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NEURAL NETWORKS IN INFINITE DOMAIN AS POSITIVE LINEAR OPERATORS

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Abstract. Neural network operators in infinite domain are interpreted as positive linear operators and related general theory applies to them. These operators are induced by a symmetrized density function deriving from the parametrized and deformed hyperbolic tangent activation function. We are acting on the space of continuous and bounded functions on the real line to the reals. We study quantitatively the rate of convergence of these neural network operators to the unit operator. Our inequalities involve the modulus of continuity of the function under approximation or its derivative.

1. Introduction

The author studied extensively the quantitative approximation of positive linear operators to the unit since 1985, see for example [1]–[3], [8], which we use in this work. He originated from the quantitative weak convergence of finite positive measures to the unit Dirac measure, having as a method the geometric moment theory, see [2], and he produced best upper bounds, leading to attained (i.e. sharp Jackson type inequalities), e.g. see [1], [2]. These studies have been gone to all possible directions, univariate and multivariate, though in this work we stay only on the univariate approach over infinite domain.

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Also, this author since 1997, he started the study of the quantitative convergence of neural network operators to the unit, and he has been written since then numerous of articles and books, e.g. see [3], [5], [8], are what we use here.

The wide range of neural network operators over $\mathbb{R}$ being treated by the author all along, are indeed by nature positive linear operators.

Here, for the first time in the literature, neural network operators in infinite domain are treated as positive linear operators.

So, methods of positive linear operators in infinite domain apply here to our infinite summation defined neural network operators, producing new and interesting pointwise and uniform results.

Of great inspiration always have been [23], [24]. For clasic studies on neural network we recommend also, [11, 13, 14, 19, 21, 22].

For newer work on neural networks we also refer the reader to [9, 10, 12, 15, 17, 18, 20, 25, 26, 27].

2. Basics

Initially we follow [5], pp. 455-460.

Our perturbed hyperbolic tangent activation function here to be used is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

Here λ is the parameter and q is the deformation coefficient.

For more details read Chapter 18 of [5]: "q-Deformed and λ-Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

The chapters 17 and 18 of [5] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our multivariate neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{2}$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$.

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \ \forall\, x \in \mathbb{R};\ q, \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \ \ \forall\, x \in \mathbb{R};\ q, \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

a key to this work.

So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} > 0, \tag{6}$$

is an even function, symmetric with respect to the y-axis.

By (18.18) of [5], we have

$$\begin{aligned} M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) &= \frac{\tanh(\lambda)}{2}, \\ M_{\frac{1}{q},\lambda}\left(-\frac{\ln q}{2\lambda}\right) &= \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \end{aligned} \tag{7}$$

sharing the same maximum at symmetric points.

By Theorem 18.1, p. 458 of [5], we have that

$$\begin{aligned} \sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) &= 1, \quad \forall\, x \in \mathbb{R},\ \lambda, q > 0, \\ \sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) &= 1, \quad \forall\, x \in \mathbb{R},\ \lambda, q > 0. \end{aligned} \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{9}$$

By Theorem 18.2, p. 459 of [5], we have that

$$\begin{aligned} \int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx &= 1, \\ \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx &= 1, \end{aligned} \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 1, \tag{11}$$

therefore Φ is a density function.

By Theorem 18.3, p. 459 of [5], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

$$\sum_{\substack{k=-\infty \\ :|nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < \tag{12}$$
$$< 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = Te^{-2\lambda n^{(1-\alpha)}},$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

$$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}. \tag{13}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ :|nx-k| \geq n^{1-\alpha}}}^{\infty} \Phi(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}, \tag{14}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

An essential property follows:

Theorem 2.1. ([7]) *Let* $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. *Then*

$$\int_{\{u \in \mathbb{R} : |nx-u| \geq n^{1-\alpha}\}} \Phi(nx-u)\, du < \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\alpha}-1)}}, \quad q, \lambda > 0. \tag{15}$$

In particular, by [7], we have that

$$\Phi(x) < \left(q + \frac{1}{q}\right) \lambda e^{-2\lambda(x-1)}, \ \ \forall\ x \geq 1. \tag{16}$$

