ESTIMATES FOR MULTIPLICATIVE FUNCTIONS, II.

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Abstract. In this paper we consider multiplicative functions f and g, $|f| \leq g$, as in Part I, and prove

$$\sum_{n\leq x} f(n) = \{1 + o(1)\}x^{ia}(1 + ia)^{-1}A_x \sum_{n\leq x} g(n)$$

if the series $\sum_p (g(p) - Ref(p)p^{-ia})p^{-1}$ converges for some $a \in \mathbb{R}$.

1. Introduction

Let $\mathcal{M}_1$ be defined by

$$\mathcal{M}_1 := \{f : \mathbb{N} \to \mathbb{C} : f \ \text{ multiplicative; } \ \text{(i), (ii), (iii)} \ \text{ hold}\}$$

where

$$\text{(i)} \quad |f(p)| \leq B \ \text{ for all primes } p,\ B \geq 1,$$

$$\text{(ii)} \quad \sum_{p,k\geq 2} \frac{|f(p^k)|}{p^k} < \infty,$$

$$\text{(iii)} \quad \sum_{\substack{p^k\leq x\\ k\geq 2}} |f(p^k)| = O\left(\frac{x}{\log x}\right).$$

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In [5] we proved asymptotic results for the behaviour of

$$M(f,x) := \sum_{n\le x} f(n) \tag{1.1}$$

as $x \to \infty$.
For example, if $f, g \in \mathcal{M}_1, |f| \le g$ and

$$\sum_{n\le x^\varepsilon} \frac{g(n)}{n} \le \delta(\varepsilon) \sum_{n\le x} \frac{g(n)}{n}, \quad \delta(\varepsilon) \to 0 \;\; as \;\; \varepsilon \to 0,$$

we showed (see [5], Theorem 5.1):
If

$$\sum_p \frac{g(p) - Ref(p)p^{-it}}{p} = \infty \;\; \text{for all} \;\; t \in \mathbb{R}, \tag{1.2}$$

then

$$M(f,x) = o\left(\frac{x}{\log x} \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right)$$

as $x \to \infty$.
In this paper we consider the case that the sum (1.2) converges for some $t = a$. For this let $\mathbb{P}_1 \subset \mathbb{P}$ be a set of primes such that, for some $\alpha > 0$,

$$\sum_{\substack{p\le x \\ p\in\mathbb{P}_1}} \frac{g(p)\log p}{p} \sim \alpha \log x \;\; as \;\; x \to \infty. \tag{1.3}$$

Define multiplicative functions $g_j, j = 1,2$ by

$$g_1(p^k) = \begin{cases} g(p^k), & p \in \mathbb{P}_1 \\ 0 & \text{otherwise}, \end{cases} \qquad g_2(p^k) = \begin{cases} g(p^k), & p \notin \mathbb{P}_1 \\ 0 & \text{otherwise}. \end{cases}$$

Then $g = g_1 * g_2$, and we prove with the notation (1.1)

Theorem 1.1. *Let $f, g \in \mathcal{M}_1, g \ge 0$ and $|f| \le g$, such that for some $a \in \mathbb{R}$*

$$\sum_p \frac{g(p) - Ref(p)p^{-ia}}{p} < \infty. \tag{1.4}$$

Assume that $g = g_1 * g_2$ with (1.3) holds. Then, as $x \to \infty$,

$$\begin{aligned} M(f,x) = \frac{x^{ia}}{1+ia} \prod_{p\le x} \left(1 + \sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)}\right) \times \\ \times \left(1 + \sum_{m=1}^{\infty} g(p^m)p^{-m}\right)^{-1} \sum_{n\le x} g(n)\{1+o(1)\}. \end{aligned} \tag{1.5}$$

