

## NEW FAMILIES OF NORMAL NUMBERS CONSTRUCTED FROM EXISTING ONES

Jean-Marie De Koninck<sup>1</sup> (Québec, Canada)

Imre Kátai (Budapest, Hungary)

Communicated by Karl-Heinz Indlekofer

(Received 14 December 2024; accepted 17 January 2025)

**Abstract.** Given an integer  $q \geq 2$ , let  $\mathcal{A}_q := \{0, 1, \dots, q-1\}$  be the set of base  $q$  digits. We say that  $\alpha = a_1 \dots a_k$ , where each  $a_i \in \mathcal{A}_q$ , is a *word* of length  $\lambda(\alpha) = k$ . Let  $H(1), H(2), \dots$  be a sequence of nonnegative integers and given a sequence of words  $\alpha_1, \alpha_2, \dots$ , we examine under which conditions the number  $0.\alpha_1^{H(1)}\alpha_2^{H(2)}\dots$  is a normal number in base  $q$  (where  $\alpha_j^r = \underbrace{\alpha_j \dots \alpha_j}_{r \text{ times}}$ ).

### 1. Introduction

Fix an integer  $q \geq 2$  and let  $\mathcal{A}_q := \{0, 1, \dots, q-1\}$  be the set of base  $q$  digits. We say that  $\alpha = a_1 \dots a_k$ , where each  $a_i \in \mathcal{A}_q$ , is a *word* of length  $\lambda(\alpha) = k$ .

Let  $\beta = \alpha_1\alpha_2\dots$  be an infinite concatenation of words  $\alpha_i$ 's, which we write as  $\beta = j_1j_2\dots$ , where each  $j_i \in \mathcal{A}_q$ . Then,  $\xi = \xi(\beta) := \sum_{\nu=1}^{\infty} \frac{j_{\nu}}{q^{\nu}}$  is a  $q$ -ary normal

---

*Key words and phrases:* Normal numbers.

*2010 Mathematics Subject Classification:* 11K16.

<sup>1</sup>The research work of the first author was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

<https://doi.org/10.71352/ac.58.170125>

Published online: 8 March 2025

number if the sequence of fractional parts  $\{q^k \xi\}_{k \geq 1}$  is uniformly distributed in  $[0, 1]$  (see Theorem 8.1 in Kuipers and Niederreiter [3]).

Over the past decade, we created several new families of normal numbers [2].

Here, we expose a new approach for creating normal numbers. To do so, we first let  $H(1), H(2), H(3), \dots$  be a sequence of natural numbers and then, given a sequence of words  $\alpha_1, \alpha_2, \dots$ , we examine under which conditions the number  $0.\alpha_1^{H(1)}\alpha_2^{H(2)}\dots$  (where  $\alpha_j^r = \underbrace{\alpha_j \dots \alpha_j}_{r \text{ times}}$ ) is a normal number in base  $q$ .

## 2. Background results

In a 1994 paper, Bassily [1] generalized to polynomials a result which originally applied only to the sum of digits function. To explain his result, we introduce additional notation. Let  $q$  and  $\mathcal{A}_q$  be as above. Then, every nonnegative integer  $n$  can be written as

$$n = \sum_{r=0}^{\infty} a_r(n) q^r, \quad \text{where each } a_r(n) \in \mathcal{A}_q.$$

Clearly, the above sum is finite, since  $a_r(n) = 0$  if  $r > (\log n)/(\log q)$ . Bassily investigated digital functions  $\alpha(n)$  which depend on the digital blocks of length  $k$ . More precisely, given  $k \in \mathbb{N}$  and a function  $F_k : \mathcal{A}_q^k \rightarrow \mathbb{R}$  which satisfies the condition  $F_k(0, \dots, 0) = 0$ , consider the function

$$\alpha(n) := \sum_{j=0}^{\infty} F_k(a_j(n), a_{j+1}(n), \dots, a_{j+k-1}(n)),$$

that is, a kind of generalisation of the sum of digits function. Further setting

$$M := \frac{1}{q^k} \sum_{(b_0, \dots, b_{k-1}) \in \mathcal{A}_q^k} F_k(b_0, \dots, b_{k-1}),$$

Bassily showed [1] that

$$\alpha(n) = (1 + o(1)) M \frac{\log n}{\log q} \quad (n \rightarrow \infty),$$

except perhaps on a set of density 0. Let  $\pi(x)$  stand for the number of primes not exceeding  $x$ . In his paper, Bassily also proved the following theorem.

