

WALSH–FEJÉR MEANS AND WEIGHTED HARDY SPACES

Ferenc Weisz (Budapest, Hungary)

Communicated by Péter Simon

(Received 5 May 2025; accepted 25 May 2025)

Abstract. In this paper we consider the A_p weights and weighted dyadic Hardy spaces. We investigate the convergence of Fejér means of Walsh–Fourier series. We prove that, under some conditions, the maximal operator of the Fejér means is bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$. This implies almost everywhere convergence of the Fejér means.

1. Introduction

We investigate the convergence of the Fejér means of Walsh–Fourier series. It was proved by Fine [3] that the Fejér means $\sigma_n(f)$ converge to f almost everywhere if $f \in L_1$. Schipp [11] obtained the same result by proving the weak type inequality of the maximal operator σ_* of the Fejér means. Next Fujii [4] showed that σ_* is bounded from the dyadic Hardy space H_1 to L_1 (see also Schipp and Simon [12]). Later the author [18] proved that σ_* is bounded from H_p to L_p for $1/2 < p < \infty$.

In this paper, we generalize these results to weighted spaces. We introduce the concept of A_p weights. With the help of the weighted $L_p(w)$ -norm of the dyadic maximal function, we introduce the weighted dyadic Hardy spaces and give their atomic decomposition. Using the atomic decomposition, we verify

Key words and phrases: Weighted martingale Hardy spaces, weighted p -atom, atomic decomposition, interpolation, Walsh functions, Fejér summability.

2010 Mathematics Subject Classification: Primary 42C10, 42B08, Secondary 60G42, 42B30.
<https://doi.org/10.71352/ac.58.250525>

First published online: 30 May 2025

a sufficient condition for an operator to be bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$.

In the next section, we prove that the maximal Fejér operator σ_* is bounded from $H_p(w)$ to $L_p(w)$ if w satisfies some conditions. Finally, we obtain the almost everywhere and norm convergence of the Fejér means to the function.

Throughout this paper, let C denote a positive constant which may vary from line to line, but is independent of the main parameters. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \leq Cg$. For any subset E of $[0, 1)$, we use $\mathbf{1}_E$ to denote its characteristic function. For $1 \leq p \leq \infty$, the dual index p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Walsh system

In this paper let w be a strictly positive integrable weight function. The weighted space $L_p(w)$ is equipped with the norm (or quasinorm)

$$\|f\|_{L_p(w)} := \left(\int_0^1 |f|^p w \, d\lambda \right)^{1/p} \quad (0 < p < \infty)$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. For a set I , we use the notation

$$(2.1) \quad w(I) := \int_I w \, d\lambda.$$

The Rademacher functions are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *Walsh system*:

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad (0 \leq n_k < 2).$$

By a dyadic interval, we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$ and $0 \leq k < 2^n$. For any given $n \in \mathbb{N}$ and $x \in [0, 1)$, denote by

$I_n(x)$ the dyadic interval of length 2^{-n} which contains x . For any $n \in \mathbb{N}$, the σ -algebra generated by the dyadic intervals $\{I_n(x) : x \in [0, 1)\}$ is denoted by \mathcal{F}_n . It is clear that, for any $n \in \mathbb{N}$, r_n is \mathcal{F}_{n+1} measurable.

Let us denote by \mathbb{E}_n the conditional expectation operator with respect to \mathcal{F}_n and the Lebesgue measure. A sequence $(f_n, n \in \mathbb{N})$ is called a dyadic martingale if f_n is \mathcal{F}_n measurable and $\mathbb{E}_{n-1}(f_n) = f_{n-1}$ ($n \in \mathbb{N}$). If $f \in L^1[0, 1)$, for any $n \in \mathbb{N}$, the number

$$\widehat{f}(n) := \int_0^1 f w_n d\lambda$$

is called the n th Walsh–Fourier coefficient of f . We can extend this definition to martingales as follows. If $f := (f_k)_{k \in \mathbb{N}}$ is a martingale, then

$$\widehat{f}(n) := \lim_{k \rightarrow \infty} \int_0^1 f_k w_n d\lambda \quad (n \in \mathbb{N}).$$

3. Weighted dyadic Hardy spaces

The definition of A_p weights for martingales was introduced by Izumisawa and Kazamaki in [8] as follows.