2. Proof of Theorem 1.1

Using the notations of [5] we put ($f, g \in \mathcal{M}_1$)

$$f = h * f_0,$$
$$g = h_1 * g_0$$

where f_0, g_0 are exponentially multiplicative. The corresponding Dirichket series are

$$F(s) = H(s)F_0(s),$$
$$G(s) = H_1(s)G_0(s)$$

where $s = \sigma + it \quad (\sigma > 1, t \in \mathbb{R})$.
Put

$$M(x) := \sum_{n \le x} f(n) - A_x \sum_{n \le x} g(n) n^{ia} \tag{2.1}$$

where

$$A_x = \frac{H(1+ia)}{H_1(1)} \exp\left(- \sum_{p \le x} \frac{g(p) - f(p)p^{-ia}}{p} \right). \tag{2.2}$$

Obviously, as $x \to \infty$,

$$A_x = (1 + o(1)) \prod_{p \le x} \left(1 + \sum_{m=1}^{\infty} f(p^m) p^{-m(1+ia)} \right) \left(1 + \sum_{m=1}^{\infty} g(p^m) p^{-m} \right)^{-1}.$$

Remark 2.1. If a sum $\sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)}$ has the value -1, then the product may be omitted. Otherwise, the product has the form $cL(\log x)$, where c is a non-zero constant and $L(y)$ a non-vanishing slowly oscillating function with $|L(y)| = 1$.

Now the following holds.

Lemma 2.1. *Let $M(x)$ be defined by (2.1). Then, as $x \to \infty$,*

$$|M(x)| \ll \frac{1}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \right) \ll$$
$$\ll \varepsilon_1^{\alpha} \frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) + \int_{x^{\varepsilon_1}}^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \right).$$

The proof follows from Remark 4.1 of [4] and Remark 5.1, (ii) of [5]. Put (cf. [4], Remark 4.1)

$$K_0 = \mathbf{1} * \Lambda_{f_0} * (f - A_x g_a)$$

where $g_a(n) = g(n)n^{ia}$. Then

$$LM = \mathbf{1} * L_0(f - A_x g_a) + R_1$$

where $R_1 = L * (f - A_x g_a)$ and

$$R_1(x) \ll \frac{x}{\log x} \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right).$$

Further

$$\mathbf{1} * L_0(f - A_x g_a) = \mathbf{1} * \Lambda_{f_0} * (f - A_x g_a) + R_2 + R_3$$

where

$$R_2 = \mathbf{1} * \Lambda_{h_1} * (f - A_x g_a),$$
$$R_3 = \mathbf{1} * A_x g_a * (\Lambda_f - \Lambda_{g_a}).$$

As in [4], page 172

$$|R_2(x)| = o\left(x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).$$

The convergence of (1.4) implies

$$\sum_p \frac{|Ref(p)p^{-ia} - g(p)|^2}{p} < \infty \tag{2.3}$$

and

$$|R_3(x)| \ll \varepsilon^\alpha x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right) + o\left(x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).$$

Thus we obtain, for $2 \le u \le x$

$$M(u) = \frac{K_0(u)}{\log u} + o\left(\frac{u}{\log u} \exp\left(\sum_{p\le u} \frac{g(p)}{p}\right)\right).$$

Then (cf. [4])

$$\begin{aligned}\int_{x^{\varepsilon_1}}^{x} \frac{|M(u)|}{u^2}du \ll \frac{1}{\varepsilon_1}\left(\frac{1}{\log x}\int_{-\infty}^{\infty} \left|\frac{F_0'(s)}{F_0(s)}\right|^2 |F(s) - A_x G(s-ia)|^2 \frac{dt}{|s|^2}\right)^{\frac{1}{2}} + \\ +o\left(x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).\end{aligned} \tag{2.4}$$

Our aim is to prove

$$\int_{-\infty}^{\infty} \Big|\frac{F_0'(s)}{F_0(s)}\Big|^2 |F(s)-A_xG(s-ia)|^2 \frac{dt}{|s|^2} =$$

(2.5)
$$= \int_{-\infty}^{\infty} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 |F(s+ia)-A_xG(s)|^2 \frac{dt}{|s+ia|^2} =$$

$$= o\left((\log x)\exp\left(2\sum_{p\le x}\frac{g(p)}{p}\right)\right) \quad as \quad x\to\infty.$$

