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STABILITY PRESERVING IN REACTION-DIFFUSION SYSTEMS

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Abstract. Reaction-diffusion systems are one of the most well-known partial differential equations used to model physical as well as chemical and biological phenomena. They consist of two parts describing different processes: the so called non-linear kinetic term which represents the driving forces and a linear diffusion term which is responsible for the spread of particles or individuals over a spatial domain. The non-linear part often depends on parameters which variation can influence the qualitative behaviour of the whole system. This paper investigates the question of how much the value of the system parameters can be changed such that the asymptotic stability of the stationary solution is maintained. For this purpose the framework of operator semigroup theory is studied and used.

1. Introduction

While modelling real-world phenomena with differential equations it is often the case that the studied phenomenon – and thus also the system for modelling it – depends on different parameters (for example environmental, physical or chemical factors). Nevertheless, the system parameters are usually partially known or completely unknown: only approximate values from measurements are available. Consequently, the qualitative behaviour of the

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model can be examined with the estimated values of the parameters. Therefore, it is a legitimate expectation that the qualitative behaviour of the model with the estimated parameters must be the same as the behaviour of the exact model. In order to verify the correspondence of the qualitative properties of the estimated and the original model we shall prove the following result. If the estimated parameters are enough close to the exact values, then the behaviour of the solutions of the system does not change. This means that the estimated model reflects faithfully the properties of the exact model, and consequently, of the considered phenomenon, too.

In case of ordinary differential equations we have answered the question of how much the parameter can be changed such that a solution remains asymptotically stable (cf. [8]). The aim of this work is to extend our previous result, i.e. to study the robustness of the asymptotic stability of a stationary solution of the autonomous parameter-dependent reaction-diffusion system

$$u_t(t,x) = D \cdot \mathbf{\Delta}_x u(t,x) + f(u(t,x);\mu) \qquad (t \geq 0, x \in \Omega), \tag{1.1}$$

subject to nonnegative initial resp. homogeneous Dirichlet or Neumann boundary conditions, where $D \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, Ω is a bounded, simply connected region in $\mathbb{R}^d$ with piecewise smooth boundary, $\mathbf{\Delta}_x$ denotes the vector Laplacian (cf. [3]), furhermore $f \in \mathfrak{C}^1(\mathbb{R}^d \times \Pi, \mathbb{R}^n)$ and $\mu \in \Pi \subset \mathbb{R}^p$ represent the parameters.

In order to examine the asymptotic stability of an equilibrium (spatially constant stationary solution) of system (1.1) denoted by $u^* = u^*(\mu)$ usually the linearized equation

$$v_t(t,x) = D \cdot \mathbf{\Delta}_x v(t,x) + A_\mu v(t,x), \qquad (t \geq 0, x \in \Omega) \tag{1.2}$$

resp. the spectral properties of the linear operator

$$D \cdot \mathbf{\Delta}_x + A_\mu : \mathcal{X} \hookrightarrow \mathcal{X}$$

are considered, where $\mathcal{X}$ is an appropriate Banach space, furthermore

$$f(u^*;\mu) \equiv 0 \quad (\mu \in \Pi) \qquad \text{and} \qquad A_\mu := \partial_u f(u^*;\mu).$$

Introducing $V(t) := v(t,\cdot)$ for $t \in \mathbb{R}_0^+$ it is easy to see that the linear system (1.2) is an abstract Cauchy problem

$$\dot{V} = AV, \qquad V(0) = V_0$$

where $A : \mathcal{X} \hookrightarrow \mathcal{X}$ is a linear operator, $\mathcal{X}$ is a Banach space with norm $\|\cdot\|$, $V_0 \in \mathcal{X}$ is the initial value and $V : [0,+\infty) \mapsto \mathcal{X}$ is the unknown function. Hence, $v(t,\cdot) \in \mathcal{X}$ holds for all $t \geq 0$.

To handle the problem, we concern ourselves with a different approach, more precisely we use a perturbation technique. Firstly, it is assumed that the approximate values μ^* of the parameters are known (e.g. from measurements or estimates) and $u^*(\mu^*)$ is asymptotically stable with respect to the norm of $\mathcal{X}$, secondly, a condition is given such that if another parameter value is enough close to μ^*, then the stability of u^* is preserved. In other words the results say that if the exact values of the parameters are enough close to the approximate values, then the stability properties of u^* do not change. Under asymptotic stability we mean that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that any other solution $v : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ of system (1.1) satisfying $\|u^*(\mu^*) - v(0,\cdot)\| < \delta$, also satisfies $\|u^*(\mu^*) - v(t,\cdot)\| < \varepsilon$ for all $t \geq 0$, furthermore there exists $\eta > 0$ such that $\lim_{t\to+\infty} \|u^*(\mu^*) - v(t,\cdot)\| = 0$ for all solutions v satisfying $\|u^*(\mu^*) - v(0,\cdot)\| < \eta$.