Definition 3.1. Let $q \in [1, \infty)$. A weight w is said to satisfy the A_q condition (briefly $w \in A_q$) if there exists a positive constant K such that, when $q \in (1, \infty)$,

$$\sup_{n \in \mathbb{N}} \mathbb{E}_n(w) \left[\mathbb{E}_n \left(w^{-\frac{1}{q-1}} \right) \right]^{q-1} \leq K$$

and, when $q = 1$,

$$\sup_{n \in \mathbb{N}} \frac{\mathbb{E}_n(w)}{w} \leq K.$$

w is said to satisfy A_∞ (briefly $w \in A_\infty$) if $w \in A_q$ for some $q \in [1, \infty)$.

Now, we give the following definition of the reverse Hölder inequality.

Definition 3.2. Let $q \in (1, \infty]$. A weight w is said to satisfy the R_q condition (briefly $w \in R_q$) if there exists a positive constant K such that, when $q \in (1, \infty)$,

$$\sup_{n \in \mathbb{N}} [\mathbb{E}_n(w^q)]^{\frac{1}{q}} [\mathbb{E}_n(w)]^{-1} \leq K$$

and, when $q = \infty$,

$$\sup_{n \in \mathbb{N}} w [\mathbb{E}_n(w)]^{-1} \leq K.$$

w is said to satisfy R_1 (briefly $w \in R_1$) if $w \in R_q$ for some $q \in (1, \infty]$.

For any $p, q \in (1, \infty)$ with $p \leq q$, we have $A_1 \subset A_p \subset A_q \subset A_\infty$ and $R_\infty \subset R_q \subset R_p \subset R_1$. So we can introduce the indices $q(w)$ and $r(w)$ as follows

$$q(w) := \inf \{q \in [1, \infty) : w \in A_q\}$$

and

$$r(w) := \sup \{q \in (1, \infty] : w \in R_q\}.$$

Let $d\hat{\lambda} := wd\lambda$. By Long [9, p. 229], we find that, for any measurable set A ,

$$\lambda(A) = 0 \iff \hat{\lambda}(A) = 0.$$

The following lemmas can be found in Doléans-Dade and Meyer [2] and Long [9] (see also Xie et al. [22]).

Lemma 3.1. *Let $J \subset I$ two dyadic intervals. If $1 \leq q < \infty$ and $w \in A_q$, then*

$$\left(\frac{\lambda(J)}{\lambda(I)}\right)^q \lesssim \frac{w(J)}{w(I)}.$$

If $1 < q \leq \infty$ and $w \in R_q$, then

$$\left(\frac{w(J)}{w(I)}\right)^{q'} \lesssim \frac{\lambda(J)}{\lambda(I)}.$$

Note that $w(I)$ is defined in (2.1). If $w \in A_q$, then w satisfies the reverse Hölder's inequality and $w \in A_{q-\varepsilon}$ for some $\varepsilon > 0$.

Lemma 3.2.

- (i) *If $1 \leq q < \infty$ and $w \in A_q$, then there exists $\varepsilon > 0$ such that $w \in R_{1+\varepsilon}$.*
- (ii) *If $1 < q < \infty$ and $w \in A_q$, then there exists $\varepsilon > 0$ such that $w \in A_{q-\varepsilon}$.*

We define the maximal operator of the dyadic martingale $f = (f_n, n \in \mathbb{N})$ by

$$f^* := \sup_{n \in \mathbb{N}} |f_n|.$$

The weighted dyadic Hardy space $H_p(w)$ ($0 < p < \infty$) consists of all dyadic martingales $f = (f_n, n \in \mathbb{N})$ for which

$$\|f\|_{H_p(w)} := \|f^*\|_{L_p(w)} < \infty.$$

For $w = 1$, we write again H_p . It is known (see e.g. Weisz [19]) that

$$H_p \sim L_p \quad (1 < p \leq \infty),$$

where \sim denotes the equivalence of spaces and norms. For weighted Hardy spaces, we can formulate this theorem as follows (see Long [9], Xie et al. [22], Weisz [21]).