**Theorem A.** (Bassily) *Let  $q \geq 2$  be a fixed integer. Let  $P(x) = c_r x^r + \dots + c_1 x + c_0$  be a polynomial with integer coefficients taking on positive values*

for all  $x > 0$  and such that  $\gcd(c_r, q) = 1$ . Then,

$$\begin{aligned} \sum_{n \leq x} \left( \alpha(P(n)) - Mr \frac{\log x}{\log q} \right)^2 &\ll x \log x, \\ \sum_{p \leq x} \left( \alpha(P(p)) - Mr \frac{\log x}{\log q} \right)^2 &\ll \pi(x) \log x. \end{aligned}$$

Let  $q$  and  $P(x)$  be as in the statement of Theorem A, and further let  $2 = p_1 < p_2 < \dots$  be the sequence of all primes. Moreover, let  $\eta_1$  and  $\eta_2$  be the real numbers whose  $q$ -ary expansions are given by

$$\begin{aligned} \eta_1 &= 0.\overline{P(1)}\overline{P(2)}\overline{P(3)}\overline{P(4)}\dots\overline{P(n)}\dots, \\ \eta_2 &= 0.\overline{P(2)}\overline{P(3)}\overline{P(5)}\overline{P(7)}\dots\overline{P(p_n)}\dots, \end{aligned}$$

where  $\overline{n}$  stands for the concatenation of the base  $q$  digits of  $n$ . Bassily proved that  $\eta_1$  and  $\eta_2$  are normal numbers in base  $q$ .

Moreover, fix  $k \in \mathbb{N}$  and a word  $\gamma \in \mathcal{A}_q^k$ , that is, a word made of  $k$  base  $q$  digits. Then, given an arbitrary word  $\beta$  made of base  $q$  digits, we define  $\omega_\gamma(\beta)$  as the number of occurrences of the subword  $\gamma$  in the word  $\beta$ . Bassily showed that if

$$(2.1) \quad \Gamma_k := \{\gamma = b_1 b_2 \dots b_k : b_i \in \mathcal{A}_q \text{ for } i = 1, \dots, k\},$$

then,

$$(2.2) \quad \sum_{n \leq x} \max_{\gamma \in \Gamma_k} \left( \omega_\gamma(P(n)) - \frac{r}{q^k} \frac{\log x}{\log q} \right)^2 \ll x \log x,$$

$$(2.3) \quad \sum_{p \leq x} \max_{\gamma \in \Gamma_k} \left( \omega_\gamma(P(p)) - \frac{r}{q^k} \frac{\log x}{\log q} \right)^2 \ll \pi(x) \log x.$$

Now, before we state our main results, we introduce two new functions as follows. Given an arithmetic function  $H : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ , consider the two functions

$$\begin{aligned} s_H(x) &:= \sum_{n \leq x} H^2(n) \Big/ \left( \sum_{n \leq x} H(n) \right)^2 \quad \text{and} \\ t_H(x) &:= \sum_{p \leq x} H^2(p) \Big/ \left( \sum_{p \leq x} H(p) \right)^2. \end{aligned}$$

### 3. The main results

**Theorem 1.** *Let  $q \geq 2$  be a fixed integer. Let  $P(x) = c_r x^r + \cdots + c_1 x + c_0$  be a polynomial with integer coefficients taking on nonnegative values for all  $x > 0$  and such that  $\gcd(c_r, q) = 1$ . Let  $H(1), H(2), H(3), \dots$  be a sequence of natural numbers satisfying the condition*

$$(3.1) \quad \lim_{x \rightarrow \infty} s_H(x) \cdot \frac{x}{\log x} = 0.$$

*Then, the number*

$$(3.2) \quad \eta_1 := 0.\overline{P(1)}^{H(1)} \overline{P(2)}^{H(2)} \overline{P(3)}^{H(3)} \overline{P(4)}^{H(4)} \dots$$