We shall show that the asymptotic stability of the zero solution of the linear system (1.2) is equivalent to the exponential stability of a certain operator semigroup. This allows us to investigate the linear asymptotic stability of the equilibrium solution of a reaction-diffusion system by applying the theory of operator semigroups and evolution families.

Our paper is organized as follows. In Section 2, firstly, we summarize some relevant results on operator semigroups and evolution families, then we recall perturbation results and prove a theorem about the robustness of exponential stability of semigroups. In Section 3, we apply our theorem to reaction-diffusion systems with a parameter dependent kinetic part.

2. Operator semigroups and evolution families

For the reader's convenience in this section we mention some well known definitions and results concerning both semigroup theory and evolution families for which we refer mainly to [5], resp. [6] (see also [1, 2, 11, 12, 14, 15]). Then we formulate and prove some lemmas and a theorem which is used to study the robustness of the stability in Section 3.

2.1. General theory

Let $(\mathcal{X}, \|\cdot\|)$ be a real or complex Banach space and denote by $\mathcal{L}(\mathcal{X})$ the Banach algebra of bounded linear operators on $\mathcal{X}$. We write $\mathcal{L}_s(\mathcal{X})$ if we endow $\mathcal{L}(\mathcal{X})$ with the strong operator topology, which is the topology of pointwise convergence on $(\mathcal{X}, \|\cdot\|)$ (cf. [5, 6]).

An operator $P \in \mathcal{L}(\mathcal{X})$ will be called a projection if $P^2 = P$ holds. The complementary projection is always denoted by $Q := I - P$ (where I is the identity operator on $\mathcal{X}$). If $\mathbf{P} : \mathbb{R} \to \mathcal{L}(\mathcal{X})$, $\mathbf{P}(t) = P$ is a projection valued function, then the function whose values are the complementary projections is denoted by $\mathbf{Q}(t) := I - \mathbf{P}(t)$ $(t \in \mathbb{R})$. In the whole section we consider

(i) one-parameter semigroups of bounded linear operators $T(t)$ $(0 \leq t \in \mathbb{R})$ on $\mathcal{X}$. By this we mean the set

$$\mathcal{T} := (T(t))_{t\geq 0} := \left\{T(t) \in \mathcal{L}(\mathcal{X}) :\ t \in \mathbb{R}_0^+\right\},$$

such that the map $\mathbb{R}_0^+ \ni t \mapsto T(t)$ is a homeomorphism from the additive group $(\mathbb{R}_0^+, +)$ into the multiplicative semigroup $(\mathcal{L}(\mathcal{X}), \cdot)$, i.e. it satisfies

$$T(t+s) = T(t)T(s) \quad (t, s \geq 0), \qquad \text{and} \qquad T(0) = I;$$

(ii) two-parameter families of bounded linear operators $U(t,s)$ $(t, s \in \mathbb{R},$ $t \geq s)$ on $\mathcal{X}$ – called evolution families –, i.e. the set

$$\mathcal{U} := (U(t,s))_{t\geq s} := \{U(t,s) \in \mathcal{L}(\mathcal{X}) :\ t, s \in \mathbb{R},\, t \geq s\},$$

such that for all $t, r, s \in \mathbb{R}$, $t \geq r \geq s$

$$U(t,s) = U(t,r)U(r,s) \qquad \text{and} \qquad U(s,s) = I$$

hold.

We say that $\mathcal{T} = (T(t))_{t\geq 0}$ is strongly continuous (C_0-semigroup), if the map $\mathbb{R}_0^+ \ni t \mapsto T(t)$ is continuous in the strong operator topology on $\mathcal{L}(\mathcal{X})$, i.e. for every $x \in \mathcal{X}$ and $\tau \in \mathbb{R}_0^+$ the limit relation

$$\lim_{t\to\tau} \|T(t)x - T(\tau)x\| = 0$$

fulfils.

To describe the exponential growth of the quantities $\|T(t)\|_{\mathcal{L}(\mathcal{X})}$ $(t \geq 0)$ resp. $\|U(t,s)\|_{\mathcal{L}(\mathcal{X})}$ $(t \geq s)$ the following important characteristic of semigroups resp. evolution families is useful.

