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A WEIGHTED PRIME NUMBER RACE THROUGH SOME PERSPECTIVE OF PROBABILITY THEORY

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Abstract. Aymone introduced the weighted prime number race, which reflects the magnitudes of the primes. We discuss the distribution of prime numbers in reside classes from some perspective of probability theory, using a weighted counting function.

1. Introduction and main result

It is known as Chebyshev's bias that there tend to be more primes of the form $4k + 3$ than of the form $4k + 1$ ($k \in \mathbb{Z}$). This phenomenon lasts up to a certain number, but in fact the following phenomenon occurs: Denote by $\pi(x, q, a)$ the number of primes $p \leq x$ such that $p \equiv a \pmod{q}$. The function

$$\pi(x, 4, 1) - \pi(x, 4, 3)$$

changes its sign infinitely many times. Let χ_4 be the real and non-principal Dirichlet character mod 4, that is,

$$\chi_4(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

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Then we have

$$\pi(x,4,1)-\pi(x,4,3)=\sum_{p\le x:\text{prime}}\chi_4(p).$$

Throughout this paper, let χ be a real and non-principal Dirichlet character modulo q. Aymone [2] discussed whether for some $0\le\sigma<1$ the weighted counting function

$$\sum_{p\le x:\text{prime}}\frac{\chi(p)}{p^\sigma}$$

changes sign only for a finite number of integers $x\ge 1$ using conditions based on the Dirichlet L-function. Furthermore, Aoki and Koyama [3] introduced the waited counting function

$$\pi_s(x,q,a)=\sum_{\substack{p\le x:\text{prime}\\ p\equiv a (\text{mod } q)}}\frac{1}{p^s}\quad (s\ge 0).$$

Note that $\pi(x,q,a)=\pi_s(x,q,a)$ if $s=0$. The sign-changes of

$$\pi(x,q,1)-\pi(x,q,l)$$

is studied in a series of papers [6, 7, 8] by Knapowski and Turán, where $l\not\equiv 1$ (mod q). Denote by $N_q(l)$ the number of solutions for $x^2\equiv l$ (mod q). Kátai [4] proved that if $L(s,\chi)\ne 0$ for $s\in(0,1)$ and if $l_1\not\equiv l_2(\text{mod } q)$ and $N_q(l_1)=$ $=N_q(l_2)$, then

$$\limsup_{x\to\infty}\frac{(\pi(x,q,l_1)-\pi(x,q,l_2))\log x}{\sqrt{x}}>0.$$

Kátai [5] also states the following fact in (4.6):

$$\pi(x+h,q,a)-\pi(x,q,a)=\frac{1}{\varphi(q)}\cdot\frac{h}{\log x}+O\left(\frac{h}{\log^2 x}\right)$$

for every fixed $q\ge 3$, and $(a,q)=1$, where φ is the Euler totient function. For these facts, we are interested in the distribution of large primes. We focus on the arithmetic functions

$$\Pi_q^+(x):=\sum_{n>x}\frac{\Lambda(n)(1+\chi(n))}{2n^\sigma\log n}, \tag{1.1}$$

$$\Pi_q^-(x):=\sum_{n>x}\frac{\Lambda(n)(1-\chi(n))}{2n^\sigma\log n} \tag{1.2}$$

for $\sigma > 1$. Here Λ denotes the von Mangoldt function which is defined as follows:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

for every integer $n \geq 1$. The meaning of (1.1) and (1.2) is as follows: Let $q = 4$ and $\chi = \chi_4$.

$$\begin{aligned} \Pi_4^+(x) - \Pi_4^-(x) &= \sum_{\substack{n>x \\ n\equiv 1 \pmod 4}} \frac{\Lambda(n)}{n^\sigma \log n} - \sum_{\substack{n>x \\ n\equiv 3 \pmod 4}} \frac{\Lambda(n)}{n^\sigma \log n} = \\ &= \sum_{n>x} \frac{\Lambda(n)\chi_4(n)}{n^\sigma \log n} = \\ &= \sum_{p>x:\text{prime}} \frac{\chi_4(p)}{p^\sigma} + \sum_{m=2}^{\infty} \sum_{\substack{p^m>x \\ p:\text{prime}}} \frac{\chi_4(p^m)}{mp^{m\sigma}}. \end{aligned}$$

