(b,c,x,y) \in \{(b,-b+1,1,1),(1,2,2,3)\}.$$

Substituting these into the earlier formulas gives the corresponding function values:

$$\begin{aligned} &\Big(b,c,x,y,F(5),F(7),F(11),F(13),F(17)\Big) \in \\ &\in \Big\{(b,-b+1,1,1,1,1,1,1,1),(1,2,2,3,5,7,11,13,17)\Big\}, \end{aligned}$$

completing the proof of Lemma 3. ■

3. Proof of Theorem 1

Proof of Theorem 1 (a). If $b = 0$ and $c \neq 0$, then $F(n^2+1) = c$ for every $n \in \mathbb{N}$, and so Theorem 1 (a) follows from Lemma 1. ∎

Proof of Theorem 1 (b). Assume that $F \in \mathcal{M}^*$ satisfies (1.1) and $bc \neq 0$. Then, by using Lemma 2 and Lemma 3, we have $bcF(2)F(3)F(5) \neq 0$. Thus, by using Lemma 3, we will prove our theorem by considering two separate cases.

Case 1: $(b, c, x, y, F(5), F(7), F(11), F(13), F(17)) = (b, -b+1, 1, 1, 1, 1, 1, 1, 1)$

Case 2: $(b, c, x, y, F(5), F(7), F(11), F(13), F(17)) = (1, 2, 2, 3, 5, 7, 11, 13, 17)$.

Proof of Case 1. In this case, we can assume that $F(n) = 1$ for every $n \leq P$, where $P > 17$. We aim to prove that $F(P) = 1$.

It is clear from our assumptions that $F(P) = 1$ if $P \notin \mathcal{P}$. So we now assume $P \in \mathcal{P}$.

(a) We first prove that $F(P) = 1$ if $P \in \mathcal{H}$.

(a1). Let $P \in \mathcal{H}$ and $\left(\frac{-1}{P}\right) = 1$.

Then $P \equiv 1 \pmod 4$ and similarly as in the proof of Lemma 1, there is $m, Q \in \mathbb{N}, m < P$ such that

$$m^2 + 1 = PQ.$$

Clearly, $Q < P$. From our assumptions, $F(Q) = 1$ and

$$F(m-1) = 1, \;\; F(m+1) = F(2)F(\frac{m+1}{2}) = 1.$$

Thus we have

$$F(m^2+1) = F(PQ) = F(P)F(Q) = F(P),$$

and from equation (1.1)

$$F(m^2+1) = bF(m-1)F(m+1) + 1 - b = b + 1 - b = 1.$$

Therefore, we proved that $F(P) = 1$ if $\left(\frac{-1}{P}\right) = 1$.

(a2). Now suppose $P \in \mathcal{H}$ and $\left(\frac{-7}{P}\right) = 1$.

Then there exist $m, Q \in \mathbb{N}, m < P$ such that

$$m^2 + m + 2 = PQ,$$

since $\left(\frac{-7}{P}\right) = 1$ and

$$m^2 + m + 2 \equiv 0 \pmod P \iff (2m+1)^2 \equiv -7 \pmod P.$$

Clearly, $m < P$, $Q < P$ and $m \neq P-1, m \neq P-2$. So we have

$$F(Q) = 1,\ F(m-1) = 1,\ F(m) = 1,\ F(m+1) = 1,\ F(m+2) = 1.$$

Now apply equation (2.3), we have

$$\begin{aligned} L_m :=& \, bF(m)F(m+1)F(m^2+m+2)- \\ & - b^2F(m-1)F(m)F(m+1)F(m+2)- \\ & - bcF(m)F(m+2) - bcF(m-1)F(m+1) + c - c^2 = 0. \end{aligned}$$

Substituting all values as 1:

$$L_m = bF(P) - b^2 - 2bc + c - c^2 = 0.$$

Recall $c = 1-b$, we have

$$L_m = bF(P) - b^2 - 2b(1-b) + (1-b) - (1-b)^2 = bF(P) - b = 0,$$

which implies $F(P) = 1$.

Therefore, the theorem is proved for all $p \in \mathcal{H}$.