Put

$$I_1(\varepsilon) := \{t: \quad |t| \le \frac{1}{\varepsilon\log x}\}$$

and

$$I_2(\varepsilon) := \{t: \quad \frac{1}{\varepsilon\log x} < |t| \le \varepsilon^{-2\alpha}\}.$$

Choose B' such that $\alpha < B'$ and $g(p) < B'$ for all p, and define (see (2.7) of [4])

$$\varphi_0 := \pi\frac{B'}{\alpha} \quad \text{and} \quad \beta = 1-\frac{\sin\varphi_0}{\varphi_0}.$$

Observe $\beta > 0$. Then the following Lemma holds.

Lemma 2.2. *For every* $\beta > 0$

(2.6)
$$|F(s+ia)-A_xG(s)| = o(1)\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right) \quad \text{for} \quad t\in I_1(\varepsilon)$$

and

(2.7)
$$\max(|F(s+ia)|,|G(s)|) \ll e^{\alpha\beta}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right) \quad \text{for} \quad t\in I_2(\varepsilon).$$

Proof. Consider

$$|F(s+ia)-A_xG(s)| = |A_xG(s)|.|\frac{F(s+ia)}{A_xG(s)}-1|$$

and

(2.8)
$$\frac{F(s+ia)}{A_xG(s)} = \frac{H(s+ia)}{H_1(s)}\cdot\frac{F_0(s+ia)}{A_xG_0(s)}.$$

In the definition of (2.8) we may assume that $f(p) = g(p) = 0$ *if* $p > x$. Then (see (2.2))

$$\begin{aligned}\frac{F_0(s+ia)}{A_x G_0(s)} &= \frac{1}{A_x}\exp\left(\sum_{p\le x}\frac{f(p)p^{-ia}-g(p)}{p^s}\right) = \\ &= \frac{H_1(1)}{H(1+ia)}\exp\left(\sum_{p\le x}\frac{f(p)p^{-ia}-g(p)}{p}\left(\frac{1}{p^{s-1}}-1\right)\right).\end{aligned}$$

Observe

$$\begin{aligned}&|\sum_{p\le x}\frac{f(p)p^{-ia}-g(p)}{p}\left(\frac{1}{p^{s-1}}-1\right)| \le \\ &\le O\left(|s-1|\sum_{p\le y}\frac{\log p}{p}\right) + 2\sum_{y<p\le x}\frac{|f(p)p^{-ia}-g(p)|}{p} =: \\ &=: \Sigma_1 + \Sigma_2.\end{aligned} \tag{2.9}$$

We know that the series (2.3) converges. Let $\frac{\log\log x}{\log x} \le \delta_0(x) = o(1)$ such that for $y = x^{\delta_0(x)}$

$$\sum_{y<p\le x}\frac{|f(p)p^{-ia}-g(p)|^2}{p} =: \delta_1(x) \le \delta_0(x)$$

as $x \to \infty$. Then

$$\begin{aligned}\Sigma_2 &\le \left(\sum_{y<p\le x}\frac{1}{p}\right)^{\frac12}\left(\sum_{y<p\le x}\frac{|f(p)p^{-ia}-g(p)|^2}{p}\right)^{\frac12} \le \\ &\le (\delta_0(x))^{\frac12}\left(\log\frac{1}{\delta_0(x)}\right)^{\frac12}\end{aligned}$$

and obviously (cf. (2.9))

$$\Sigma_1 + \Sigma_2 \ll |s-1|\delta_0(x)\log x + (\delta_0(x))^{\frac12}(\log\frac{1}{\delta_0(x)})^{\frac12}$$

which proves (2.6).