Hence we investigate the asymptotic behavior of

$$\sum_{n>x} \frac{\Lambda(n)\chi(n)}{n^\sigma \log n}.$$

We introduce the Möbius function μ: $\mu(1) = 1$. For $n > 1$, we write $n = = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ denote distinct prime numbers, then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we write

$$A(n) = \sum_{d|n} \chi(d), \quad B(n) = \sum_{d|n} \mu(d)\chi(d).$$

We note that $A(n)$ and $B(n)$ are nonnegative and multiplicative and nonnegativity will be proved in the proof of Lemma 3.1 of Section 3. The Dirichlet series $\sum_{n=1}^{\infty} A(n)n^{-s}$ and $\sum_{n=1}^{\infty} B(n)n^{-s}$ are calculated in Corollaries 2.1 and 2.2 of Section 2. We state our main theorem:

Theorem 1.1. *Assume $\sigma > 1$.*

(i) *There is a positive constant C_0 such that*

$$C_0 \sum_{n>x} \frac{1}{n^\sigma} - \sum_{n>x} \frac{B(n)}{n^\sigma} \leq \sum_{n>x} \frac{\Lambda(n)\chi(n)}{n^\sigma \log n} \leq \sum_{n>x} \frac{A(n)}{n^\sigma} - C_0 \sum_{n>x} \frac{1}{n^\sigma}$$

for all positive integers x.

(ii) *There is a positive constant* C_1 *such that*

$$C_1 \sum_{n>x} \frac{A(n)}{n^\sigma} - \sum_{n>x} \frac{1}{n^\sigma} \leq \sum_{n>x} \frac{\Lambda(n)\chi(n)}{n^\sigma \log n} \leq \sum_{n>x} \frac{1}{n^\sigma} - C_1 \sum_{n>x} \frac{B(n)}{n^\sigma}$$

for all positive integers x.

Corollary 1.1. *Assume* $\sigma > 1$. *We have*

$$\begin{aligned} \left(\frac{C_0}{\sigma - 1} + o(1)\right) \frac{1}{x^{\sigma-1}} - \sum_{n>x} \frac{B(n)}{n^\sigma} \quad &\leq \quad \sum_{\substack{p>x \\ p:\text{prime}}} \frac{\chi(p)}{p^\sigma} \leq \\ &\leq \quad \sum_{n>x} \frac{A(n)}{n^\sigma} - \left(\frac{C_0}{\sigma - 1} + o(1)\right) \frac{1}{x^{\sigma-1}} \end{aligned}$$

as $x \to \infty$. *Here* C_0 *is a constant in Theorem 1.1.*

We remark that there are a lot of studies on the tails of infinitely divisible distributions. For example, see [10], [11], [13] and [14]. Furthermore, see Sato's book [12] for infinitely divisible distributions. Terminology follows Sato's book [12].

2. Dirichlet L-function

We consider the Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $s = \sigma + it$ with $\sigma > 1$. Note that $L(s, \chi) \neq 0$ for $\sigma > 1$.

Lemma 2.1. *Assume* $\sigma > 1$. *We have*

$$\begin{aligned} \frac{L(\sigma - it, \chi)}{L(\sigma, \chi)} \quad &= \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma L(\sigma, \chi)} \, e^{it \log n} = \\ &= \quad \exp \left[\sum_{n=2}^{\infty} (e^{it \log n} - 1) \frac{\chi(n)\Lambda(n)}{n^\sigma \log n} \right]. \end{aligned}$$

Remark 2.1. If $\chi(n)$ takes negative values, $\frac{L(\sigma - it, \chi)}{L(\sigma, \chi)}$ is not a characteristic function of an infinitely divisible distribution.

Proof. The first equation is obvious. We see from Example 2 of [1, p.239] that

$$L(\sigma - it, \chi) = \exp\left[\sum_{n=2}^{\infty} e^{it\log n} \frac{\chi(n)\Lambda(n)}{n^\sigma \log n}\right]$$

for all $t \in \mathbb{R}$. Hence the second equation holds. ■

The following lemma holds in the same way as Lemma 2.1.

