

ESTIMATES FOR MULTIPLICATIVE FUNCTIONS, II.

Karl-Heinz Indlekofer (Paderborn, Germany)

Erdener Kaya (Cologne, Germany)

Communicated by Imre Kátai

(Received 21 May 2025; accepted 10 June 2025)

Abstract. In this paper we consider multiplicative functions f and g , $|f| \leq g$, as in Part I, and prove $\sum_{n \leq x} f(n) = \{1 + o(1)\} x^{ia} (1 + ia)^{-1} A_x \sum_{n \leq x} g(n)$ if the series $\sum_p (g(p) - \operatorname{Ref}(p) p^{-ia}) p^{-1}$ converges for some $a \in \mathbb{R}$.

1. Introduction

Let $\mathcal{M}_1$ be defined by

$$\mathcal{M}_1 := \{f : \mathbb{N} \rightarrow \mathbb{C} : f \text{ multiplicative; (i), (ii), (iii) hold}\}$$

where

- (i) $|f(p)| \leq B$ for all primes p , $B \geq 1$,
- (ii) $\sum_{p, k \geq 2} \frac{|f(p^k)|}{p^k} < \infty$,
- (iii) $\sum_{\substack{p^k \leq x \\ k \geq 2}} |f(p^k)| = O\left(\frac{x}{\log x}\right).$

Key words and phrases: Convolution of arithmetical functions, multiplicative functions, asymptotic results on arithmetic functions.

2010 Mathematics Subject Classification: 11A25, 11K65, 11N37.

<https://doi.org/10.71352/ac.58.100625>

Published online: 15 June 2025

In [5] we proved asymptotic results for the behaviour of

$$(1.1) \quad M(f, x) := \sum_{n \leq x} f(n)$$

as $x \rightarrow \infty$.

For example, if $f, g \in \mathcal{M}_1, |f| \leq g$ and

$$\sum_{n \leq x^\varepsilon} \frac{g(n)}{n} \leq \delta(\varepsilon) \sum_{n \leq x} \frac{g(n)}{n}, \quad \delta(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

we showed (see [5], Theorem 5.1):

If

$$(1.2) \quad \sum_p \frac{g(p) - \operatorname{Ref}(p)p^{-it}}{p} = \infty \text{ for all } t \in \mathbb{R},$$

then

$$M(f, x) = o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right)$$

as $x \rightarrow \infty$.

In this paper we consider the case that the sum (1.2) converges for some $t = a$.

For this let $\mathbb{P}_1 \subset \mathbb{P}$ be a set of primes such that, for some $\alpha > 0$,

$$(1.3) \quad \sum_{\substack{p \leq x \\ p \in \mathbb{P}_1}} \frac{g(p) \log p}{p} \sim \alpha \log x \text{ as } x \rightarrow \infty.$$

Define multiplicative functions $g_j, j = 1, 2$ by

$$g_1(p^k) = \begin{cases} g(p^k), & p \in \mathbb{P}_1 \\ 0 & \text{otherwise,} \end{cases} \quad g_2(p^k) = \begin{cases} g(p^k), & p \notin \mathbb{P}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g = g_1 * g_2$, and we prove with the notation (1.1)

Theorem 1.1. *Let $f, g \in \mathcal{M}_1, g \geq 0$ and $|f| \leq g$, such that for some $a \in \mathbb{R}$*

$$(1.4) \quad \sum_p \frac{g(p) - \operatorname{Ref}(p)p^{-ia}}{p} < \infty.$$

*Assume that $g = g_1 * g_2$ with (1.3) holds. Then, as $x \rightarrow \infty$,*

$$(1.5) \quad \begin{aligned} M(f, x) &= \frac{x^{ia}}{1 + ia} \prod_{p \leq x} \left(1 + \sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)}\right) \times \\ &\times \left(1 + \sum_{m=1}^{\infty} g(p^m)p^{-m}\right)^{-1} \sum_{n \leq x} g(n)\{1 + o(1)\}. \end{aligned}$$

2. Proof of Theorem 1.1

Using the notations of [5] we put $(f, g \in \mathcal{M}_1)$

$$\begin{aligned} f &= h * f_0, \\ g &= h_1 * g_0 \end{aligned}$$

where f_0, g_0 are exponentially multiplicative. The corresponding Dirichket series are

$$\begin{aligned} F(s) &= H(s)F_0(s), \\ G(s) &= H_1(s)G_0(s) \end{aligned}$$

where $s = \sigma + it$ ($\sigma > 1, t \in \mathbb{R}$).