We need,

Definition 2.1. In this article we study in an alternative way the smooth approximation properties of the following interpolation neural network operators acting on $f \in C_B(\mathbb{R})$ (continuous and bounded functions):

(i) the basic ones

$$B_N(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right) \Phi(Nx-k), \ \forall\, x \in \mathbb{R},\, N \in \mathbb{N}, \tag{17}$$

(ii) the Kantorovich type operators

$$C_N(f,x) := \sum_{k=-\infty}^{\infty} \left(N \int_{\frac{k}{N}}^{\frac{k+1}{N}} f(t)\, dt \right) \Phi(Nx-k), \ \forall\, x \in \mathbb{R},\, N \in \mathbb{N}, \tag{18}$$

(iii) let $\theta \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{\theta} w_r = 1$, $k \in \mathbb{Z}$, and

$$\delta_{Nk}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{N} + \frac{r}{N\theta}\right), \tag{19}$$

we consider also the quadrature type operators

$$D_N(f,x) := \sum_{k=-\infty}^{\infty} \delta_{Nk}(f)\, \Phi(Nx-k), \ \forall\, x \in \mathbb{R},\, N \in \mathbb{N}. \tag{20}$$

Notice that $B_N(1) = C_N(1) = D_N(1) = 1$. Clearly B_N, C_N, D_N, $N \in \mathbb{N}$, are positive linear operators over $C_B(\mathbb{R})$.

We will be using the first modulus of continuity:

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \ \ \delta > 0, \tag{21}$$

where $f \in C(\mathbb{R})$ which is bounded and or uniformly continuous.

3. Auxilliary results

We need the following important results.

Lemma 3.1. *Let here $x \in \mathbb{R}$, $j = 1, 2, n, n+1$, $n \in \mathbb{N}$; $0 < \beta < 1$, and $N \in \mathbb{N}$ with $N^{1-\beta} > 2$; $q, \lambda > 0$. Then*

$$0 < B_N\left(|\cdot - x|^j\right)(x) \leq \frac{1}{N^{\beta j}} + \frac{1}{N^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(N^{1-\beta}-1\right)} < +\infty. \tag{22}$$

Clearly, $\lim_{N\to\infty} B_N\left(|\cdot - x|^j\right)(x) = 0$.

Proof. We are based on [6].

$$B_N\left(|\cdot - x|^j\right)(x) = \sum_{k=-\infty}^{\infty} \Phi(Nx-k)\left|\frac{k}{N} - x\right|^j =$$

$$= \sum_{\substack{k=-\infty \\ :\left|x-\frac{k}{N}\right| < \frac{1}{N^\beta}}}^{\infty} \Phi(Nx-k)\left|\frac{k}{N} - x\right|^j +$$

$$+ \sum_{\substack{k=-\infty \\ :\left|x-\frac{k}{N}\right| \geq \frac{1}{N^\beta}}}^{\infty} \Phi(Nx-k)\left|\frac{k}{N} - x\right|^j \leq$$

$$\leq \frac{1}{N^{\beta j}} + \frac{1}{N^j}\sum_{\substack{k=-\infty \\ :|Nx-k| \geq N^{1-\beta}}}^{\infty} \Phi(|Nx-k|)|Nx-k|^j \leq \tag{23}$$

$$\leq \frac{1}{N^{\beta j}} + \frac{1}{N^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(N^{1-\beta}-1\right)} < +\infty. \qquad \blacksquare$$

Lemma 3.2. *All as in Lemma 3.1 and $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$. Then*

$$0 < C_N\left(|\cdot - x|^j\right)(x) \leq \tag{24}$$

$$\leq \left(\frac{1}{N} + \frac{1}{N^\beta}\right)^j + \frac{2^{j-1}}{N^j}\left[\frac{T}{e^{2\lambda}N^{1-\beta}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(N^{1-\beta}-1\right)}\right] < +\infty.$$

Clearly, $\lim_{N\to\infty} C_N\left(|\cdot - x|^j\right)(x) = 0$.

Proof. We are based again on [6].