Lemma 2.2. *Assume* $\sigma > 1$. *We have*

$$\begin{aligned} \frac{L(\sigma,\chi)}{L(\sigma - it,\chi)} &= \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)L(\sigma,\chi)}{n^\sigma} e^{it\log n} = \\ &= \exp\left[\sum_{n=2}^{\infty} (e^{it\log n} - 1) \frac{-\chi(n)\Lambda(n)}{n^\sigma \log n}\right]. \end{aligned}$$

Let $\sigma > 1$. As stated in [15], the Riemann zeta distribution Ψ is an infinitely divisible distribution whose characteristic function is represented as

$$\begin{aligned} \hat{\Psi}(t) &= \int_{\mathbb{R}} e^{itu}\Psi(du) = \\ &= \exp\left[\sum_{n=2}^{\infty} (e^{it\log n} - 1)\frac{\Lambda(n)}{n^\sigma \log n}\right] = \\ &= \exp\left[\int_0^\infty (e^{itu} - 1)\Pi(du)\right], \end{aligned} \tag{2.1}$$

where the Lévy measure Π is as follows:

$$\Pi(du) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} \delta_{\log n}(du).$$

Here δ_a represents the probability measure concentrated at a, that is,

$$\delta_a(B) = \begin{cases} 1 & \text{if } a \in B, \\ 0 & \text{otherwise} \end{cases}$$

for any Borel set B in $\mathbb{R}$. Using the Riemann zeta function $\zeta(s)$, we have

$$\hat{\Psi}(t) = \frac{\zeta(\sigma - it)}{\zeta(\sigma)}, \quad t \in \mathbb{R}.$$

Hence we have

$$\hat{\Psi}(t) = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{1}{n^\sigma} e^{it \log n}.$$

This means that

$$\Psi(dx) = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \delta_{\log n}(dx).$$

See Lin and Hu [9] for the Riemann zeta distribution.

In this section, we investigate the following two functions:

$$\hat{\Psi}_+(t) := \hat{\Psi}(t) \cdot \frac{L(\sigma - it, \chi)}{L(\sigma, \chi)},$$
$$\hat{\Psi}_-(t) := \hat{\Psi}(t) \cdot \frac{L(\sigma, \chi)}{L(\sigma - it, \chi)}.$$

We show that these become characteristic functions of infinitely divisible distributions. The convolution for two finite measures H_1 and H_2 on $\mathbb{R}$ is defined by

$$H_1 * H_2(B) = \int_{\mathbb{R}} H_1(B - x) H_2(dx)$$

for any Borel set B in $\mathbb{R}$.

Theorem 2.1. *Assume $\sigma > 1$. We have*

$$\begin{aligned} \hat{\Psi}_+(t) &= \int_0^\infty e^{itx} \frac{1}{\zeta(\sigma) L(\sigma, \chi)} \sum_{n=1}^{\infty} \frac{A(n)}{n^\sigma} \delta_{\log n}(dx) = \\ &= \exp\left[\sum_{n=2}^{\infty} (e^{it \log n} - 1) \frac{(1 + \chi(n))\Lambda(n)}{n^\sigma \log n} \right]. \end{aligned}$$

Remark 2.2. The probability distribution Ψ_+ defined by $\hat{\Psi}_+(t)$ is an infinitely divisible distribution.

Proof. From (2.1) and Lemma 2.1, it follows that

$$\hat{\Psi}_+(t) = \exp\left[\sum_{n=2}^{\infty} (e^{it \log n} - 1) \frac{(1 + \chi(n))\Lambda(n)}{n^\sigma \log n} \right].$$

Set

$$\nu_+(dx) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma L(\sigma,\chi)} \delta_{\log n}(dx)$$

for any Borel set B in $\mathbb{R}$. Then it follows that

$$\hat{\Psi}(t) \cdot \frac{L(\sigma - it,\chi)}{L(\sigma,\chi)} = \int_{\mathbb{R}} e^{itx} \Psi * \nu_+(dx).$$

Here we have

$$\begin{aligned}
\Psi * \nu_+(B) &= \int_{\mathbb{R}} \nu_+(B-x)\Psi(dx) = \\
&= \frac{1}{\zeta(\sigma)} \sum_{m=1}^{\infty} \frac{1}{m^\sigma} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma L(\sigma,\chi)} \int_{\mathbb{R}} \delta_{\log n}(B-x)\delta_{\log m}(dx) = \\
&= \frac{1}{\zeta(\sigma)} \sum_{m=1}^{\infty} \frac{1}{m^\sigma} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma L(\sigma,\chi)} \delta_{\log n + \log m}(B).
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
\hat{\Psi}_+(t) &= \hat{\Psi}(t) \cdot \frac{L(\sigma - it,\chi)}{L(\sigma,\chi)} = \\
&= \frac{1}{\zeta(\sigma)} \sum_{m=1}^{\infty} \frac{1}{m^\sigma} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma L(\sigma,\chi)} \int_{\mathbb{R}} e^{itx} \delta_{\log n + \log m}(dx) = \\
&= \frac{1}{\zeta(\sigma)L(\sigma,\chi)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{(mn)^\sigma} e^{it\log(mn)} = \\
&= \frac{1}{\zeta(\sigma)L(\sigma,\chi)} \sum_{k=1}^{\infty} \sum_{n|k} \frac{\chi(n)}{k^\sigma} e^{it\log k}.
\end{aligned}$$

This implies the first equality holds. ∎

Corollary 2.1. *We have*

$$\sum_{n=1}^{\infty} \frac{A(n)}{n^s} = \zeta(s)L(s,\chi)$$

for $s = \sigma + it$ with $\sigma > 1$.