*is a  $q$ -normal number.*

**Theorem 2.** *Let  $q$ ,  $P(x)$  and  $H(n)$  be as in Theorem 1. Assuming that*

$$(3.3) \quad \lim_{x \rightarrow \infty} t_H(x) \cdot \frac{x}{\log^2 x} = 0,$$

*then,*

$$(3.4) \quad \eta_2 := 0.\overline{P(2)}^{H(2)} \overline{P(3)}^{H(3)} \overline{P(5)}^{H(5)} \overline{P(7)}^{H(7)} \dots$$

*is a  $q$ -normal number.*

### 4. Some preliminary results

We start with a classic result in analytic number theory. As is common,  $\zeta(s)$  stands for the Riemann Zeta Function.

**Proposition 1.** *Let  $f$  be an arithmetic function and let  $t$  be a positive real number. Assume that, for every real  $s > 1$ ,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta^t(s) \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

*where  $g$  is such that  $\sum_{n=1}^{\infty} \frac{g(n)}{n}$  converges absolutely. Then,*

$$\sum_{n \leq x} f(n) = (c + o(1)) x \log^{t-1} x \quad (x \rightarrow \infty),$$

$$\text{where } c = \frac{1}{\Gamma(t)} \sum_{n=1}^{\infty} \frac{g(n)}{n}.$$

**Proof.** This is a particular case of Theorem 2 in the 1954 paper of Atle Selberg [5]. ■

Let  $\phi$  stand for the Euler totient function. The following is often called the *prime number theorem for arithmetic progressions*.

**Proposition 2.** *Given two coprime integers  $k \geq 1$  and  $\ell$ , let  $\pi(x; k, \ell) := \#\{p \leq x : p \equiv \ell \pmod{k}\}$ . Then,*

$$\pi(x; k, \ell) = (1 + o(1)) \frac{1}{\phi(k)} \frac{x}{\log x} \quad (x \rightarrow \infty).$$

**Proof.** See Theorems 5.11 and 5.14 in the book of Narkiewicz [4]. ■

We now recall a 1961 result of Wirsing [6] which we state as a proposition.

**Proposition 3.** *Let  $f(n)$  be a real valued non negative multiplicative function satisfying  $f(p^\nu) \leq c_1 c_2^\nu$  for all prime powers  $p^\nu$ ,  $\nu \geq 2$ , for some positive constant  $c_1$  and  $c_2$ , with  $c_2 < 2$ . Assuming that there exists a positive constant  $\tau$  for which*

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x} \quad (x \rightarrow \infty),$$

*then, as  $x \rightarrow \infty$ ,*

$$\sum_{n \leq x} f(n) = \left( \frac{e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots \right),$$

*where  $\gamma$  stands for the Euler-Mascheroni constant and  $\Gamma(r)$  is the Gamma function.*

The next proposition is an immediate application of Proposition 3.

**Proposition 4.** *Let  $f$  be a completely multiplicative function such that  $f(p) \in \{0, 1\}$  for all primes  $p$ . Assume that there exists a real number  $\delta \in (0, 1)$  such that*

$$\sum_{p \leq x} f(p) = (\delta + o(1)) \frac{x}{\log x} \quad (x \rightarrow \infty).$$

*Then, as  $x \rightarrow \infty$ ,*

$$\sum_{n \leq x} f(n) = \left( \frac{e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 - \frac{f(p)}{p} \right)^{-1}.$$

## 5. Proof of Theorem 1

Letting  $\Gamma_k$  be the set defined in (2.1), in order to prove that  $\eta_1$  is a  $q$ -normal number, it is sufficient to prove that

$$(5.1) \quad \sum_{n \leq x} H(n) \cdot \max_{\gamma \in \Gamma_k} \left| \omega_\gamma(P(n)) - \frac{r}{q^k} \frac{\log x}{\log q} \right| = o\left((\log x) \cdot \sum_{n \leq x} H(n)\right) \quad (x \rightarrow \infty).$$

Indeed, we need to show that for most integers  $n \leq x$ , the number of occurrences of  $P(n)$  in the words  $\gamma = b_1 \dots b_k \in \mathcal{A}_q^k$  is asymptotic to  $\frac{r}{q^k} \left\lfloor \frac{\log n}{\log q} \right\rfloor$  (which for large  $n \leq x$  is asymptotic to  $\frac{r}{q^k} \frac{\log x}{\log q}$ ). By proving (5.1), we will have shown that the largest possible difference between  $\omega_\gamma(P(n))$  and  $\frac{r}{q^k} \frac{\log x}{\log q}$  is on average  $o(\log x)$ , which is sufficient to show that  $\eta_1$  is a  $q$ -normal number.