Definition 2.1. By the exponential growth bound of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ resp. an evolution family $\mathcal{U} = (U(t,s))_{t\geq s}$ we understand the number

$$\omega(\mathcal{T}) := \inf\left\{w \in \mathbb{R} :\ \exists\, M \geq 1 \text{ with } \|T(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{wt} \text{ for } t \geq 0\right\}$$

resp.

$$\omega(\mathcal{U}) := \inf\left\{w \in \mathbb{R} :\ \exists\, M \geq 1 \text{ with } \|U(t,s)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{w(t-s)} \text{ for } t \geq s\right\}.$$

The following properties are essential for semigroups.

Definition 2.2. A semigroup $\mathcal{T} := (T(t))_{t\geq 0}$ is called

1. (uniformly) exponentially bounded, if $\omega(\mathcal{T}) < +\infty$ holds;

2. (uniformly) exponentially stable, if $\omega(\mathcal{T}) < 0$,

3. exponential dichotomic ($\mathcal{T}$ possesses an exponential dichotomy), if there exists a projection $P \in \mathcal{L}(\mathcal{X})$ and two constants $K \geq 1$ and $\delta > 0$ such that

   (a) $T(t)P = PT(t)$ $(t \geq 0)$,

   (b) the restriction $T(t)_{|} : \mathfrak{Ker}(P) \to \mathfrak{Ker}(P)$ is an isomorphism $(t \geq 0)$,

   (c) the following estimations hold:

$$\begin{aligned} \|T(t)x\| &\leq K \cdot e^{-\delta t}\|x\| \qquad (t \geq 0,\, x \in \mathfrak{Im}(P)), \\ \|T(t)x\| &\geq \frac{1}{K} \cdot e^{\delta t}\|x\| \qquad (t \geq 0,\, x \in \mathfrak{Ker}(P)). \end{aligned}$$

We remark (cf. [1]) that

(i) the semigroup $(T(t))_{t\geq 0}$ is strongly continuous if and only if for all $x \in \mathcal{X}$ the map

$$\xi_x : \mathbb{R}_0^+ \to \mathcal{X}, \qquad \xi_x(t) := T(t)x \tag{2.1}$$

is continuous;

(ii) a strongly continuous semigroup is exponentially bounded.

In [12] a characterization of the exponential stability is to find in terms of the exponential dichotomy.

Theorem 2.1. (cf. [12]) *The C_0-semigroup $(T(t))_{t\geq 0}$ is exponentially stable if and only if it possesses an exponential dichotomy with $P = I$.*

For the last result of this section we shall use some properties of the evolution families. Hence, firstly, we recall the needed notions from the theory of evolution families.

Definition 2.3. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s}$ is said to be

1. (uniformly) exponentially bounded, if $\omega(\mathcal{U}) < +\infty$ holds,

2. (uniformly) exponentially stable, if $\omega(\mathcal{U}) < 0$ holds,

3. exponential dichotomic (or $\mathcal{U}$ possesses an exponential dichotomy), if there exists a family of projections $(\mathbf{P}(t))_{t\in\mathbb{R}}$ and two constants $K \geq 1$ and $\delta > 0$ such that

   (a) $U(t,s)\mathbf{P}(s) = \mathbf{P}(t)U(t,s)$ $(t \geq s)$,

   (b) the restriction $U(t,s)_{|} : \mathfrak{Ker}(\mathbf{P}(s)) \to \mathfrak{Ker}(\mathbf{P}(t))$ is an isomorphism $(t \geq s)$,

(c) the following estimations hold:

$$\begin{aligned} \|U(t,s)x\| &\leq K\cdot e^{-\delta(t-s)}\cdot\|x\| \qquad (t\geq s,\ x\in\mathfrak{Im}(\mathbf{P}(s))),\\ \|U(t,s)x\| &\leq \frac{1}{K}\cdot e^{\delta(t-s)}\cdot\|x\| \qquad (t\geq s,\ x\in\mathfrak{Ker}(\mathbf{P}(s))). \end{aligned}$$

Clearly, if $(T(t))_{t\geq 0}$ is a semigroup, then

$$\mathcal{U}^T := (U^T(t,s))_{t\geq s}, \qquad U^T(t,s) := T(t-s) \quad (t\geq s)$$

defines an evolution family. Using the definition of $\mathcal{U}^T$ we can formulate

Lemma 2.1. *The semigroup* $(T(t))_{t\geq 0}$

1. *possesses an exponential dichotomy if and only if $\mathcal{U}^T$ possesses an exponential dichotomy;*
2. *is exponentially stable if and only if the evolution family $\mathcal{U}^T$ is exponentially stable.*

Proof. For the first statement compare [13]. The second statement follows directly from the definition of $\mathcal{U}^T$. ■

2.2. Perturbation results

The linear operator A defined by

$$\mathrm{dom}(A) := \left\{x\in\mathcal{X}\ :\ \lim_{t\to 0+}\frac{T(t)x-x}{t}\ \text{exists}\right\}$$

and

$$Ax := \lim_{t\to 0+}\frac{T(t)x-x}{t} \qquad (x\in\mathrm{dom}(A))$$

is called the generator of $\mathcal{T}$ (cf. [11]).