Theorem 3.1.

(i) If $w \in A_\infty$ and $q(w) < p < \infty$, then

$$H_p(w) \sim L_p(w).$$

(ii) If $w \in A_1$ and $f \in L_1(w)$, then

$$\sup_{\rho>0} \rho w(f^* > \rho) \leq C \|f\|_{L_1(w)}.$$

Here $w(f^* > \rho)$ denotes $w(\{x \in [0, 1] : f^*(x) > \rho\})$ (see (2.1)). The atomic decomposition provides a useful characterization of Hardy spaces. First we introduce the concept of atoms.

Definition 3.3. A bounded function a is a weighted dyadic p -atom if there exists a dyadic interval $I \subset [0, 1)$ such that

- (a) $\text{supp } a \subset I$,
- (b) $\|a\|_\infty \leq w(I)^{-1/p}$,
- (c) $\int_I a(x) dx = 0$.

The atomic decomposition of the Hardy space $H_p(w)$ means that every function (more exactly, martingale) from the Hardy space can be decomposed into the sum of atoms. The following theorem was shown in Weisz [21].

Theorem 3.2. Let $0 < p \leq 1$ and $w \in A_\infty$. A martingale f is in $H_p(w)$ if and only if there exist a sequence (μ_k) of real numbers and a sequence (a^k) of weighted dyadic p -atoms such that

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k \quad \text{a.e.,} \quad n \in \mathbb{N}.$$

Moreover,

$$\|f\|_{H_p(w)} \sim \inf \left(\sum_{k \in \mathbb{Z}} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f as above.

The next result gives a sufficient condition for an operator to be bounded from $H_p(w)$ to $L_p(w)$ (see Weisz [21]). For the unweighted case see Weisz [19] and, for $p_0 = 1$, Schipp, Wade, Simon and Pál [13] and Móricz, Schipp and Wade [10].

Let X be a martingale space and Y a measurable function space. An operator $T : X \rightarrow Y$ is said to be a σ -sublinear operator if, for any complex number α ,

$$\left| T \left(\sum_{k=1}^{\infty} f_k \right) \right| \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha| |T(f)|.$$

Theorem 3.3. *Let $w \in A_\infty$. Suppose that V is a σ -sublinear operator and*

$$(3.1) \quad \int_{[0,1] \setminus I} |Va|^{p_0} w \, d\lambda \leq C_{p_0}$$

for all weighted dyadic p_0 -atoms a and for some fixed $0 < p_0 \leq 1$, where the interval I is the support of the atom. If V is bounded from $L_{p_1}(w)$ to $L_{p_1}(w)$ for some $1 < p_1 \leq \infty$, then

$$\|Vf\|_{L_p(w)} \leq C_p \|f\|_{H_p(w)} \quad (f \in H_p(w))$$

for all $p_0 \leq p \leq p_1$. Moreover, if $w \in A_1$ and $p_0 < 1$, then the operator V is of weak type $(1, 1)$, i.e., if $f \in L_1(w)$, then

$$\sup_{\rho > 0} \rho w(|Vf| > \rho) \leq C \|f\|_{L_1(w)}.$$

4. Fejér summability of Walsh–Fourier series

Denote by $s_n f$ the n th partial sum of the Walsh–Fourier series of a martingale f , namely,

$$s_n(f) := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \quad (n \in \mathbb{N}).$$

If $f \in L_1$, then

$$s_n(f)(x) = \int_0^1 f(t) D_n(x \dot{+} t) \, dt \quad (n \in \mathbb{N}),$$

where $\dot{+}$ denotes the dyadic addition and

$$D_n(u) := \sum_{k=0}^{n-1} w_k(u)$$

is the n th Walsh–Dirichlet kernel (see, for example, Schipp, Wade, Simon and Pál [13]).