We have that

$$C_N\left(|\cdot - x|^j\right)(x) = \sum_{k=-\infty}^{\infty} \Phi(Nx-k)\left(N\int_0^{\frac{1}{N}}\left|t+\frac{k}{N}-x\right|^j dt\right) \leq$$

$$\leq \sum_{k=-\infty}^{\infty} \Phi(Nx-k)\left(N\int_0^{\frac{1}{N}}\left(\left|\frac{k}{N}-x\right|+|t|\right)^j dt\right) \leq$$

$$\leq \sum_{k=-\infty}^{\infty} \Phi(Nx-k)\left(\left|\frac{k}{N}-x\right|+\frac{1}{N}\right)^j =$$

$$= \sum_{\substack{k=-\infty \\ :\left|\frac{k}{N}-x\right|<\frac{1}{N^\beta}}}^{\infty} \Phi(Nx-k)\left(\left|\frac{k}{N}-x\right|+\frac{1}{N}\right)^j +$$

$$+ \sum_{\substack{k=-\infty \\ :\left|\frac{k}{N}-x\right|\geq\frac{1}{N^\beta}}}^{\infty} \Phi(Nx-k)\left(\left|\frac{k}{N}-x\right|+\frac{1}{N}\right)^j \leq \tag{25}$$

$$\leq \left(\frac{1}{N^\beta}+\frac{1}{N}\right)^j + \frac{1}{N^j}\sum_{\substack{k=-\infty \\ :|Nx-k|\geq N^{1-\beta}}}^{\infty} \Phi(|Nx-k|)(|Nx-k|+1)^j \leq$$

$$\leq \left(\frac{1}{N^\beta}+\frac{1}{N}\right)^j + \frac{2^{j-1}}{N^j}\left(\sum_{\substack{k=-\infty \\ :|Nx-k|\geq N^{1-\beta}}}^{\infty} \Phi(|Nx-k|)\left(1+|Nx-k|^j\right)\right)$$

$$\leq \left(\frac{1}{N}+\frac{1}{N^\beta}\right)^j + \tag{26}$$

$$+\frac{2^{j-1}}{N^j}\left[\frac{T}{e^{2\lambda N^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(N^{1-\beta}-1\right)}\right] < +\infty. \qquad \blacksquare$$

Lemma 3.3. *All here are as in Lemma 3.2. Then*

$$0 < D_N \left(|\cdot - x|^j \right)(x) \le \tag{27}$$

$$\le \left(\frac{1}{N} + \frac{1}{N^{\beta}} \right)^j + \frac{2^{j-1}}{N^j} \left[\frac{T}{e^{2\lambda} N^{1-\beta}} + \frac{\left(q + \frac{1}{q} \right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda \left(N^{1-\beta} - 1 \right)} \right] < +\infty.$$

Clearly, $\lim_{N \to \infty} D_N \left(|\cdot - x|^j \right)(x) = 0.$

Proof. As similar to Lemma 3.2 is omitted, it is based on [6]. ∎

We need the following Hölder's type inequality for positive linear operators.

Theorem 3.1. *Let L be a positive linear operator from $C(\mathbb{R})$ into $C_B(\mathbb{R})$, and $f, g \in C(\mathbb{R})$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L((|f(\cdot)|^p))(s_*)$, $L((|g(\cdot)|^q))(s_*) > 0$ for some $s_* \in \mathbb{R}$. Then*

$$L(|f(\cdot) g(\cdot)|)(s_*) \le \left(L((|f(\cdot)^p|))(s_*) \right)^{\frac{1}{p}} \left(L((|g(\cdot)^q|))(s_*) \right)^{\frac{1}{q}}. \tag{28}$$

Proof. Let $a, b \ge 0$, $p.q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. The Young's inequality says

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{29}$$

Then

$$\frac{|f(s)|}{\left(L((|f(\cdot)^p|))(s_*) \right)^{\frac{1}{p}}} \cdot \frac{|g(s)|}{\left(L((|g(\cdot)^q|))(s_*) \right)^{\frac{1}{q}}} \le$$

$$\le \frac{|f(s)|^p}{p \left(L((|f(\cdot)^p|))(s_*) \right)} + \frac{|g(s)|^q}{q \left(L((|g(\cdot)^q|))(s_*) \right)}, \quad \forall\, s \in \mathbb{R}. \tag{30}$$