In what follows, we assume that the linear operator $A:\mathcal{X}\hookrightarrow\mathcal{X}$ is a generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and examine the perturbation of semigroups and evolution families.

It follows from the Bounded Perturbation Theorem (cf. [6]) that if $B\in\mathcal{L}(\mathcal{X})$ then $A+B$ generates a strongly continuous semigroup, denoted by $(T_{A+B}(t))_{t\geq 0}$. For the sake of the completeness we recall a theorem from the book of Engel and Nagel on the perturbation of an evolution family $(U(t,s))_{t\geq s}$.

Theorem 2.2. (cf. [6], Ch. VI., 9.20. Corollary) *Let $(U(t,s))_{t\geq s}$ be an exponentially bounded evolution family on the Banach space $\mathcal{X}$ and let $B\in$ $\in\mathfrak{C}_b(\mathbb{R},\mathcal{L}_s(\mathcal{X}))$. Then there exists a unique bounded $(U_B(t,s))_{t\geq s}$ evolution family on $\mathcal{X}$, for which*

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)\,\mathrm{d}\tau \qquad (t\geq s, x\in\mathcal{X})$$

holds. Furthermore for all $x \in \mathcal{X}$ and $s \in \mathbb{R}$, we have $U_B(t,s)x \in \mathrm{dom}(B(t))$ for almost all $t > s$, the function $BU_B(\cdot, s)x$ is locally integrable on $[s, +\infty)$, finally

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)\,\mathrm{d}\tau \qquad (t \geq s, x \in \mathcal{X}) \tag{2.2}$$

holds.

In order to study the perturbed semigroup $(T_{A+B}(t))_{t\geq 0}$ we apply the above Theorem 2.2 to the evolution family $(U^T(t,s))_{t\geq s}$.

Lemma 2.2. *Let $B \in \mathcal{L}_s(\mathcal{X})$ and consider the evolution family*

$$U^T(t,s) := T(t-s) \qquad (t \geq s).$$

Then for the evolution family $(U_B(t,s))_{t\geq s}$ coming from Theorem 2.2

$$U_B(t,s) = T_{A+B}(t-s) \qquad (t \geq s)$$

holds.

Proof. Let $(\hat{T}(t))_{t\geq 0}$ be an arbitrary C_0-semigroup. Substituting

$$U(t,s) = U^T(t,s) = T(t-s), \qquad U_B(t,s) = \hat{T}(t-s) \quad (t \geq s)$$

and finally $s = 0$ into the equation (2.2) gives

$$\hat{T}(t)x = T(t)x + \int_0^t T(t-\tau)B\hat{T}(\tau)\,\mathrm{d}\tau \qquad (x \in \mathcal{X}).$$

The above equation holds for the perturbed semigroup $(T_{A+B}(t))_{t\geq 0}$, too (cf. [6], Ch. III. 1.7 Corollary). Hence, following from the uniqueness of the semigroups we have

$$T_{A+B}(t) = \hat{T}(t) \qquad (t \geq 0),$$

therefore

$$U_B(t,s) = T_{A+B}(t-s) \qquad (t \geq s)$$

holds. ■

The following theorem states the exponential stability of the perturbed evolution family $(U_B(t,s))_{t\geq s}$.

Theorem 2.3. (cf. [6], Ch. VI., 9.25. Corollary) *Let $(U(t,s))_{t\geq s}$ be an exponentially bounded evolution family on $\mathcal{X}$ and assume that it possesses an exponential dichotomy with constants K, $\delta > 0$, furthermore let $B \in \mathfrak{C}_b(\mathbb{R}, \mathcal{L}_s(X))$. If*

$$\omega(\mathcal{U}) < 0 \qquad \text{and} \qquad \sup_{t\in\mathbb{R}}\|B(t)\| < \frac{\delta}{K},$$

then $\omega(\mathcal{U}_B) < 0$ holds.

Now, we can formulate a theorem which is a very useful tool in the study of exponential stability of solutions of reaction-diffusion systems.