It is easy to see that, for any martingale $f = (f_n)$,

$$s_{2^n}(f) = f_n \quad (n \in \mathbb{N}).$$

Hence, by the martingale convergence theorem, we know that, for $1 \leq p < \infty$ and $f \in L_p$,

$$\lim_{n \rightarrow \infty} s_{2^n}(f) = f \quad \text{in the } L_p\text{-norm.}$$

This result was generalized (see e.g. Schipp et al. [13, Theorem 4.1]). More precisely, it was proved that, for any $1 < p < \infty$ and $f \in L_p$,

$$\lim_{n \rightarrow \infty} s_n(f) = f \quad \text{in the } L_p\text{-norm.}$$

The generalization of this theorem to the $L_p(w)$ spaces can be found in [21, 22].

Theorem 4.1. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n \in \mathbb{N}} \|s_n(f)\|_{L_p(w)} \leq C_p \|f\|_{L_p(w)} \quad (f \in L_p(w)).$$

Corollary 4.1. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n \rightarrow \infty} s_n(f) = f \quad \text{in the } L_p(w)\text{-norm.}$$

The inequality of Theorem 4.1 and the convergence in Corollary 4.1 do not hold for $p = 1$ or $p = \infty$. With the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [16], Butzer and Nessel [1], Trigub and Belinsky [17], Grafakos [7] and Weisz [19, 20] and the references therein.

In this paper we consider the Fejér means defined by

$$\sigma_n(f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x) = \sum_{j=0}^n \left(1 - \frac{j}{n}\right) \widehat{f}(j) w_j(x) = \int_0^1 f(x-u) K_n(u) du,$$

where the Fejér kernels are given by

$$K_n(u) := \sum_{j=0}^n \left(1 - \frac{j}{n}\right) w_j(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u).$$

It is known (see [13, Theorem 1.16]) that

$$D_{2^k}(x) = \begin{cases} 2^k & \text{when } x \in [0, 2^{-k}), \\ 0 & \text{when } x \in [2^{-k}, 1) \end{cases}$$

and

$$(4.1) \quad |K_n(x)| \leq \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} [D_{2^i}(x) + D_{2^i}(x + 2^{-j-1})],$$

where $x \in [0, 1)$, $n \in \mathbb{N}$ and N is a positive integer such that $2^{N-1} \leq n < 2^N$.

Theorem 4.1 and Corollary 4.1 imply the next results.

Corollary 4.2. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n \in \mathbb{N}} \|\sigma_n(f)\|_{L_p(w)} \leq C_p \|f\|_{L_p(w)} \quad (f \in L_p(w)).$$

Corollary 4.3. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n \rightarrow \infty} \sigma_n(f) = f \quad \text{in the } L_p(w)\text{-norm.}$$

Now let us investigate maximal operator of the Fejér means:

$$\sigma_*(f) := \sup_{n \in \mathbb{N}} |\sigma_n(f)|.$$

Applying Theorem 3.3, we can generalize the previous results to the $L_p(w)$ spaces and to the maximal operator.

Theorem 4.2. *Suppose that $w \in A_\infty$ is a weight and*

$$\max \left\{ \frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'} \right] \right\} < 1.$$

If

$$(4.2) \quad \max \left\{ \frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'} \right] \right\} < p < \infty,$$

then, for any $f \in H_p(w)$,

$$\|\sigma_*(f)\|_{L_p(w)} \leq C \|f\|_{H_p(w)}.$$

Moreover, if $w \in A_1$ and $f \in L_1(w)$, then

$$\sup_{\rho > 0} \rho w(\sigma_*(f) > \rho) \leq C \|f\|_{L_1(w)}.$$

Proof. For any $f \in L_\infty(w)$,

$$(4.3) \quad \|\sigma_*(f)\|_{L_\infty(w)} \leq \|f\|_{L_\infty(w)} \sup_{n \in \mathbb{N}} \|K_n\|_1 = \|f\|_{L_\infty(w)}.$$

Note that $\|f\|_{L_\infty(w)} = \|f\|_{L_\infty}$.