Put

$$(2.1) \quad M(x) := \sum_{n \leq x} f(n) - A_x \sum_{n \leq x} g(n)n^{ia}$$

where

$$(2.2) \quad A_x = \frac{H(1+ia)}{H_1(1)} \exp \left(- \sum_{p \leq x} \frac{g(p) - f(p)p^{-ia}}{p} \right).$$

Obviously, as $x \rightarrow \infty$,

$$A_x = (1 + o(1)) \prod_{p \leq x} \left(1 + \sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)} \right) \left(1 + \sum_{m=1}^{\infty} g(p^m)p^{-m} \right)^{-1}.$$

Remark 2.1. If a sum $\sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)}$ has the value -1 , then the product may be omitted. Otherwise, the product has the form $cL(\log x)$, where c is a non-zero constant and $L(y)$ a non-vanishing slowly oscillating function with $|L(y)| = 1$.

Now the following holds.

Lemma 2.1. *Let $M(x)$ be defined by (2.1). Then, as $x \rightarrow \infty$,*

$$\begin{aligned} |M(x)| &\ll \frac{1}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + o \left(\frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right) \ll \\ &\ll \varepsilon_1^\alpha \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) + \int_{x^{\varepsilon_1}}^x \frac{|M(u)|}{u^2} du + o \left(\frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right). \end{aligned}$$

The proof follows from Remark 4.1 of [4] and Remark 5.1, (ii) of [5]. Put (cf. [4], Remark 4.1)

$$K_0 = \mathbf{1} * \Lambda_{f_0} * (f - A_x g_a)$$

where $g_a(n) = g(n)n^{ia}$. Then

$$LM = \mathbf{1} * L_0(f - A_x g_a) + R_1$$

where $R_1 = L * (f - A_x g_a)$ and

$$R_1(x) \ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right).$$

Further

$$\mathbf{1} * L_0(f - A_x g_a) = \mathbf{1} * \Lambda_{f_0} * (f - A_x g_a) + R_2 + R_3$$

where

$$\begin{aligned} R_2 &= \mathbf{1} * \Lambda_{h_1} * (f - A_x g_a), \\ R_3 &= \mathbf{1} * A_x g_a * (\Lambda_f - \Lambda_{g_a}). \end{aligned}$$

As in [4], page 172

$$|R_2(x)| = o \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

The convergence of (1.4) implies

$$(2.3) \quad \sum_p \frac{|Ref(p)p^{-ia} - g(p)|^2}{p} < \infty$$

and

$$|R_3(x)| \ll \varepsilon^\alpha x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) + o \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

Thus we obtain, for $2 \leq u \leq x$

$$M(u) = \frac{K_0(u)}{\log u} + o \left(\frac{u}{\log u} \exp \left(\sum_{p \leq u} \frac{g(p)}{p} \right) \right).$$

Then (cf. [4])

$$(2.4) \quad \int_{x^{\varepsilon_1}}^x \frac{|M(u)|}{u^2} du \ll \frac{1}{\varepsilon_1} \left(\frac{1}{\log x} \int_{-\infty}^{\infty} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 |F(s) - A_x G(s - ia)|^2 \frac{dt}{|s|^2} \right)^{\frac{1}{2}} + o \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

Our aim is to prove

$$(2.5) \quad \begin{aligned} & \int_{-\infty}^{\infty} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 |F(s) - A_x G(s - ia)|^2 \frac{dt}{|s|^2} = \\ & = \int_{-\infty}^{\infty} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 |F(s + ia) - A_x G(s)|^2 \frac{dt}{|s + ia|^2} = \\ & = o \left((\log x) \exp \left(2 \sum_{p \leq x} \frac{g(p)}{p} \right) \right) \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Put

$$I_1(\varepsilon) := \{t : |t| \leq \frac{1}{\varepsilon \log x}\}$$

and

$$I_2(\varepsilon) := \{t : \frac{1}{\varepsilon \log x} < |t| \leq \varepsilon^{-2\alpha}\}.$$

Choose B' such that $\alpha < B'$ and $g(p) < B'$ for all p , and define (see (2.7) of [4])

$$\varphi_0 := \pi \frac{B'}{\alpha} \quad \text{and} \quad \beta = 1 - \frac{\sin \varphi_0}{\varphi_0}.$$

Observe $\beta > 0$. Then the following Lemma holds.