Theorem 2.4. *Assume that the C_0-semigroup $(T(t))_{t\geq 0}$ is exponentially stable with positive constants M, $w > 0$, i.e.*

$$\|T(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{-wt} \qquad (t \geq 0)$$

holds. If an operator $B \in \mathcal{L}_s(\mathcal{X})$ satisfies the condition

$$\|B\|_{\mathcal{L}(\mathcal{X})} < \frac{w}{M},$$

then the semigroup $(T_{A+B}(t))_{t\geq 0}$ is exponentially stable.

Proof. Following from Lemma 2.1 the evolution family $\mathcal{U}^T$ is exponentially stable, i.e. $\omega(\mathcal{U}^T) < 0$. Based on the previous Theorem 2.3 we can see that the perturbed evolution family $\mathcal{U}_B$ is also exponentially stable. Then we get from Proposition 2.2 that

$$U_B(t,s) = T_{A+B}(t-s) \qquad (t \geq s),$$

therefore based on Proposition 2.1 the perturbed semigroup $(T_{A+B})_{t\geq 0}$ is exponentially stable. ■

We should mention that the above result is also a consequence of the Bounded Perturbation Theorem. Nevertheless we believe that our method contains interesting and useful ideas, such as the use of the evolution family U^T or the characterization of the exponential stability via the exponential dichotomy.

In addition, it is worth noting that using two different ways gives the same upper bound on the exponential stability of operator semigroups.

3. Reaction-diffusion systems

In this section, we are going to apply the abstract result proved in Theorem 2.4 to obtain some information of stability preserving in reaction-diffusion systems.

As it was mentioned, the autonomous system of reaction-diffusion equations

$$u_t(t,x) = D \cdot \mathbf{\Delta}_x u(t,x) + f(u(t,x);\mu) \qquad (t \geq 0, x \in \Omega) \tag{3.1}$$

is considered with positive diagonal matrix $D \in \mathbb{R}^{n\times n}$, where $\Omega \subset \mathbb{R}^d$ is a bounded, simply connected region with piecewise smooth boundary, $f \in \in \mathfrak{C}^1(\mathbb{R}^d \times \Pi, \mathbb{R}^n)$ and $\mu \in \Pi$ represents the parameters, $\Pi \subset \mathbb{R}^p$ is an open subset, $p \in \mathbb{N}$. We focus on the stability of solutions in two types of boundary conditions: firstly we are interested in solutions $\Phi : \overline{\Omega} \times \mathbb{R}_0^+ \to \mathbb{R}^n$ that satisfy the Dirichlet boundary condition

$$u(t, x) = 0 \qquad (t, x) \in (\mathbb{R}_0^+ \times \partial\Omega),$$

and secondly the no-flux or homogeneous Neumann boundary condition

$$(\mathbf{n} \cdot \nabla)u(t, x) = 0 \qquad (t, x) \in (\mathbb{R}_0^+ \times \partial\Omega).$$

In what follows $u^* \in \mathbb{R}^n$ denotes an equilibrium of the kinetic system

$$\dot{\widetilde{u}} = f(\widetilde{u}, \mu).$$

It is easy to see that u^* is a stationary solution of the reaction-diffusion system (3.1), too. Linearizing system (3.1) around u^* we obtain the linear system

$$v_t(t, x) = \mathcal{A}_\mu v(t, x) \qquad (t \geq 0, x \in \Omega) \tag{3.2}$$

where

$$\mathcal{A}_\mu := D \cdot \mathbf{\Delta}_x + A_\mu : \mathcal{X} \hookrightarrow \mathcal{X} \qquad \text{and} \qquad A_\mu := \partial_u f(u^*; \mu).$$

The equilibrium u^* is an asymptotically stable solution of the original system (3.1) if the zero solution is an asymptotically stable solution of the linearized system (3.2) (cf. [4]). For both types of boundary conditions, a Banach space $\mathcal{X}$ and a domain of the operator $\mathcal{A}_\mu$ are defined, such that $\mathcal{A}_\mu$ generates a strongly continuous semigroup. The following result describes the correspondence between the asymptotic stability of the zero solution of the linearized system (3.2) and the exponential stability of the semigroup generated by the operator defining the right-hand side of system (3.2).

Theorem 3.1. *Let $\mu \in \Pi$ be an arbitrary fixed value of the parameter, and suppose that the coefficient operator of the linear system* (3.2) *generates a strongly continuous semigroup on some Banach space $\mathcal{X}$* (*equipped with norm $\| \cdot \|$*) *denoted by $\mathcal{T} := (T(t))_{t \geq 0}$. Then, the asymptotic stability of the zero solution of system* (3.2) *is equivalent to the uniform exponential stability of the semigroup $\mathcal{T}$.*

Proof.