In addition to the conditions of the theorem, suppose that $p \leq 1$. We have to prove (3.1) for $\sigma_*(a)$, for $p_0 = p$ and for all weighted dyadic p -atoms a .

Denote the support of a by I and let $\lambda(I) = 2^{-K}$ ($K \in \mathbb{N}$). Without loss of generality we can suppose that $I = [0, 2^{-K})$. It is easy to see that $\hat{a}(n) = 0$ if $n < 2^K$, so, in this case, $\sigma_n(a) = 0$. Therefore we assume that $n \geq 2^K$.

Let $x \in I^c := [0, 1) \setminus I$. If $j \geq K$, then $x + 2^{-j-1} \notin I$. Hence, for $x \notin I$ and $i \geq j \geq K$, we have

$$a(t)D_{2^i}(x + t) = a(t)D_{2^i}(x + t + 2^{-j-1}) = 0.$$

If $2^N > n \geq 2^{N-1}$, then $N - 1 \geq K$. Henceforth, for $x \notin I$,

$$\begin{aligned}
 |\sigma_n(a)(x)| &\leq \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} \int_I |a(t)| (D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})) dt \leq \\
 &\leq Cw(I)^{-1/p} 2^{-K} \sum_{j=0}^{K-1} 2^j \sum_{i=j}^{K-1} \int_I (D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})) dt + \\
 &\quad + Cw(I)^{-1/p} \sum_{j=0}^{K-1} 2^j \sum_{i=K}^{\infty} 2^{-i} \int_I (D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})) dt =: \\
 &=: A_1(x) + A_2(x).
 \end{aligned}$$

Note that the right hand side is independent of n . Using (4.1), we can verify that for $x \notin I$,

$$\int_I D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1}) dt = 2^{i-K} \mathbf{1}_{[2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i}]}(x)$$

if $j \leq i \leq K - 1$,

$$\int_I D_{2^i}(x \dot{+} t) dt = 2^{i-K} \mathbf{1}_{[2^{-K}, 2^{-i}]}(x)$$

if $i \in \mathbb{N}$ and

$$\int_I D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1}) dt = \mathbf{1}_{[2^{-j-1}, 2^{-j-1} \dot{+} 2^{-K}]}(x)$$

if $i \geq K$. Therefore, for $x \notin I$,

$$\begin{aligned}
 \int_{I^c} A_1(x)^p w(x) dx &\lesssim \\
 &\lesssim 2^{-2Kp} w(I)^{-1} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=j}^{K-1} 2^{ip} \left[w([2^{-K}, 2^{-i})) + w([2^{-j-1}, 2^{-j-1} + 2^{-i})) \right].
 \end{aligned}$$

Let L, J, K three dyadic intervals such that $L, J \subset K$. Moreover, let $\tau > q(w)$ and $\rho < r(w)$. By Lemma 3.1, we conclude that

$$(4.4) \quad \frac{w(L)}{w(J)} = \frac{w(K)}{w(J)} \frac{w(L)}{w(K)} \lesssim \left[\frac{\lambda(K)}{\lambda(J)} \right]^\tau \left[\frac{\lambda(L)}{\lambda(K)} \right]^{\frac{1}{\rho'}}.$$

Let $L = [2^{-K+s}, 2^{-i}]$ or $L = [2^{-j-1}, 2^{-j-1} + 2^{-i}]$, $J = I$ and $K = [0, 2^{-j}]$. From this it follows that

$$\frac{w([2^{-K}, 2^{-i}))}{w(I)} + \frac{w([2^{-j-1}, 2^{-j-1} + 2^{-i}))}{w(I)} \lesssim 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}},$$

where $\tau > q(w)$ and $\rho < r(w)$. Consequently,

$$\begin{aligned} \int_{I^c} A_1(x)^p w(x) dx &\lesssim 2^{-2Kp} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=j}^{K-1} 2^{ip} 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}} \lesssim \\ &\lesssim 2^{K(\tau-2p)} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \sum_{i=j}^{K-1} 2^{i(p-1/\rho')}. \end{aligned}$$