Proof. The corollary is obvious from Theorem 2.1. ∎

Theorem 2.2. *Assume $\sigma > 1$. We have*

$$\begin{aligned}\hat{\Psi}_-(t) &= \int_0^\infty e^{itx}\frac{L(\sigma,\chi)}{\zeta(\sigma)}\sum_{n=1}^\infty \frac{B(n)}{n^\sigma}\delta_{\log n}(dx) = \\ &= \exp\left[\sum_{n=2}^\infty (e^{it\log n}-1)\frac{(1-\chi(n))\Lambda(n)}{n^\sigma \log n}\right].\end{aligned}$$

Remark 2.3. The probability distribution Ψ_- defined by $\hat{\Psi}_-(t)$ is an infinitely divisible distribution.

Proof. From (2.1) and Lemma 2.2, it follows that

$$\hat{\Psi}_-(t) = \exp\left[\sum_{n=2}^\infty (e^{it\log n}-1)\frac{(1-\chi(n))\Lambda(n)}{n^\sigma \log n}\right].$$

Set

$$\nu_-(B) = L(\sigma,\chi)\sum_{n=1}^\infty \frac{\mu(n)\chi(n)}{n^\sigma}\delta_{\log n}(B)$$

for any Borel set B in $\mathbb{R}$. Similarly to the proof of Theorem 2.1, we obtain that

$$\begin{aligned}\Psi * \nu_-(B) &= \frac{L(\sigma,\chi)}{\zeta(\sigma)}\sum_{m=1}^\infty \frac{1}{m^\sigma}\sum_{n=1}^\infty \frac{\mu(n)\chi(n)}{n^\sigma}\delta_{\log n+\log m}(B) = \\ &= \frac{L(\sigma,\chi)}{\zeta(\sigma)}\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\mu(n)\chi(n)}{(mn)^\sigma}\delta_{\log(nm)}(B) = \\ &= \frac{L(\sigma,\chi)}{\zeta(\sigma)}\sum_{k=1}^\infty\sum_{n|k} \frac{\mu(n)\chi(n)}{k^\sigma}\delta_{\log k}(B).\end{aligned}$$

Hence the first equality holds. ■

Corollary 2.2. *We have*

$$\sum_{n=1}^\infty \frac{B(n)}{n^s} = \zeta(s)L(s,\chi)^{-1}$$

for $s = \sigma + it$ with $\sigma > 1$.

Proof. The corollary is obvious form Theorem 2.2. ■

3. Proof of Theorem 1.1

First, we make preparations for proving Theorem 1.1.

Lemma 3.1. *For any positive integer x, we have*

$$\Pi_q^+(x) \le \sum_{n>x} \frac{A(n)}{2n^\sigma} \quad \text{and} \quad \Pi_q^-(x) \le \sum_{n>x} \frac{B(n)}{2n^\sigma}.$$

Proof. Let p be prime. Notice that

$$\begin{aligned} A(p^m) &= \sum_{d|p^m} \chi(d) = 1 + \sum_{n=1}^{m} \chi(p)^n, \\ B(p^m) &= \sum_{d|p^m} \mu(d)\chi(d) = \mu(1)\chi(1) + \mu(p)\chi(p) = 1 - \chi(p) \end{aligned}$$

for $m \ge 1$. Since $B(n)$ are multiplicative, we see from this that $B(n) \ge 0$. The fact that $A(n) \ge 0$ is stated in Theorem 6.19 of [1]. First, we prove the first inequality. If $\chi(p) = 0$, then $A(p^m) = 1 = 1 + \chi(p)^m$. If $\chi(p) = 1$, then $A(p^m) = m + 1 \ge 1 + \chi(p)^m$. Let $\chi(p) = -1$. If m is odd, then $A(p^m) = 0 = = 1 + \chi(p)^m$. If m is even, then

$$A(p^m) = 1 = \frac{1 + \chi(p)^m}{2} \ge \frac{1 + \chi(p)^m}{m}.$$

These imply that

$$\frac{(1 + \chi(p^m))\Lambda(p^m)}{p^{m\sigma} \log p^m} = \frac{1 + \chi(p)^m}{p^{m\sigma} m} \le \frac{A(p^m)}{p^{m\sigma}}.$$