Hence it holds

$$\frac{L(|f(\cdot)|\,|g(\cdot)|)(s_*)}{\left(L((|f(\cdot)^p|))(s_*) \right)^{\frac{1}{p}} \left(L((|g(\cdot)^q|))(s_*) \right)^{\frac{1}{q}}} \le$$

$$\le \frac{L((|f(\cdot)^p|))(s_*)}{p \left(L((|f(\cdot)^p|))(s_*) \right)} + \frac{L((|g(\cdot)^q|))(s_*)}{q \left(L((|g(\cdot)^q|))(s_*) \right)} = \frac{1}{p} + \frac{1}{q} = 1, \tag{31}$$

for $s_* \in \mathbb{R}$, proving the claim. ∎

4. Main results

We present the following general approximation result.

Theorem 4.1. *Let $n \in \mathbb{N}$ and $f, f^{(n)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider L_N a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $N \in \mathbb{N}$, such that $L_N(1) = 1$. Assume $L_N\left(|\cdot - x|^{n+1}\right)(x) > 0$, and $f^{(i)}(x) = 0$, for $i = 1, ..., n$.*

Then

$$|L_N(f)(x) - f(x)| \leq \omega_1\left(f^{(n)}, \left(\left(L_N\left(|\cdot - x|^{n+1}\right)\right)(x)\right)^{\frac{1}{n+1}}\right) \leq$$

$$\leq \left[L_N\left(|\cdot - x|^n\right)(x) + \frac{\left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right] < +\infty, \quad \forall\, N \in \mathbb{N}. \tag{32}$$

Given that $\lim_{N \to +\infty} L_N\left(|\cdot - x|^{n+1}\right)(x) = 0$, *then* $\lim_{N \to +\infty} L_N(f)(x) = f(x)$.

If $n = 1$, we derive

$$|L_N(f)(x) - f(x)| \leq \omega_1\left(f', \left(\left(L_N\left((\cdot - x)^2\right)\right)(x)\right)^{\frac{1}{2}}\right) \leq$$

$$\leq \left[L_N\left(|\cdot - x|\right)(x) + \frac{\left(L_N\left((\cdot - x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right] < +\infty, \quad \forall\, N \in \mathbb{N}. \tag{33}$$

Given that $\lim_{N \to +\infty} L_N\left((\cdot - x)^2\right)(x) = 0$, *then* $\lim_{N \to +\infty} L_N(f)(x) = f(x)$.

Proof. Let $x, y \in \mathbb{R}$, by Taylor's formula we have

$$f(y) = \sum_{i=0}^{n} f^{(i)}(x) \frac{(y-x)^i}{i!} + \tag{34}$$
$$+ \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \left(f^{(n)}(t) - f^{(n)}(x)\right) dt.$$

We call the remainder in (34) as

$$R_n(x, y) = \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \left(f^{(n)}(t) - f^{(n)}(x)\right) dt, \quad \forall\, x, y \in \mathbb{R}. \tag{35}$$

By [2], p. 217, and [4], p. 194, Chapter 7, (7.27) there, we get

$$|R_n(x,y)| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}|x-y|^n\left(1+\frac{|x-y|}{(n+1)\delta}\right), \quad \forall\, x,y\in\mathbb{R},\ \delta>0. \tag{36}$$

We may write (36) as

$$|R_n(x,y)| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right], \quad \forall\, x,y\in\mathbb{R},\ \delta>0. \tag{37}$$

That is

$$\left|f(y)-\sum_{i=0}^{n}f^{(i)}(x)\frac{(y-x)^i}{i!}\right| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right], \tag{38}$$

$\forall\, x,y\in\mathbb{R}$, $\delta>0$.

Furthermore it holds

$$|f(y)-f(x)| \leq \tag{39}$$

$$\leq \sum_{i=0}^{n}\left|f^{(i)}(x)\right|\frac{|y-x|^i}{i!}+\frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right],$$

$\forall\, x,y\in\mathbb{R}$, $\delta>0$.