Lemma 2.2. *For every $\beta > 0$*

$$(2.6) \quad |F(s + ia) - A_x G(s)| = o(1) \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \quad \text{for } t \in I_1(\varepsilon)$$

and

$$(2.7) \quad \max(|F(s + ia)|, |G(s)|) \ll e^{\alpha\beta} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \quad \text{for } t \in I_2(\varepsilon).$$

Proof. Consider

$$|F(s + ia) - A_x G(s)| = |A_x G(s)| \cdot \left| \frac{F(s + ia)}{A_x G(s)} - 1 \right|$$

and

$$(2.8) \quad \frac{F(s + ia)}{A_x G(s)} = \frac{H(s + ia)}{H_1(s)} \cdot \frac{F_0(s + ia)}{A_x G_0(s)}.$$

In the definition of (2.8) we may assume that $f(p) = g(p) = 0$ if $p > x$. Then (see (2.2))

$$\begin{aligned} \frac{F_0(s + ia)}{A_x G_0(s)} &= \frac{1}{A_x} \exp \left(\sum_{p \leq x} \frac{f(p)p^{-ia} - g(p)}{p^s} \right) = \\ &= \frac{H_1(1)}{H(1 + ia)} \exp \left(\sum_{p \leq x} \frac{f(p)p^{-ia} - g(p)}{p} \left(\frac{1}{p^{s-1}} - 1 \right) \right). \end{aligned}$$

Observe

$$\begin{aligned} & \left| \sum_{p \leq x} \frac{f(p)p^{-ia} - g(p)}{p} \left(\frac{1}{p^{s-1}} - 1 \right) \right| \leq \\ (2.9) \quad & \leq O \left(|s-1| \sum_{p \leq y} \frac{\log p}{p} \right) + 2 \sum_{y < p \leq x} \frac{|f(p)p^{-ia} - g(p)|}{p} =: \\ & =: \Sigma_1 + \Sigma_2. \end{aligned}$$

We know that the series (2.3) converges. Let $\frac{\log \log x}{\log x} \leq \delta_0(x) = o(1)$ such that for $y = x^{\delta_0(x)}$

$$\sum_{y < p \leq x} \frac{|f(p)p^{-ia} - g(p)|^2}{p} =: \delta_1(x) \leq \delta_0(x)$$

as $x \rightarrow \infty$. Then

$$\begin{aligned} \Sigma_2 &\leq \left(\sum_{y < p \leq x} \frac{1}{p} \right)^{\frac{1}{2}} \left(\sum_{y < p \leq x} \frac{|f(p)p^{-ia} - g(p)|^2}{p} \right)^{\frac{1}{2}} \leq \\ &\leq (\delta_0(x))^{\frac{1}{2}} \left(\log \frac{1}{\delta_0(x)} \right)^{\frac{1}{2}} \end{aligned}$$

and obviously (cf. (2.9))

$$\Sigma_1 + \Sigma_2 \ll |s-1| \delta_0(x) \log x + (\delta_0(x))^{\frac{1}{2}} \left(\log \frac{1}{\delta_0(x)} \right)^{\frac{1}{2}}$$

which proves (2.6).

Next put

$$|G(s)| = |G_1(s)||G_2(s)|.$$

Then

$$(2.10) \quad |G_2(s)| \ll \exp \left(\sum_{p \leq x} \frac{g_2(p)}{p} \right)$$

and

$$(2.11) \quad |G_1(s)| \ll \exp \left(\sum_{p \leq x^\varepsilon} \frac{g_1(p)}{p} \right) \exp \left(\sum_{x^\varepsilon < p \leq x} \frac{g_1(p) \cos(t \log p)}{p} \right).$$

By [4], Lemma 2.3

$$(2.12) \quad |G_1(s)| \ll \varepsilon^{\alpha\beta} \exp \left(\sum_{p \leq x} \frac{g_1(p)}{p} \right) \quad \text{for } t \in I_2(\varepsilon)$$

and (2.10), (2.11) and (2.12) give the estimate (2.7) for $G(s)$.