If $p < \frac{1}{r(\varphi_1)'}$, then we can choose ρ near enough to $r(w)$ such that $p < \frac{1}{\rho'} < \frac{1}{r(w)'}$. Since $p > q(w)/2$, we can choose τ such that $p > \tau/2$. Then

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim 2^{K(\tau-2p)} \sum_{j=0}^{K-1} 2^{j(2p-\tau)} \lesssim C.$$

On the other hand, if $p \geq \frac{1}{r(\varphi_1)'}$, then we can choose ρ such that $\frac{1}{\rho'} < p$. Since $p > q(w) - \frac{1}{r(w)'}$, we can choose τ and ρ such that $p > \tau - \frac{1}{\rho'}$. Then

$$\int_{I^c} A_1(x)^p w(x) dx \lesssim 2^{K(\tau-p-1/\rho')} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \lesssim C.$$

Similarly, for $x \notin I$,

$$\int_{I^c} A_2(x)^p w(x) dx \lesssim w(I)^{-1} \sum_{j=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} w([2^{-j-1}, 2^{-j-1} + 2^{-K})).$$

Using (4.4) for $L = [2^{-j-1}, 2^{-j-1} + 2^{-K})$, $J = I$ and $K = [0, 2^{-j}]$, we have

$$\frac{w([2^{-j-1}, 2^{-j-1} + 2^{-K}))}{w(I)} \lesssim 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}},$$

where $\tau > q(w)$ and $\rho < r(w)$. Therefore,

$$\begin{aligned} \int_{I^c} A_2(x)^p w(x) dx &\lesssim \sum_{j=0}^{K-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} 2^{-\tau(j-K)} 2^{(j-i)\frac{1}{\rho'}} \lesssim \\ &\lesssim 2^{-K(p-\tau+1/\rho')} \sum_{j=0}^{K-1} 2^{j(p-\tau+1/\rho')} \lesssim C. \end{aligned}$$

In the last step we have used that we can choose τ and ρ such that $p > \tau - \frac{1}{\rho'}$ because of the condition $p > q(w) - \frac{1}{r(w)'}$.

Combining the above inequalities, we can establish that

$$\int_{I^c} |\sigma_*(a)|^p w \, d\lambda \leq C_p.$$

Now Theorem 3.3 finishes the proof for

$$\max \left\{ \frac{q(w)}{2}, \left[q(w) - \frac{1}{r(w)'} \right] \right\} < p \leq 1.$$

For $1 < p < \infty$, the theorem follows from (4.3) and from interpolation. ■

The first inequality of Theorem 4.2 was proved by Fujii [4] for $w = p = 1$ (see also Schipp and Simon [12]) and for $w = 1$ and $1/2 < p < \infty$ by the author [18, 19]. Unfortunately, the second condition of (4.2) is missing in [21]. Theorem 4.2 implies the following convergence results.

Corollary 4.4. *Besides the conditions of Theorem 4.2, if $f \in H_p(w)$, then $\sigma_n(f)$ converges almost everywhere on $[0, 1)$ as well as in the $L_p(w)$ -norm as $n \rightarrow \infty$. Moreover, if $w \in A_1$ and $f \in L_1(w)$, then*

$$\lim_{n \rightarrow \infty} \sigma_n(f) = f \quad \text{a.e.}$$

References

- [1] Butzer, P. L. and R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser Verlag, Basel, 1971.
<https://doi.org/10.1007/978-3-0348-7448-9>
- [2] Doléans-Dade, C. and P. A. Meyer, Inegalites de normes avec poids. In: Dellacherie, C., Meyer, P.A., Weil, M. (eds), *Séminaire de Probabilités XIII. Lecture Notes in Mathematics*, vol 721. Springer, Berlin, Heidelberg, pp 313–331. <https://doi.org/10.1007/BFb0070873>
- [3] Fine, N. J., Cesàro summability of Walsh–Fourier series, *Proc. Nat. Acad. Sci. USA*, **41** (1955), 558–591.
<https://doi.org/10.1073/pnas.41.8.588>
- [4] Fujii, N., A maximal inequality for H^1 -functions on a generalized Walsh–Paley group, *Proc. Amer. Math. Soc.*, **77** (1979), 111–116.
<https://doi.org/10.1090/S0002-9939-1979-0539641-9>
- [5] Gát, G. and U. Goginava, The weak type inequality for the maximal operator of the (C, α) -means of the Fourier series with respect to the Walsh–Kaczmarz system, *Acta Math. Hungar.*, **125** (2009) 65–83.
<https://doi.org/10.1007/s10474-009-8217-8>