Now, using the Cauchy–Schwarz inequality, the Bassily result (2.2) and finally the condition (3.1), we obtain that

$$\begin{aligned} \sum_{n \leq x} H(n) \cdot \max_{\gamma \in \Gamma_k} \left| \omega_\gamma(P(n)) - \frac{r}{q^k} \frac{\log x}{\log q} \right| &\leq \sqrt{\sum_{n \leq x} H^2(n) \cdot x \log x} \leq \\ &\leq \sqrt{x \log x \cdot s_H(x) \cdot \left( \sum_{n \leq x} H(n) \right)^2} = \\ &= \sqrt{x \log x} \cdot \sqrt{s_H(x)} \cdot \sum_{n \leq x} H(n) = \\ &= o\left((\log x) \cdot \sum_{n \leq x} H(n)\right) \quad (x \rightarrow \infty), \end{aligned}$$

thus establishing (5.1) and completing the proof of Theorem 1. ■

## 6. Proof of Theorem 2

Theorem 2 will follow if we can prove that

$$(6.1) \quad \sum_{p \leq x} H(p) \max_{j \in \Gamma_k} \left| \omega_j(P(p)) - \frac{r}{q^k} \frac{\log x}{\log q} \right| = o\left(\log x \cdot \sum_{p \leq x} H(p)\right) \quad (x \rightarrow \infty).$$

Proceeding essentially as in the proof of Theorem 1, one can see that (6.1) is a consequence of (3.3), from which the proof of Theorem 2 follows. ■

## 7. Applications

We now identify some particular types of functions  $H(n)$  for which Theorems 1 and 2 apply.

### 7.1. Choosing $H(n)$ to be a polynomial

Let  $Q(x) \in \mathbb{R}[x]$  be a polynomial of degree  $d \geq 1$ , and set  $H(n) := \lfloor |Q(n)| \rfloor$  for each  $n \in \mathbb{N}$ . In this case,

$$\sum_{n \leq N} H(n) \approx N^{d+1} \quad \text{and} \quad \sum_{n \leq x} H^2(n) \approx \sum_{n \leq x} n^{2d} \sim x^{2d+1}.$$

Hence,

$$\left( \sum_{n \leq x} H(n) \right)^2 \approx \left( \sum_{n \leq x} n^d \right)^2 \approx (x^{d+1})^2 = x^{2d+2},$$

from which it follows that condition (3.1) is satisfied, implying that the corresponding real number defined by (3.2) is indeed a normal number in base  $q$ .

Following essentially the same kind of reasoning, one easily sees that the conditions of Theorem 2 are also satisfied and therefore that the corresponding real number defined by (3.4) is a normal number in base  $q$ .

### 7.2. Choosing $H(n)$ to be a multiplicative function

Let  $H_0$  be an arithmetic multiplicative function such that  $H_0(p) = w$  for all primes  $p$ , where  $1 < w < 2$ , and also such that  $H_0(p^r) < r + 1$  for each prime power  $p^r$ . Then, set  $H(n) := \lfloor H_0(n) \rfloor$  for every  $n \in \mathbb{N}$ . In order to show that condition (3.1) holds, we will prove that

$$(7.1) \quad \lim_{x \rightarrow \infty} S_{H_0}(x) \cdot \frac{x}{\log x} = 0.$$

and

$$(7.2) \quad \lim_{x \rightarrow \infty} \frac{s_{H_0}(x)}{s_H(x)} = 1.$$

First observe that for  $\Re s > 1$ ,

$$\sum_{n=1}^{\infty} \frac{H_0(n)}{n^s} = \prod_p \left( 1 + \frac{w}{p^s} + \sum_{r \geq 2} \frac{H_0(p^r)}{p^{rs}} \right) = \zeta(s)^w A_1(s),$$