Next put

$$|G(s)| = |G_1(s)||G_2(s)|.$$

Then

$$|G_2(s)| \ll \exp\left(\sum_{p\le x}\frac{g_2(p)}{p}\right) \tag{2.10}$$

and

$$(2.11)\qquad |G_1(s)| \ll \exp\left(\sum_{p\le x^{\varepsilon}} \frac{g_1(p)}{p}\right) \exp\left(\sum_{x^{\varepsilon}<p\le x} \frac{g_1(p)\cos(t\log p)}{p}\right).$$

By [4], Lemma 2.3

$$(2.12)\qquad |G_1(s)| \ll \varepsilon^{\alpha\beta} \exp\left(\sum_{p\le x} \frac{g_1(p)}{p}\right) \quad \text{for} \quad t\in I_2(\varepsilon)$$

and (2.10), (2.11) and (2.12) give the estimate (2.7) for $G(s)$.
Let

$$|F(s+ia)| = |F_1(s+ia)||F_2(s+ia)|.$$

Then

$$(2.13)\qquad |F_2(s+ia)| \ll \exp\left(Re\sum_{p\le x} \frac{f_2(p)p^{-ia}}{p^{1+it}}\right) \le \exp\left(\sum_{p\le x} \frac{g_2(p)}{p}\right).$$

Further

$$(2.14)\qquad |F_1(s+ia)| \ll \exp\left(\sum_{p\le x^{\varepsilon}} \frac{g_1(p)}{p} + \sum_{x^{\varepsilon}<p\le x} \frac{Ref_1(p)p^{-ia}p^{-it}}{p}\right).$$

Observe the estimate (see G. Tenenbaum [7], (4.35))

$$Ref_1(p)p^{-ia}e^{i\vartheta} \le |f_1(p)|\cos\vartheta + \sqrt{2|f_1(p)|(|f_1(p)| - Ref_1(p)p^{-ia})}$$

and obtain for the last sum in (2.14)

$$\sum_{x^{\varepsilon}<p\le x} \frac{Ref_1(p)p^{-ia}p^{-it}}{p} \le$$

$$(2.15)\qquad \le \sum_{x^{\varepsilon}<p\le x} \frac{f_1(p)}{p}\cos(t\log p) + \sum_{x^{\varepsilon}<p\le x} \frac{\sqrt{2B}}{p}|g_1(p) - Ref_1(p)p^{-ia}|^{\frac{1}{2}} \le$$

$$\le \sum_{x^{\varepsilon}<p\le x} \frac{g_1(p)}{p}\cos(t\log p) + O(1).$$

Thus, by (2.13) and (2.15), the estimate (2.7) holds for $F(s+ia)$.
For the estimate (2.5) we split the interval $(-\infty,\infty)$ into $I_1(\varepsilon)\cup I_2(\varepsilon)$ and $\{t:\ \varepsilon^{-2\alpha} < |t|\ \}$.

In the first case we obtain by Lemma 2.2

$$\int\limits_{|t|\le\varepsilon^{-2\alpha}} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 |F(s+ia)-A_xG(s)|^2 \frac{dt}{|s+ia|^2} \ll \tag{2.16}$$
$$\ll \max\{\varepsilon^{2\alpha}, e^{2\alpha\beta}\} \exp\left(2\sum_{p\le x}\frac{g(p)}{p}\right) \cdot \int\limits_{|t|\le e^{-2\alpha}} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 \frac{dt}{|s+ia|^2}$$

whereas

$$\int\limits_{|t|\ge\varepsilon^{-2\alpha}} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 |F(s+ia)-A_xG(s)|^2 \frac{dt}{|s+ia|^2} \ll \tag{2.17}$$
$$\ll \exp\left(2\sum_{p\le x}\frac{g(p)}{p}\right) \int\limits_{|t|\ge e^{-2\alpha}} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 \frac{dt}{|s+ia|^2}.$$

The Integrals in (2.16) and (2.17) can be estimated by

$$\sum_{\substack{k\in\mathbb{Z}\\ |k|\le\varepsilon^{-2\alpha}}} \frac{1}{k^2+1} \int\limits_{|t-k|\le\frac12} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 dt$$

and

$$\sum_{\substack{k\in\mathbb{Z}\\ |k|>\varepsilon^{-2\alpha}}} \frac{1}{k^2+1} \int\limits_{|t-k|\le\frac12} \Big|\frac{F_0'(s+ia)}{F_0(s+ia)}\Big|^2 dt.$$