Step 1. The uniform exponential stability of $\mathcal{T}$ implies that there exists $\varepsilon > 0$ such that $\lim_{t \to +\infty} e^{\varepsilon t} \|T(t)x\| = 0$ holds for all $x \in \mathcal{X}$ (cf. [6]). This guarantees the asymptotic stability of the zero solution of the linear system (3.2), because every solution u of (3.2) can be expressed in the form

$$v(t, \cdot) = T(t)v(0, \cdot) =: T(t)v_0 \qquad (t \geq 0), \tag{3.3}$$

using the semigroup $\mathcal{T}$, where $v_0 := v(0, \cdot) \in \operatorname{dom}(\mathcal{A}_\mu)$.

Step 2. Assume that the zero solution of the linear system (3.2) is asymptotically stable, hence there exists a number $\delta > 0$ such that for all solutions $u : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ of system (3.2)

$$\lim_{t\to+\infty} \|v(t,\cdot)\| = 0 \tag{3.4}$$

holds, provided $\|v(0,\cdot)\| < \delta$ satisfies. Therefore, using notation (3.3) and limit relation (3.4) we have

$$0 = \lim_{t\to+\infty} \|v(t,\cdot)\| = \lim_{t\to+\infty} \|T(t)v_0\|. \tag{3.5}$$

Due to the estimate $\|T(t)v_0\| \le \|T(t)\|_{\mathcal{L}(\mathcal{X})} \cdot \|v_0\|$, the above inequality (3.5) implies that there exists a $t_0 > 0$, for which $\|T(t_0)\|_{\mathcal{L}(\mathcal{X})} < 1$ holds. This is equivalent to the fact that $\mathcal{T}$ is uniformly exponentially stable (cf. [6], Ch. V., Prop. 1.7), which completes the proof. ■

3.1. Dirichlet boundary condition

Hereinafter let $\mathcal{X} := L^2(\Omega, \mathbb{R}^n)$. In this subsection we shall show that the Dirichlet Laplacian, or in other words the coefficient operator of the linear system (3.2) defined on an appropriate domain generates an analytic semigroup.

Theorem 3.2. *The operator*

$$\mathrm{dom}(\mathcal{A}_\mu) := H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n), \qquad \mathcal{A}_\mu := D \cdot \mathbf{\Delta}_x + A_\mu$$

generates an analytic semigroup in $L^2(\Omega, \mathbb{R}^n)$ *for all* $\mu \in \Pi$.

Proof. The one-dimensional Laplacian operator

$$\Delta_x u := \sum_{i=1}^{d} \partial_{ii} u \qquad (u \in H^2(\Omega) \cap H_0^1(\Omega))$$

is sectorial (cf. [10]). Then we define $\mathcal{D} := d \cdot \Delta_x$, $\mathrm{dom}(\mathcal{D}) := \mathrm{dom}(\Delta_x)$, where $d > 0$ is the diffusion coefficient. This operator is also sectorial, because $\sigma(\mathcal{D}) = \sigma(d \cdot \Delta_x) = \sigma(\Delta_x)$ (cf. [6] Ch. II., section 2/a, subsection 2.2), furthermore $R(\lambda, \Delta_x) \le M/|\lambda|$ implies

$$\|R(\lambda, \mathcal{D})\| = \|(\lambda - \mathcal{D})^{-1}\| = \frac{1}{d} \cdot \left\| \left(\frac{\lambda}{d} - \Delta_x \right)^{-1} \right\| \le \frac{1}{d} \cdot \frac{M}{|\lambda/d|} = \frac{M}{|\lambda|}.$$

Then, the n-dimensional diffusional operator is defined by

$$\mathbf{D} := D \cdot \mathbf{\Delta}_x := (d_1 \cdot \Delta_x, \ \dots, \ d_n \cdot \Delta_x), \qquad \mathrm{dom}(\mathbf{D}) := \mathrm{dom}(\Delta_x) \times \cdots \times \mathrm{dom}(\Delta_x),$$

where $D \in \mathbb{R}^{n\times n}$, $D = \mathrm{diag}(d_0, \ \dots, \ d_n)$ contains the positive diffusion coefficients. Based on [10] the operator $\mathbf{D}$ is sectorial, too, therefore it generates an analytic semigroup (cf. [6], Ch. II., Theorem 4.6). Finally, since A_μ is bounded, the operator $\mathcal{A}_\mu$ also generates an analytic semigroup on $L^2(\Omega, \mathbb{R}^n)$ (cf. [11], Ch. 3. Corollary 2.2). ■

Next, the asymptotic stability of $u^*(\mu)$ is examined, using the spectral bound of $\mathbf{D}$:

$$s(\mathbf{D}) := \sup\{\Re(\lambda) \ : \ \lambda \in \sigma(\mathbf{D})\}$$

where $\sigma(\mathbf{D}) \subset \mathbb{C}$ denotes the spectrum of the operator $\mathbf{D}$. Clearly, $s(\mathbf{D})$ is well defined, because the spectrum of $\mathbf{D}$ is contained in a left half-plane.