$$\sum_{n=1}^{\infty} \frac{H_0^2(n)}{n^s} = \zeta(s)^{w^2} A_2(s),$$

where  $A_1(s)$  and  $A_2(s)$  are two holomorphic functions bounded in the half plane  $\Re s > 1 - \delta$ , where  $\delta > 0$  is a suitable small number, with  $A_1(1) \neq 0$  and  $A_2(1) \neq 0$ . Then, it follows from Proposition 1 that there exist positive constants  $c_1$  and  $c_2$  such that, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \sum_{n \leq x} H_0(n) &= (1 + o(1)) c_1 x \log^{w-1} x, \\ \sum_{n \leq x} H_0^2(n) &= (1 + o(1)) c_2 x \log^{w^2-1} x, \end{aligned}$$

from which it follows that

$$\begin{aligned} (7.3) \quad s_{H_0}(x) &= (1 + o(1)) \frac{c_2 x \log^{w^2-1} x}{c_1^2 x^2 \log^{2(w-1)} x} = \\ &= (1 + o(1)) \frac{c_2 \log^{(w-1)^2} x}{c_1^2 x} \quad (x \rightarrow \infty). \end{aligned}$$

Since  $(w-1)^2 < 1$ , it is clear that (7.3) implies condition (7.1). On the other hand, since  $H(n) = H_0(n) + O(1)$ , it follows that

$$H^2(n) = H_0^2(n) + O(H_0(n)) + O(1),$$

which combined with (7.3) implies (7.2). From this, condition (3.1) is proved.

Hence, it follows that the conditions of Theorem 1 is satisfied, so that the corresponding real number defined by (3.2) is a normal number in base  $q$ .

Similarly, one can show that the conditions of Theorem 2 are satisfied and therefore that the corresponding real number defined by (3.4) is also a normal number in base  $q$ .

### 7.3. Choosing $H(n)$ as completely multiplicative functions

#### 7.3.1. Considering sets of integers whose prime factors are all congruent to 1 mod $D$

Fix an integer  $D \geq 3$  and consider the completely multiplicative function  $H(n)$  defined on primes  $p$  by

$$H(p) := \begin{cases} 1 & \text{if } p \equiv 1 \pmod{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Proposition 4 with  $f(n) = H(n)$ , we find that, as  $x \rightarrow \infty$ ,

$$(7.4) \quad \sum_{n \leq x} H(n) = \left( \frac{e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 - \frac{f(p)}{p} \right)^{-1}.$$

Now, setting  $\wp_0 := \{p \in \wp : f(p) = H(p) = 1\}$ , we have that, in light of Proposition 2,

$$(7.5) \quad \sum_{\substack{p \leq x \\ p \in \wp_0}} 1 = \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{D}}} 1 = (1 + o(1)) \frac{1}{\phi(D)} \frac{x}{\log x},$$

so that, as  $x \rightarrow \infty$ ,

$$(7.6) \quad \sum_{\substack{p \leq x \\ p \in \wp_0}} \frac{1}{p} = \frac{1}{\phi(D)} \log \log x + C_1 + o(1)$$

for some constant  $C_1$ . Hence, using (7.6), we have, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \prod_{p \leq x} \left( 1 - \frac{f(p)}{p} \right)^{-1} &= \prod_{\substack{p \leq x \\ p \in \wp_0}} \left( 1 - \frac{1}{p} \right)^{-1} = \exp \left\{ - \sum_{\substack{p \leq x \\ p \in \wp_0}} \log \left( 1 - \frac{1}{p} \right) \right\} = \\ &= \exp \left\{ \sum_{\substack{p \leq x \\ p \in \wp_0}} \frac{1}{p} + C_2 + o(1) \right\} = \\ &= \exp \left\{ \frac{1}{\phi(D)} \log \log x + C_3 + o(1) \right\} = \\ (7.7) \quad &= (C_4 + o(1)) (\log x)^{1/\phi(D)} \end{aligned}$$

for some constants  $C_2$ ,  $C_3$  and  $C_4 > 0$ .