Let

$$|F(s + ia)| = |F_1(s + ia)||F_2(s + ia)|.$$

Then

$$(2.13) \quad |F_2(s + ia)| \ll \exp \left(\operatorname{Re} \sum_{p \leq x} \frac{f_2(p)p^{-ia}}{p^{1+it}} \right) \leq \exp \left(\sum_{p \leq x} \frac{g_2(p)}{p} \right).$$

Further

$$(2.14) \quad |F_1(s + ia)| \ll \exp \left(\sum_{p \leq x^\varepsilon} \frac{g_1(p)}{p} + \sum_{x^\varepsilon < p \leq x} \frac{\operatorname{Re} f_1(p)p^{-ia}p^{-it}}{p} \right).$$

Observe the estimate (see G. Tenenbaum [7], (4.35))

$$\operatorname{Re} f_1(p)p^{-ia}e^{i\vartheta} \leq |f_1(p)| \cos \vartheta + \sqrt{2|f_1(p)|(|f_1(p)| - \operatorname{Re} f_1(p)p^{-ia})}$$

and obtain for the last sum in (2.14)

$$(2.15) \quad \begin{aligned} & \sum_{x^\varepsilon < p \leq x} \frac{\operatorname{Re} f_1(p)p^{-ia}p^{-it}}{p} \leq \\ & \leq \sum_{x^\varepsilon < p \leq x} \frac{f_1(p)}{p} \cos(t \log p) + \sum_{x^\varepsilon < p \leq x} \frac{\sqrt{2B}}{p} |g_1(p) - \operatorname{Re} f_1(p)p^{-ia}|^{\frac{1}{2}} \leq \\ & \leq \sum_{x^\varepsilon < p \leq x} \frac{g_1(p)}{p} \cos(t \log p) + O(1). \end{aligned}$$

Thus, by (2.13) and (2.15), the estimate (2.7) holds for $F(s + ia)$.

For the estimate (2.5) we split the interval $(-\infty, \infty)$ into $I_1(\varepsilon) \cup I_2(\varepsilon)$ and $\{t : \varepsilon^{-2\alpha} < |t|\}$.

In the first case we obtain by Lemma 2.2

$$(2.16) \quad \int_{|t| \leq \varepsilon^{-2\alpha}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 |F(s + ia) - A_x G(s)|^2 \frac{dt}{|s + ia|^2} \ll \\ \ll \max\{\varepsilon^{2\alpha}, e^{2\alpha\beta}\} \exp\left(2 \sum_{p \leq x} \frac{g(p)}{p}\right) \cdot \int_{|t| \leq e^{-2\alpha}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 \frac{dt}{|s + ia|^2}$$

whereas

$$(2.17) \quad \int_{|t| \geq \varepsilon^{-2\alpha}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 |F(s + ia) - A_x G(s)|^2 \frac{dt}{|s + ia|^2} \ll \\ \ll \exp\left(2 \sum_{p \leq x} \frac{g(p)}{p}\right) \int_{|t| \geq e^{-2\alpha}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 \frac{dt}{|s + ia|^2}.$$

The Integrals in (2.16) and (2.17) can be estimated by

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \leq \varepsilon^{-2\alpha}}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 dt$$

and

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| > \varepsilon^{-2\alpha}}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{F'_0(s + ia)}{F_0(s + ia)} \right|^2 dt.$$

Further we have

$$\frac{F'_0(s + ia)}{F_0(s + ia)} = - \sum_p \frac{f(p)p^{-ia}}{p^s}$$

and

$$- \sum_p \frac{\log p}{p^s} = \frac{\zeta'(s)}{\zeta(s)} + O(1) \text{ for } \sigma > 1.$$

Since $|f(p)| \leq B$, we use $|a_n| \leq Bb_n$ in the proof of Lemma 4.10 in Chapter III. 4. of [8] and conclude

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{F'_0}{F_0} \left(1 + \frac{1}{\log x} + i(a + k) + it\right) \right|^2 dt \leq 3B^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\zeta'}{\zeta} \left(1 + \frac{1}{\log x} + it\right) \right|^2 dt \ll \log x.$$

Thus, by (2.16) and (2.17) the estimate (2.5) holds.

Then, by Lemma 2.1, (2.4) and (2.5)

$$M(f, x) = A_x \sum_{n \leq x} g(n) n^{ia} + o \left(\frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right)$$

as $x \rightarrow \infty$.