Theorem 3.3. *Assume that for a certain parameter value $\mu^* \in \Pi$ the equilibrium solution $u^*(\mu^*)$ is an asymptotically stable stationary solution of the nonlinear system (3.1). Furthermore, suppose that the inequality*

$$\|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})} < -s(\mathbf{D}) \tag{3.6}$$

holds, and the function $A(\mu)$ is Lipschitz-continuous, i.e. there exists a positive constant $L > 0$ such that for all $\mu_1, \mu_2 \in \Pi$

$$\|A(\mu_1) - A(\mu_2)\|_{\mathcal{L}(\mathcal{X})} \leq L\|\mu_1 - \mu_2\|_{\mathbb{R}^p}$$

is fulfilled. Then, there exists $R > 0$ such that, for all parameter values $\widetilde{\mu} \in \Pi$ satisfying the condition

$$\|\widetilde{\mu} - \mu^*\|_{\mathbb{R}^p} < R, \tag{3.7}$$

the stationary solution $u^*(\widetilde{\mu})$ is asymptotically stable.

Proof. Denote by $\mathcal{T}_\mathbf{D} = (T_\mathbf{D}(t))_{t\geq 0}$ the analytic semigroup generated by the operator $\mathbf{D}$. The spectral bound of $\mathbf{D}$ has the property $s(\mathbf{D}) = \omega(\mathcal{T}_\mathbf{D})$, because $\mathbf{D}$ is sectorial. Furthermore, the spectrum $\sigma(\mathbf{D})$ consists of isolated negative (real) eigenvalues, consequently $\mathcal{T}_\mathbf{D}$ is exponentially stable with constants $M > 0$ and $w := s(\mathbf{D})$, i.e. $\omega(\mathcal{T}_\mathbf{D}) < 0$ holds. Hence, if condition (3.6) fulfils, then the semigroup $(T_{\mu^*}(t))_{t\geq 0}$ generated by $\mathcal{A}_{\mu^*}$ is exponentially stable with some constants $M \geq 1$ and $w = s(\mathbf{D}) + \|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})}$, because applying the Bounded Perturbation Theorem from [6] the estimate

$$\|T_{\mu^*}(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{(s(\mathbf{D})+\|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})})t} \qquad (t \geq 0) \tag{3.8}$$

is valid, moreover $|\omega(\mathcal{T}_{\mu^*})| > |s(\mathbf{D}) + \|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})}|$. Let us define

$$R := \frac{\left|\ s(\mathbf{D}) + \|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})}\right|}{L \cdot M}. \tag{3.9}$$

Writing the operator $\mathcal{A}_{\widetilde{\mu}} = D \cdot \mathbf{\Delta}_x + A(\widetilde{\mu})$ as

$$D \cdot \mathbf{\Delta}_x + A(\widetilde{\mu}) = (D \cdot \mathbf{\Delta}_x + A(\mu^*)) + (A(\widetilde{\mu}) - A(\mu^*)),$$

then condition (3.7) (with R defined in (3.9)) implies that the norm of the operator $B := A(\widetilde{\mu}) - A(\mu^*) \in \mathcal{L}(\mathcal{X})$ can be estimated as

$$\|B\|_{\mathcal{L}(\mathcal{X})} = \|A(\widetilde{\mu}) - A(\mu^*)\|_{\mathcal{L}(\mathcal{X})} \leq L \cdot \|\widetilde{\mu} - \mu^*\|_{\mathbb{R}^p} < \frac{\left|s(\mathbf{D}) + \|A_{\mu^*}\|_{\mathcal{L}(\mathcal{X})}\right|}{M}.$$