- [6] **Goginava, U.**, Maximal operators of Fejér means of double Walsh-Fourier series, *Acta Math. Hungar.*, **115** (2007), 333–340.
<https://doi.org/10.1007/s10474-007-5268-6>
- [7] **Grafakos, L.**, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.
- [8] **Izumisawa, M. and N. Kazamaki**, Weighted norm inequalities for martingales, *Tohoku Math. J. (2)*, **29** (1977), 115–124.
<https://doi.org/10.2748/tmj/1178240700>
- [9] **Long, R.**, *Martingale Spaces and Inequalities*. Peking University Press and Vieweg Publishing, 1993.
<https://doi.org/10.1007/978-3-322-99266-6>
- [10] **Móricz, F., F. Schipp, and W. R. Wade**, Cesàro summability of double Walsh-Fourier series, *Trans. Amer. Math. Soc.*, **329** (1992), 131–140. <https://doi.org/10.1090/S0002-9947-1992-1030510-8>
- [11] **Schipp, F.**, Über gewissen Maximaloperatoren. *Ann. Univ. Sci. Budapest., Sect. Math.*, **18** (1975), 189–195.
- [12] **Schipp, F. and P. Simon**, On some (H, L_1) -type maximal inequalities with respect to the Walsh-Paley system, In: *Functions, Series, Operators, Proc. Conf. in Budapest, 1980*, volume 35 of *Coll. Math. Soc. J. Bolyai*, pages 1039–1045. North Holland, Amsterdam, 1981.
- [13] **Schipp, F., W. R. Wade, P. Simon, and J. Pál**, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*. Adam Hilger, Bristol, New York, 1990.
- [14] **Simon, P.**, Cesàro summability with respect to two-parameter Walsh systems, *Monatsh. Math.*, **131** (2000), 321–334.
<https://doi.org/10.1007/s006050070004>
- [15] **Simon, P. and F. Weisz**, Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series, *J. Approx. Theory*, **151** (2008), 1–19.
<https://doi.org/10.1016/j.jat.2007.05.004>
- [16] **Stein, E. M. and G. Weiss**, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.
<https://doi.org/10.1515/9781400883899>
- [17] **Trigub, R. M. and E. S. Belinsky**, *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.
- [18] **Weisz, F.**, Cesàro summability of one- and two-dimensional Walsh-Fourier series. *Anal. Math.*, **22** (1996), 229–242.
<https://doi.org/10.1007/BF02205221>
- [19] **Weisz, F.**, *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
<https://doi.org/10.1007/978-94-017-3183-6>

- [20] **Weisz, F.**, *Convergence and Summability of Fourier Transforms and Hardy Spaces*, Applied and Numerical Harmonic Analysis. Springer, Birkhäuser, Basel, 2017.
<https://doi.org/10.1007/978-3-319-56814-0>
- [21] **Weisz, F.**, Weighted Hardy spaces and the Fejér means of Walsh–Fourier series. *Ann. Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 411–424.
<https://doi.org/10.71352/ac.49.411>
- [22] **Xie, G., F. Weisz, D. Yang, and Y. Jiao**, New martingale inequalities and applications to Fourier analysis, *Nonlinear Analysis*, **182** (2019), 143–192. <https://doi.org/10.1016/j.na.2018.12.011>

Ferenc Weisz

<https://orcid.org/0000-0002-7766-2745>

ELTE Eötvös Loránd University

Department of Numerical Analysis

H-1117 Budapest, Pázmány P. sétány 1/C.

Hungary

`weisz@inf.elte.hu`