Hence the first inequality holds. Secondly, we prove the second inequality. We obtain that

$$\begin{aligned} \frac{(1 - \chi(p^m))\Lambda(p^m)}{p^{m\sigma} \log p^m} &= \frac{(1 - \chi(p))(1 + \chi(p) + \cdots + \chi(p)^{m-1})}{p^{m\sigma} m} = \\ &= \frac{1 - \chi(p)}{p^{m\sigma}} \cdot \frac{1 + \chi(p) + \cdots + \chi(p)^{m-1}}{m} \le \\ &\le \frac{1 - \chi(p)}{p^{m\sigma}} = \frac{B(p^m)}{p^{m\sigma}}, \end{aligned}$$

Hence the second inequality holds. ■

Here we set

$$\Pi_+(du) = \sum_{n=2}^{\infty} \frac{(1+\chi(n))\Lambda(n)}{n^\sigma \log n}\delta_{\log n}(du),$$
$$\Pi_-(du) = \sum_{n=2}^{\infty} \frac{(1-\chi(n))\Lambda(n)}{n^\sigma \log n}\delta_{\log n}(du).$$

In addition, we set $\lambda_1 = \Pi_+((0,\infty))$, $\lambda_2 = \Pi_-((0,\infty))$, $\rho_+(du) = \lambda_1^{-1}\Pi_+(du)$, and $\rho_-(du) = \lambda_2^{-1}\Pi_-(du)$. Let $K > 0$ and define finite measures $R^+_{\lambda_1}$ and $R^-_{\lambda_2}$ by

$$R^+_{\lambda_1}(du) = K^{-1}e^{-\lambda_1}\sum_{n=1}^{\infty}\frac{\lambda_1^n}{n!}\rho_+^{n*}(du), \tag{3.1}$$

$$R^-_{\lambda_2}(du) = K^{-1}e^{-\lambda_2}\sum_{n=1}^{\infty}\frac{\lambda_2^n}{n!}\rho_-^{n*}(du) \tag{3.2}$$

on $(0,\infty)$. Here ρ_+^{n*} and ρ_-^{n*} denote the n-fold convolutions of ρ_+ and ρ_-, respectively. For any finite measure H, we denote by $\overline{H}$ the tail of H, that is,

$$\overline{H}(x) = H(\{u : u > x\}).$$

Lemma 3.2. *For $x \geq 0$, we have*

$$K\overline{R^+_{\lambda_1}}(x) = \overline{\Psi_+}(x) \quad \textit{and} \quad K\overline{R^-_{\lambda_2}}(x) = \overline{\Psi_-}(x).$$

Proof. By virtue of Theorems 2.1 and 2.2, we obtain that

$$\Psi_+(du) = e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\lambda_1^n}{n!}\rho_+^{n*}(du),$$
$$\Psi_-(du) = e^{-\lambda_2}\sum_{n=0}^{\infty}\frac{\lambda_2^n}{n!}\rho_-^{n*}(du),$$

where ρ_+^{0*} and ρ_-^{0*} are understood to be δ_0. The equations (3.1) and (3.2) tell us that the lemma holds. ■

Take λ_0 such that $\lambda_0 > \max\{\lambda_1,\ \lambda_2,\ \log 2\}$ and take $K > 0$ such that $Ke^{\lambda_0} < 1$. We define new finite measures $R^+_{\lambda_0}$ and $R^-_{\lambda_0}$ by

$$R^+_{\lambda_0}(du) = K^{-1}e^{-\lambda_0}\sum_{n=1}^{\infty}\frac{\lambda_0^n}{n!}\rho_+^{n*}(du),$$
$$R^-_{\lambda_0}(du) = K^{-1}e^{-\lambda_0}\sum_{n=1}^{\infty}\frac{\lambda_0^n}{n!}\rho_-^{n*}(du)$$

on $(0,\infty)$. We can obtain similar lemmas to Lemmas 2.3 and 2.4 of [15]:

Lemma 3.3. *For $k \ge 1$ and $l \ge 1$, we have*

$$\begin{aligned}\overline{(R^+_{\lambda_0})^{(k+l)*}}(x) &\ge (K^{-1}(1-e^{-\lambda_0}))^l \cdot \overline{(R^+_{\lambda_0})^{k*}}(x),\\ \overline{(R^-_{\lambda_0})^{(k+l)*}}(x) &\ge (K^{-1}(1-e^{-\lambda_0}))^l \cdot \overline{(R^-_{\lambda_0})^{k*}}(x)\end{aligned}$$

and

$$\begin{aligned}\overline{(R^+_{\lambda_0})^{k*}}(x) &\ge \left(K^{-1}(1-e^{-\lambda_0})\right)^{k-1} \cdot \overline{R^+_{\lambda_0}}(x),\\ \overline{(R^-_{\lambda_0})^{k*}}(x) &\ge \left(K^{-1}(1-e^{-\lambda_0})\right)^{k-1} \cdot \overline{R^-_{\lambda_0}}(x).\end{aligned}$$

Lemma 3.4. *Let $m \ge 2$. For $x \ge 0$, we have*

$$\begin{aligned}\lambda_0\overline{\rho_+}(x) &= \sum_{k=1}^{m} \frac{(Ke^{\lambda_0})^{2k-1}}{2k-1}\overline{(R^+_{\lambda_0})^{(2k-1)*}}(x) + \sum_{k=1}^{\infty} \frac{(Ke^{\lambda_0})^{2k}}{2(m+k)-1} S_+(k,x),\\ \lambda_0\overline{\rho_-}(x) &= \sum_{k=1}^{m} \frac{(Ke^{\lambda_0})^{2k-1}}{2k-1}\overline{(R^-_{\lambda_0})^{(2k-1)*}}(x) + \sum_{k=1}^{\infty} \frac{(Ke^{\lambda_0})^{2k}}{2(m+k)-1} S_-(k,x),\end{aligned}$$

where

$$\begin{aligned}S_+(k,x) &= (Ke^{\lambda_0})^{2m-1}\overline{(R^+_{\lambda_0})^{(2(m+k)-1)*}}(x) - \frac{2(m+k)-1}{2k}\overline{(R^+_{\lambda_0})^{(2k)*}}(x),\\ S_-(k,x) &= (Ke^{\lambda_0})^{2m-1}\overline{(R^-_{\lambda_0})^{(2(m+k)-1)*}}(x) - \frac{2(m+k)-1}{2k}\overline{(R^-_{\lambda_0})^{(2k)*}}(x).\end{aligned}$$

The following theorem can be proved in the same way as Theorem 1.1 of [15]. The proof is omitted.

Theorem 3.1. *Assume $\sigma > 1$. There is a positive constant C_2 such that*

$$C_2 \sum_{n>x} \frac{A(n)}{n^\sigma} \le \Pi_q^+(x) = \sum_{n>x} \frac{(1+\chi(n))\Lambda(n)}{2n^\sigma \log n} \le \sum_{n>x} \frac{A(n)}{2n^\sigma}, \tag{3.3}$$

$$C_2 \sum_{n>x} \frac{B(n)}{n^\sigma} \le \Pi_q^-(x) = \sum_{n>x} \frac{(1-\chi(n))\Lambda(n)}{2n^\sigma \log n} \le \sum_{n>x} \frac{B(n)}{2n^\sigma} \tag{3.4}$$

for all positive integers x.

Now we proof Theorem 1.1. From (3.3) and (3.4) it follows that

$$\begin{aligned}\sum_{n>x} \frac{\Lambda(n)}{n^\sigma \log n} - \sum_{n>x} \frac{B(n)}{n^\sigma} &\le \sum_{n>x} \frac{\chi(n)\Lambda(n)}{n^\sigma \log n} \le \sum_{n>x} \frac{A(n)}{n^\sigma} - \sum_{n>x} \frac{\Lambda(n)}{n^\sigma \log n},\\ 2C_2 \sum_{n>x} \frac{A(n)}{n^\sigma} - \sum_{n>x} \frac{\Lambda(n)}{n^\sigma \log n} &\le \sum_{n>x} \frac{\chi(n)\Lambda(n)}{n^\sigma \log n} \le \\ &\le \sum_{n>x} \frac{\Lambda(n)}{n^\sigma \log n} - 2C_2 \sum_{n>x} \frac{B(n)}{n^\sigma}.\end{aligned}$$

Using Theorem 1.1 of [15], we obtain the theorem. ■

Finally, we prove Corollary 1.1. We have

$$\sum_{n>x} \frac{\Lambda(n)\chi(n)}{n^\sigma \log n} = \sum_{p>x} \frac{\chi(p)}{p^\sigma} + \sum_{m=2}^{\infty} \sum_{p^m>x} \frac{\chi(p^m)}{mp^{m\sigma}}.$$