Observe

$$\begin{aligned} \sum_{n \leq x^\varepsilon} g_1(n) \sum_{m \leq \frac{x}{n}} g_2(m) &\ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g_2(p)}{p} \right) \sum_{n \leq x^\varepsilon} \frac{g_1(n)}{n} \ll \\ &\ll \varepsilon^\alpha \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right). \end{aligned}$$

Then

$$\begin{aligned} \sum_{n \leq x} g(n) n^{ia} &= \\ (2.18) \quad &= \sum_{m \leq x^{1-\varepsilon}} g_2(m) m^{ia} \sum_{n \leq \frac{x}{m}} g_1(n) n^{ia} + \varepsilon^\alpha O \left(\frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right). \end{aligned}$$

Put $y = \frac{x}{m}$, $x^\varepsilon \leq y \leq x$. Since (see [9])

$$\sum_{n \leq y} g_1(n) \sim \frac{cyL(\log y)}{(\log y)^{1-\alpha}} \quad (y \rightarrow \infty)$$

where $L(\log y)$ is slowly oscillating, we obtain by partial summation

$$\begin{aligned} \sum_{n \leq y} g_1(n) n^{ia} &= \frac{cy^{1+ia} L(\log y)}{(\log y)^{1-\alpha}} - ia \int_2^y \frac{cu^{ia} L(\log u)}{(\log u)^{1-\alpha}} du + o \left(\sum_{n \leq y} g_1(n) \right) = \\ (2.19) \quad &= \frac{cy^{1+ia} L(\log y)}{(\log y)^{1-\alpha}} \left(1 - \frac{ia}{1+ia} \right) + o \left(\sum_{n \leq y} g_1(n) \right) = \\ &= \frac{y^{ia}}{1+ia} \sum_{n \leq y} g_1(n) + o \left(\sum_{n \leq y} g_1(n) \right). \end{aligned}$$

Combining (2.18) and (2.19) gives

$$\sum_{n \leq x} g(n) n^{ia} = \{1 + o(1)\} \frac{x^{ia}}{1+ia} \sum_{n \leq x} g(n).$$

Since

$$A_x = (1 + o(1)) \prod_{p \leq x} \left(1 + \sum_{m=1}^{\infty} f(p^m) p^{-m(1+ia)} \right) \left(1 + \sum_{m=1}^{\infty} g(p^m) p^{-m} \right)^{-1}$$

and

$$\sum_{n \leq x} g(n) \geq \sum_{m \leq x^{1/2}} g_2(m) \sum_{n \leq \frac{x}{m}} g_1(n) \gg \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right)$$

the assertion (1.5) of Theorem 1.1 holds. ■

Remark 2.2 Theorem 1.1 contains the corresponding results of P.D.T.A. Elliott [1], K.-H. Indlekofer, I. Kátai, R. Wagner [3], E. Kaya [6] and E. Wirsing [9]. If $g = 1$ and $|f(n)| \leq 1$ for all n we refer to [2].

References

- [1] **Elliott, P.D.T.A.**, Multiplicative function mean values: Asymptotic estimates, *Funct. Approx. Comment. Math.*, **56(2)** (2017), 217–238.
<https://doi.org/10.7169/facm/1608>
- [2] **Indlekofer, K.-H.**, Identities in the convolution arithmetic of number theoretical functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **28** (2008), 203–325.
- [3] **Indlekofer, K.-H., I. Kátai and R. Wagner**, A comparative result for multiplicative functions, *Lithuanian Mathematical Journal*, **41** (2001), 143–157. <https://doi.org/10.1023/A:1011624300084>
- [4] **Indlekofer, K.-H.**, A new proof of a result of Wirsing for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **55** (2023), 151–174.
<https://doi.org/10.71352/ac.55.151>
- [5] **Indlekofer, K.-H. and E. Kaya**, Estimates for multiplicative functions, I., *Annales Univ. Sci. Budapest., Sect. Comp.*, **56** (2024), 249–261.
<https://doi.org/10.71352/ac.56.249>
- [6] **Kaya, E.**, On some results of Indlekofer for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **54** (2023), 199–209.
<https://doi.org/10.71352/ac.54.199>
- [7] **Tenenbaum, G.**, Moyennes effectives de fonctions multiplicatives complexes, *Ramanujan Journal*, **44(3)** (2017), 641–701.
<https://doi.org/10.1007/s11139-017-9949-7>

- [8] **Tenenbaum, G.**, *Introduction to Analytic and Probabilistic Number Theory*, Graduate Studies in Mathematics, Vol. 163., American Math. Soc., 2015. <https://doi.org/10.1090/gsm/163>
- [9] **Wirsing, E.**, Das asymptotische Verhalten von Summen über multiplikative Funktionen II, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 411–467. <https://doi.org/10.1007/BF02280301>

Karl-Heinz Indlekofer

Faculty of Computer Science
Electrical Engineering and Mathematics
University of Paderborn
Warburger Straße 100
D-33098 Paderborn, Germany
k-heinz@math.uni-paderborn.de

Erdener Kaya

kayaerdener@mersin.edu.tr