Here $f^{(i)}(x)=0$, for $i=1,...,n$; and for a specific $x\in\mathbb{R}$, we get

$$|f(y)-f(x)| \leq \frac{\omega_1\left(f^{(n)},\delta\right)}{n!}\left[|x-y|^n+\frac{|x-y|^{n+1}}{(n+1)\delta}\right], \quad \forall y\in\mathbb{R},\ \delta>0. \tag{40}$$

In case of $n=1$, we derive

$$|f(y)-f(x)| \leq |f'(x)|\,|y-x|+\omega_1(f',\delta)\left[|x-y|+\frac{(x-y)^2}{2\delta}\right], \tag{41}$$

$\forall\, x,y\in\mathbb{R}$, $\delta>0$.

If $n=1$ and $f'(x)=0$, for a specific $x\in\mathbb{R}$, we get that

$$|f(y)-f(x)| \leq \omega_1(f',\delta)\left[|x-y|+\frac{(x-y)^2}{2\delta}\right], \quad \forall y\in\mathbb{R},\ \delta>0. \tag{42}$$

Let $L_N: C_B(\mathbb{R})\to C_B(\mathbb{R})$ a sequence of positive linear operators, $N\in\mathbb{N}$.

Let $f, g \in C_B(\mathbb{R})$, we have

$$f = f - g + g \leq |f - g| + g,$$

hence

$$L_N(f) \leq L_N(|f - g|) + L_N(g), \tag{43}$$

and

$$L_N(f) - L_N(g) \leq L_N(|f - g|). \tag{44}$$

Similarly, it holds

$$L_N(g) - L_N(f) \leq L_N(|f - g|). \tag{45}$$

That is

$$|L_N(f) - L_N(g)| \leq L_N(|f - g|). \tag{46}$$

Next, we assume that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then

$$|L_N(f)(x) - f(x)| = |L_N(f)(x) - f(x) L_N(1)(x)| =$$

$$= |L_N(f)(x) - L_N(f(x))(x)| = |L_N(f(\cdot) - f(x))(x)| \overset{(46)}{\leq}$$

$$\overset{(46)}{\leq} L_N(|f(\cdot) - f(x)|)(x) \overset{(40)}{\leq} \tag{47}$$

$$\overset{(40)}{\leq} \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!}\left[L_N\left(|\cdot - x|^n\right)(x) + \frac{L_N\left(|\cdot - x|^{n+1}\right)(x)}{(n+1)\delta}\right],$$

$\forall\ x \in \mathbb{R}$, $\delta > 0$.

Here we assume and choose

$$\delta = \left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{1}{n+1}} > 0. \tag{48}$$

Therefore we get

$$|L_N(f)(x) - f(x)| \leq \omega_1\left(f^{(n)}, \left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right) \leq$$

$$\leq \left[L_N\left(|\cdot - x|^n\right)(x) + \frac{\left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right], \tag{49}$$

$\forall\ N \in \mathbb{N}$

If $n=1$, we find that

$$|L_N(f)(x)-f(x)| \leq \omega_1\left(f', \left(L_N\left((\cdot-x)^2\right)(x)\right)^{\frac{1}{2}}\right) \leq$$

$$\leq \left[L_N(|\cdot-x|)(x)+\frac{\left(L_N\left((\cdot-x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right], \quad \forall\, N \in \mathbb{N}. \tag{50}$$

Using (48) and Theorem 3.1, for $g=1$ and L_N such that $L_N(1)=1$, we obtain

$$L_N(|\cdot-x|^n)(x) \leq \left(L_N\left(|\cdot-x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}. \tag{51}$$

In case of $n=1$, we derive

$$L_N(|\cdot-x|)(x) \leq \sqrt{\left(L_N\left((\cdot-x)^2\right)(x)\right)}. \tag{52}$$ ■

An application of Theorem 4.1 follows.

Call Z_N any of the operators B_N, C_N, D_N, $N \in \mathbb{N}$.

Theorem 4.2. *Let $n \in \mathbb{N}$ and $f, f^{(n)} \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Assume $f^{(i)}(x)=0$, $i=1,...,n$. Then*

$$|Z_N(f)(x)-f(x)| \leq \omega_1\left(f^{(n)}, \left(\left(Z_N\left(|\cdot-x|^{n+1}\right)\right)(x)\right)^{\frac{1}{n+1}}\right) \leq$$

$$\leq \left[Z_N(|\cdot-x|^n)(x)+\frac{\left(Z_N\left(|\cdot-x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right] < +\infty, \quad \forall\, N \in \mathbb{N}. \tag{53}$$

Here $\lim_{N\to+\infty} Z_N(f)(x)=f(x)$.