The second term is calculated as follows:

$$\begin{aligned}
\left| \sum_{m=2}^{\infty} \sum_{p^m>x} \frac{\chi(p^m)}{mp^{m\sigma}} \right| &\leq \sum_{m=2}^{\infty} \sum_{p>[x^{1/m}]} \frac{1}{mp^{m\sigma}} = \\
&= \sum_{m=2}^{[\log_2 x]} \sum_{p>[x^{1/m}]} \frac{1}{mp^{m\sigma}} + \sum_{m=[\log_2 x]+1}^{\infty} \sum_{p>[x^{1/m}]} \frac{1}{mp^{m\sigma}}
\end{aligned}$$

for $x \geq 4$. Then we obtain that

$$\begin{aligned}
\sum_{m=2}^{[\log_2 x]} \sum_{p>[x^{1/m}]} \frac{1}{mp^{m\sigma}} &\leq \sum_{m=2}^{[\log_2 x]} \frac{1}{m} \sum_{n=[x^{1/m}]+1}^{\infty} \frac{1}{n^{m\sigma}} \leq \\
&\leq \sum_{m=2}^{[\log_2 x]} \frac{1}{m} \left\{ \frac{1}{([x^{1/m}]+1)^{m\sigma}} + \int_{[x^{1/m}]+1}^{\infty} \frac{du}{u^{m\sigma}} \right\} \leq \\
&\leq \sum_{m=2}^{[\log_2 x]} \frac{1}{m} \left\{ \frac{1}{x^\sigma} + \frac{1}{m\sigma - 1} \cdot \frac{1}{([x^{1/m}]+1)^{m\sigma-1}} \right\} \leq \\
&\leq \sum_{m=2}^{[\log_2 x]} \frac{1}{m} \cdot \frac{m\sigma}{m\sigma - 1} \frac{1}{x^{\sigma - 2^{-1}}} \leq \\
&\leq \frac{\sigma}{\sigma - 2^{-1}} \cdot \frac{\log \log_2 x}{x^{\sigma - 2^{-1}}}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{m=[\log_2 x]+1}^{\infty} \sum_{p>[x^{1/m}]} \frac{1}{mp^{m\sigma}} &\leq \sum_{m=[\log_2 x]+1}^{\infty} \sum_{p:prime} \frac{1}{p^{m\sigma}} \leq \\
&\leq \sum_{m=[\log_2 x]+1}^{\infty} \left\{ \frac{1}{2^{m\sigma}} + \int_2^{\infty} \frac{du}{u^{m\sigma}} \right\} = \\
&= \sum_{m=[\log_2 x]+1}^{\infty} \frac{m\sigma + 1}{m\sigma - 1} \cdot \frac{1}{2^{m\sigma}} \leq \\
&\leq \frac{\sigma + 1}{\sigma - 1} \cdot \frac{1}{1 - 2^{-\sigma}} \cdot \frac{1}{x^\sigma}.
\end{aligned}$$

Hence we obtain that

$$\left|\sum_{m=2}^{\infty}\sum_{p^m>x}\frac{\chi(p^m)}{mp^{m\sigma}}\right| \le \left\{\frac{\sigma}{\sigma-1}\cdot\frac{\log\log_2 x}{x^{1/2}}+\frac{\sigma+1}{(\sigma-1)(1-2^{-\sigma})}\cdot\frac{1}{x}\right\}\frac{1}{x^{\sigma-1}}$$

for $x \ge 4$. It follows from Theorem 1.1 (i) that

$$\sum_{p>x:prime}\frac{\chi(p)}{p^{\sigma}} \ge C_0\sum_{n>x}\frac{1}{n^{\sigma}}-\sum_{n>x}\frac{B(n)}{n^{\sigma}}-\sum_{m=2}^{\infty}\sum_{p^m>x}\frac{\chi(n)\Lambda(n)}{n^{\sigma}\log n},$$

$$\sum_{p>x:prime}\frac{\chi(p)}{p^{\sigma}} \le \sum_{n>x}\frac{A(n)}{n^{\sigma}}-C_0\sum_{n>x}\frac{1}{n^{\sigma}}-\sum_{m=2}^{\infty}\sum_{p^m>x}\frac{\chi(n)\Lambda(n)}{n^{\sigma}\log n}.$$