Using (7.7) in (7.4), we obtain that

$$\sum_{n \leq x} H(n) = \left( \frac{C_4 e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{(\log x)^{1-1/\phi(D)}} \quad (x \rightarrow \infty),$$

from which it follows, setting  $C = \frac{C_4 e^{\gamma r}}{\Gamma(r)}$  and using the trivial fact that in this case  $H(n)^2 = H(n)$ , that

$$s_H(x) = \frac{\sum_{n \leq x} H^2(n)}{\left( \sum_{n \leq x} H(n) \right)^2} = (1 + o(1)) \frac{(\log x)^{1-1/\phi(D)}}{Cx} = o\left( \frac{\log x}{x} \right) \quad (x \rightarrow \infty),$$

implying that condition (3.1) of Theorem 1 is satisfied. On the other hand, to see that the conditions of Theorem 2 are satisfied, we first observe that  $H(p)^2 = H(p)$  for all primes  $p$  and that it follows from (7.5) that

$$\sum_{p \leq x} H(p) = (1 + o(1)) \frac{1}{\phi(D)} \frac{x}{\log x} \quad (x \rightarrow \infty),$$

from which we have that

$$t_H(x) = \frac{1}{\sum_{p \leq x} H(p)} = (1 + o(1)) \frac{\phi(D) \log x}{x} = o\left(\frac{\log^2 x}{x}\right) \quad (x \rightarrow \infty),$$

as required. It is then easy to conclude that the conditions of Theorem 2 are satisfied.

The conditions of both Theorems 1 and 2 having been verified for this particular function  $H$ , we may therefore conclude that the corresponding real numbers defined by (3.2) and (3.4) are normal numbers in base  $q$ .

### 7.3.2. Considering sets of integers generated by particular subsets of primes

Let  $\wp$  be a set of primes for which there exists a positive constant  $\delta < 1$  satisfying

$$\#\{p \leq x : p \in \wp\} = (\delta + o(1)) \frac{x}{\log x} \quad (x \rightarrow \infty).$$

Then, consider the completely multiplicative function  $H(n)$  defined on primes  $p$  by

$$H(p) := \begin{cases} 1 & \text{if } p \in \wp, \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding in a manner similar to the one used in the preceding subsection, we obtain that

$$\sum_{n \leq x} H(n) = (A + o(1)) \frac{x}{(\log x)^{1-\delta}} \quad (x \rightarrow \infty),$$

where  $A$  is a computable constant which depends on  $\delta$ .

Again, proceeding as in preceding subsection, one will reach the conclusion that with this particular function  $H(n)$ , the corresponding real numbers defined by (3.2) and (3.4) are normal numbers in base  $q$ .

**Acknowledgment.** The authors are grateful to the referee for pointing out some discrepancies and for making various valuable suggestions.

## References

- [1] **Bassily, N.L.**, Distribution of  $q$ -ary digits in some sequences of integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **14** (1994), 13–22.
- [2] **De Koninck, J.-M. and I. Kátai**, Nineteen papers on normal numbers, <https://www.jeanmariedekoninck.mat.ulaval.ca/en/documents>
- [3] **Kuipers, L. and H. Niederreiter**, *Uniform Distribution of Sequences*, Dover Books on Mathematics, N.Y., 2006.
- [4] **Narkiewicz, W.**, *The Development of Prime Number theory, From Euclid to Hardy and Littlewood*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. <https://doi.org/10.1007/978-3-662-13157-2>
- [5] **Selberg, A.**, Note on a paper by L.G. Sathe, *J. Indian Math. Soc.*, **18** (1954), 83–87.
- [6] **Wirsing, E.**, Das asymptotische Verhalten von Summen über multiplikative Funktionen, *Math. Ann.*, **143** (1961), 75–102. <https://doi.org/10.1007/BF01351892>

**Jean-Marie De Koninck**

<https://orcid.org/0000-0003-4506-959X>

Dép. de mathématiques et de statistique

Université Laval

Québec

Québec G1V 0A6

Canada

[jmdk@mat.ulaval.ca](mailto:jmdk@mat.ulaval.ca)

**Imre Kátai**

Department of Computer Algebra

Faculty of Informatics

Eötvös Loránd University

H-1117 Budapest

Pázmány Péter sétány 1/C

Hungary

[katai@inf.elte.hu](mailto:katai@inf.elte.hu)