(b) Now we prove that $F(P) = 1$ if $P \in \mathcal{K}$.

To apply Conjecture 2, by using $P > p_2 = 3$, we first verify the special cases:

$$F(P) = 1 \quad \text{if} \quad P \in \{p_{11} = 31, p_{15} = 47, p_{39} = 167\}.$$

(b1) If $P = p_{11} = 31$, then from E_9 and E_{32}, we compute:

$$F(41) = \frac{bF(2)^4F(5)+c}{F(2)} = b + c = 1$$

and

$$F(31) = \frac{F(5)^2F(41) - c}{bF(3)F(11)} = \frac{1-c}{b} = 1.$$

(b2) If $P = p_{15} = 47$, then $F(n) = 1$ for every $n < 47$. Thus we infer from E_{27} and E_{46} that

$$F(73) = \frac{bF(2)^3F(13)F(7)+c}{F(2)F(5)} = b + c = 1$$

and

$$F(47) = \frac{F(29)F(73) - c}{bF(3)^2F(5)} = \frac{1-c}{b} = 1.$$

Consequently

$$F(P) = 1 \quad \text{for} \quad P = 47.$$

(b3) If $P = p_{39} = 167$, then $F(n) = 1$ for every $n < 167$. Thus we infer from E_{107}, E_{8351} that

$$E(107) = F(2)F(5)^2F(229) - bF(2)^3F(53)F(3)^3 - c = F(229) - 1 = 0$$

and

$$\begin{aligned} E(8351) = {} & F(53)F(2)F(13)^2F(17)F(229)- \\ & - bF(2)F(5)^2F(167)F(29)F(2)^5F(3)^2 - c = \\ = {} & F(229) - bF(167) - c = 0. \end{aligned}$$

Consequently

$$F(229) = 1 \quad \text{and} \quad F(167) = F(P) = 1.$$

Now we use Conjecture 2. Since $F(n) = 1$ for every $n \le p_{39} = 167$, we infer from Conjecture 2 that there exists a positive integer ℓ such that

$$P|\ell^2 - 1 \quad \text{and} \quad P\Big(\frac{(\ell^2+1)(\ell^2-1)}{P}\Big) < P.$$

Thus, it follows from our assumptions that

$$F(\ell^2+1) = 1, F(\ell^2-1) = F\Big(\frac{\ell^2-1}{P}\Big)F(P) = F(P),$$

which with (1.1) imply that

$$1 = F(\ell^2+1) = bF(\ell^2-1) + c = bF(P) + c = bF(P) + 1 - b.$$

Therefore $F(P) = 1$ and we proved that $(F,\ b,\ c) = (\mathbb{E},\ b,\ -b+1)$.

Proof of Case 2. Assume that

$$\Big(b, c, x, y, F(5), F(7), F(11), F(13), F(17)\Big) = (1, 2, 2, 3, 5, 7, 11, 13, 17).$$

Similar way as in the proof of Case 1, we can deduce that $F(n) = n$ for every $n \in \mathbb{N}$. Thus, we have $(F,\ b,\ c) = (\mathbb{I},\ 1,\ 2)$.

This completes the proof of Theorem 1. ■

4. Remark

Conjecture 3. *Let H be a complex valued function defined on the set of Gaussian integers satisfying the equations*

$$H(\alpha\beta) = H(\alpha)H(\beta) \quad \textit{for every} \quad \alpha, \beta \in \mathbb{Z}[i]$$

$$H(\alpha^2+1) = F(\alpha^2-1) + 2 \quad \textit{for every} \quad \alpha \in \mathbb{Z}[i].$$

Then either $H(\alpha) = \alpha$ or $H(\alpha) = \overline{\alpha}$ for every $\alpha \in \mathbb{Z}[i]$, where $\overline{\alpha}$ is the conjugate of α.

This conjecture is weaker than the assertion (b) in Theorem 1.

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Imre Kátai, Bui Minh Mai Khanh and Bui Minh Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest,
Pázmány Péter sétány 1/C, Hungary
katai@inf.elte.hu
mbuiminh@yahoo.com
bui@inf.elte.hu