Hence the corollary holds. ∎

4. Remark on Theorem 3.1

In [15], we discussed about the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

for $s=\sigma+it$ with $\sigma>\sigma_a$. Then we assumed that $f(n)\ge 0$ for any n and $f(n)>0$ for some $n\ge 2$. Now we suppose that $F(s)$ converges absolutely for $\sigma>\sigma_a=1$. If $F(s)\neq 0$ for $\sigma>\sigma_0\ge 1$, then Theorem 11.14 of [1] tells us that

$$\text{(4.1)}\qquad \frac{F(\sigma-it)}{F(\sigma)} = \exp\left[\sum_{n=2}^{\infty}(e^{it\log n}-1)\frac{(f'\circ f^{-1})(n)}{n^{\sigma}\log n}\right]$$

for $\sigma>\sigma_0$. Here f^{-1} is the Dirichlet inverse of f and $f'(n)=f(n)\log n$ and $f'\circ f^{-1}$ denotes the Dirichlet convolution of f' and f^{-1}. Our method depends on the characteristic function of an infinitely divisible distribution for (4.1). So we required $f(n)$ to be completely multiplicative in Theorem 3.1 of [15]. If $f(n)$ is not completely multiplicative, there exists an example such that (4.1) is not a characteristic function of an infinitely divisible distribution as follows: Let $d(n)$ be the number of divisors of n, that is, $d(n)=\sum_{d|n}1$. We consider $f(n)=d(n)^3$. Then $f(n)$ is multiplicative, but it's not completely multiplicative. Furthermore, it follows from Theorem 13.12 of [1] that $f(n)=o(n^{\delta})$ for every $\delta>0$. Hence the Dirichlet series $F(s)$ converges absolutely for $\sigma>1$.

Lemma 4.1. *If $f(n)=d(n)^3$, then we have $f'\circ f^{-1}(4)<0$.*

Proof. We see from Theorem 2.8 of [1] that

$$f^{-1}(2) \;=\; -\frac{1}{f(1)} \sum_{\substack{d|2 \\ d<2}} f\left(\frac{2}{d}\right) f^{-1}(d) = -f(2) = -8.$$

Hence we obtain that

$$f' \circ f^{-1}(4) \;=\; \sum_{d|4} (f(d)\log d) f^{-1}\left(\frac{4}{d}\right) = -10\log 2 < 0.$$

The lemma holds. ■

Set

$$\Pi(B) = \int_B \sum_{m=2}^{\infty} \frac{f' \circ f^{-1}(n)}{n^\sigma \log n} \delta_{\log n}(du)$$

for any Borel set B in $\mathbb{R}$. Lemma 4.1 implies that if $f(n) = d(n)^3$, Π is not a measure. Hence the following proposition holds:

Proposition 4.1. *If $f(n) = d(n)^3$, then $F(\sigma - it)F(\sigma)^{-1}$ is not a characteristic function of an infinitely divisible distribution.*

Remark. The arithmetic function $d(n)^3$ is not completely multiplicative.

In this paper, we discuss two cases

$$F(s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s} \quad \text{and} \quad F(s) = \sum_{n=1}^{\infty} \frac{B(n)}{n^s}.$$

See Theorems 2.1 and 2.2, Corollaries 2.1 and 2.2.

Proposition 4.2. *The arithmetic functions $A(n)$ and $B(n)$ are not completely multiplicative.*

Proof. Since χ is non-principal, there is a prime number p such that $\chi(p) = -1$. Then

$$A(p^2) = 1 + \chi(p) + \chi(p)^2 \neq (1 + \chi(p))^2 = A(p)^2,$$
$$B(p^2) = 1 - \chi(p) \neq (1 - \chi(p))^2 = B(p)^2.$$

Hence the lemma holds. ■

Proposition 4.2 insists that we do not know whether $(f' \circ f^{-1})(n) \geq 0$ for any $n \geq 2$. Hence Theorems 2.1 and 2.2 do not tell us that

$$(A' \circ A^{-1})(n) = 1 + \chi(n), \tag{4.2}$$

$$(B' \circ B^{-1})(n) = 1 - \chi(n) \tag{4.3}$$

for $n \geq 2$ from the point of view of probability theory. However, from the point of view of Dirichlet series, we can obtain (4.2). Indeed, we have

$$\sum_{n=2}^{\infty} \frac{(1+\chi(n))\Lambda(n)}{n^s \log n} = \sum_{n=2}^{\infty} \frac{(f' \circ f^{-1})(n)}{n^s \log n}$$

for $s = \sigma + it$ with $\sigma > 1$. It follows from Theorem 11.3 of [1] that (4.2) holds. Similarly, (4.3) holds.

Hence, in this paper, we deal with an example that could not be handled in Theorem 3.1 of [15].

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