Further we have

$$\frac{F_0'(s+ia)}{F_0(s+ia)} = -\sum_p \frac{f(p)p^{-ia}}{p^s}$$

and

$$-\sum_p \frac{\log p}{p^s} = \frac{\zeta'(s)}{\zeta(s)} + O(1) \quad for \quad \sigma>1.$$

Since $|f(p)|\le B$, we use $|a_n|\le Bb_n$ in the proof of Lemma 4.10 in Chapter III. 4. of [8] and conclude

$$\int_{-\frac12}^{\frac12} \Big|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+i(a+k)+it)\Big|^2 dt \le 3B^2 \int_{-\frac12}^{\frac12} \Big|\frac{\zeta'}{\zeta}(1+\frac{1}{\log x}+it)\Big|^2 dt \ll \log x.$$

Thus, by (2.16) and (2.17) the estimate (2.5) holds.
Then, by Lemma 2.1, (2.4) and (2.5)

$$M(f,x) = A_x \sum_{n\le x} g(n)n^{ia} + o\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right)$$

as $x\to\infty$.

Observe

$$\sum_{n\le x^{\varepsilon}} g_1(n) \sum_{m\le \frac{x}{n}} g_2(m) \ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g_2(p)}{p}\right)\sum_{n\le x^{\varepsilon}}\frac{g_1(n)}{n} \ll$$
$$\ll \varepsilon^{\alpha}\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right).$$

Then

$$\sum_{n\le x} g(n)n^{ia} =$$
$$= \sum_{m\le x^{1-\varepsilon}} g_2(m)m^{ia}\sum_{n\le \frac{x}{m}} g_1(n)n^{ia} + \varepsilon^{\alpha} O\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right). \tag{2.18}$$

Put $y = \frac{x}{m}$, $x^{\varepsilon}\le y\le x$. Since (see [9])

$$\sum_{n\le y} g_1(n) \sim \frac{cyL(\log y)}{(\log y)^{1-\alpha}} \quad (y\to\infty)$$

where $L(\log y)$ is slowly oscillating, we obtain by partial summation

$$\sum_{n\le y} g_1(n)n^{ia} = \frac{cy^{1+ia}L(\log y)}{(\log y)^{1-\alpha}} - ia\int_2^y \frac{cu^{ia}L(\log u)}{(\log u)^{1-\alpha}}du + o\left(\sum_{n\le y} g_1(n)\right) =$$
$$= \frac{cy^{1+ia}L(\log y)}{(\log y)^{1-\alpha}}(1-\frac{ia}{1+ia}) + o\left(\sum_{n\le y} g_1(n)\right) = \tag{2.19}$$
$$= \frac{y^{ia}}{1+ia}\sum_{n\le y} g_1(n) + o\left(\sum_{n\le y} g_1(n)\right).$$

Combining (2.18) and (2.19) gives

$$\sum_{n\le x} g(n)n^{ia} = \{1+o(1)\}\frac{x^{ia}}{1+ia}\sum_{n\le x} g(n).$$

Since

$$A_x = (1+o(1))\prod_{p\le x}\left(1+\sum_{m=1}^{\infty} f(p^m)p^{-m(1+ia)}\right)\left(1+\sum_{m=1}^{\infty} g(p^m)p^{-m}\right)^{-1}$$

and

$$\sum_{n\le x} g(n) \ge \sum_{m\le x^{1/2}} g_2(m)\sum_{n\le \frac{x}{m}} g_1(n) \gg \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)$$

the assertion (1.5) of Theorem 1.1 holds. ∎

Remark 2.2 Theorem 1.1 contains the corresponding results of P.D.T.A. Elliott [1], K.-H. Indlekofer, I. Kátai, R. Wagner [3], E. Kaya [6] and E. Wirsing [9]. If $g = 1$ and $|f(n)| \leq 1$ for all n we refer to [2].

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