If $n=1$, we obtain

$$|Z_N(f)(x)-f(x)| \leq \omega_1\left(f', \left(\left(Z_N\left((\cdot-x)^2\right)\right)(x)\right)^{\frac{1}{2}}\right) \leq$$

$$\leq \left[Z_N(|\cdot-x|)(x)+\frac{\left(Z_N\left((\cdot-x)^2\right)(x)\right)^{\frac{1}{2}}}{2}\right] < +\infty, \quad \forall\, N \in \mathbb{N}. \tag{54}$$

Again it holds $\lim_{N\to+\infty} Z_N(f)(x)=f(x)$.

Proof. By Theorem 4.1 and Lemmas 3.1–3.3. ■

We mention the following result.

Theorem 4.3. ([16]) *Let $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Consider L_N a sequence of positive linear operators from $C_B(\mathbb{R})$ into itself, $N \in \mathbb{N}$, such that $L_N(1) = 1$.*

Assume that $L_N(|\cdot - x|)(x) > 0$. Then

$$|L_N(f)(x) - f(x)| \leq 2\omega_1(f, L_N(|\cdot - x|)(x)), \quad \forall\ N \in \mathbb{N}. \tag{55}$$

Given that $\lim_{N\to+\infty} L_N(|\cdot - x|)(x) = 0$, and f is also uniformly continuous, we obtain $\lim_{N\to+\infty} L_N(f)(x) = f(x)$.

An application of Theorem 4.3 comes next.

Theorem 4.4. *Let $f \in C_B(\mathbb{R})$, $x \in \mathbb{R}$. Here $Z_N = B_N$, C_N, D_N. Then*

$$|Z_N(f)(x) - f(x)| \leq 2\omega_1(f, Z_N(|\cdot - x|)(x)) < +\infty, \tag{56}$$

$\forall\ N \in \mathbb{N} : N^{1-\beta} > 2$, $0 < \beta < 1$.

If f is also uniformly continuous we derive $\lim_{N\to+\infty} Z_N(f)(x) = f(x)$.

Proof. Direct application of Theorem 4.3 and Lemmas 3.1–3.3. ■

Corollary 4.1. *Let $f \in C_B(\mathbb{R})$. Then*

$$\|B_N(f) - f\|_\infty \leq \tag{57}$$

$$\leq 2\omega_1\left(f, \left(\frac{1}{N^\beta} + \frac{1}{N}\frac{\left(q + \frac{1}{q}\right)}{\lambda}2e^{2\lambda}e^{-\lambda\left(N^{1-\beta}-1\right)}\right)\right) < +\infty,$$

$\forall\ N \in \mathbb{N} : N^{1-\beta} > 2$, $0 < \beta < 1$; $q, \lambda > 0$.

If f is also uniformly continuous, we derive that $\lim_{N\to+\infty} B_N(f) = f$, pointwise and uniformly.

Proof. By Lemma 3.1 and Theorem 4.4. ■

We finish with

Corollary 4.2. *Let $f \in C_B(\mathbb{R})$. Then*

$$\left\{ \begin{array}{c} \|C_N(f) - f\|_\infty \\ \|D_N(f) - f\|_\infty \end{array} \right\} \leq \tag{58}$$

$$\leq 2\omega_1\left(f, \left[\left(\frac{1}{N}+\frac{1}{N^\beta}\right)+\frac{1}{N}\left(\frac{T}{e^{2\lambda N^{1-\beta}}}\frac{\left(q+\frac{1}{q}\right)}{\lambda}2e^{2\lambda}e^{-\lambda\left(N^{1-\beta}-1\right)}\right)\right]\right) < +\infty,$$

$\forall\ N \in \mathbb{N} : N^{1-\beta} > 2,\ 0 < \beta < 1;\ q, \lambda > 0,$ *and* $T = 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}.$

If f is also uniformly continuous, we obtain that

$$\lim_{N\to+\infty} C_N(f) = \lim_{N\to+\infty} D_N(f) = f,$$

pointwise and uniformly.

Proof. By Lemmas 3.2, 3.3 and Theorem 4.4. ■

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