Hence, the inequality

$$\|B\|_{\mathcal{L}(\mathcal{X})} < \frac{|\omega(\mathcal{T}_{\widetilde{\mu}})|}{M}$$

also holds. Thus, based on Theorem 2.4, the semigroup $\mathcal{T}_{\widetilde{\mu}}$ is exponentially stable, hence, the zero solution of the linearized system

$$v_t = (D \cdot \mathbf{\Delta}_x + A(\mu^*) + B)v = (D \cdot \mathbf{\Delta}_x + A(\widetilde{\mu}))v$$

is asymptotically stable due to Thm. 3.1. Then, $u^*(\widetilde{\mu})$ is an asymptotically stable solution of the reaction-diffusion system (3.1). ■

Example 3.4. Consider the case $n = 2$ and $p = 1$, i.e. the given system consists of two equations, which depend on one parameter. Assume that the matrix $A(\mu) := A_\mu$ in (3.2) is diagonal, more precisely it has the form

$$A(\mu) := \begin{bmatrix} a & 0 \\ 0 & \mu \end{bmatrix} \qquad (\mu \in \Pi),$$

where $a \in \mathbb{R}$ is a fixed value (not a parameter) and $\mu \in \Pi$ is the parameter, $\Pi \subset \mathbb{R}$ is an open interval. We assume furthermore that $\mu^* \in \mathbb{R}$ is a parameter value for which $|\mu^*| > |a|$ holds and the stationary solution $u^*(\mu^*) \in \mathbb{R}^2$ of the reaction-diffusion system (3.1) is asymptotically stable. It is easy to see that the equality

$$\|A(\mu_1) - A(\mu_2)\|_2 = |\mu_1 - \mu_2|$$

holds for all μ_1, $\mu_2 \in \Pi$, which means that A is Lipschitz continuous with $L = 1$, where $\|\cdot\|_2$ denotes the usual euclidean, resp. the spectral norm in $\mathbb{R}^2$, resp. in $\mathbb{R}^{2\times 2}$. Consequently, the stationary solution is asymptotically stable with parameter value $\widetilde{\mu} \in \mathbb{R}$, provided

$$|\mu^* - \widetilde{\mu}| < \frac{|s(\mathbf{D}) + \|A(\mu^*)\|_2|}{M}$$

fulfils with some positive constants M. Because $\|A(\mu^*)\|_2 = |\mu^*|$, the above inequality means that in case of

$$\mu^* - \frac{|s(\mathbf{D}) + |\mu^*|\,|}{M} < \widetilde{\mu} < \mu^* + \frac{|s(\mathbf{D}) + |\mu^*|\,|}{M} \tag{3.10}$$

the semigroup $(T_{\widetilde{\mu}}(t))_{t\geq 0}$ is exponentially stable, resp. the equilibrium solution $u^*(\widetilde{\mu})$ is asymptotically stable.

Following from the inequality (3.8) it is easy to see that in case of

$$\|A(\widetilde{\mu})\|_2 < -s(\mathbf{D}) \tag{3.11}$$

the semigroup $(T_{\widetilde{\mu}}(t))_{t\geq 0}$ and hence the stationary solution is asymptotically stable. But it can be easily seen that our result in (3.10) gives a completely different condition on the parameter value $\widetilde{\mu}$ as (3.11).

3.2. Neumann boundary condition

After the Dirichlet-boundary condition we focus on the case of the homogeneous Neumann boundary condition, which means in the case of applications that there is no migration across the boundary of the given domain Ω. Consider the operator

$$\mathbf{D}v := D \cdot \mathbf{\Delta}_x v \qquad (v \in \mathrm{dom}(\mathbf{D}))$$

with domain

$$\mathrm{dom}(\mathbf{D}) := \left\{\phi \in H^2(\Omega, \mathbb{R}^n) \subset \mathcal{X} \;:\; (\mathbf{n} \cdot \nabla_x)\, \phi\,(x,t) = 0 \;\left((x,t) \in \partial\Omega \times \mathbb{R}_0^+\right)\right\}$$

where $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$.

Theorem 3.5. *For each parameter value $\mu \in \Pi$ the coefficient operator*

$$D \cdot \mathbf{\Delta}_x + A_\mu = \mathbf{D} + A_\mu$$

of the linear reaction-diffusion system (3.2) generates a C_0-semigroup.

Proof. The operator $\mathbf{D}$ generates a $(T(t))_{t\geq 0}$ C_0-semigroup (cf. [9]), and there exist constants $M_{\mathbf{\Delta}_x} \geq 0$ and $w_{\mathbf{\Delta}_x} \in \mathbb{R}$ such that

$$\|T(t)\|_{\mathcal{L}(\mathcal{X})} \leq M_{\mathbf{\Delta}_x} e^{w_{\mathbf{\Delta}_x} t} \qquad (t \geq 0)$$

holds (cf. [6]). Considering A_μ as a linear operator acting on $\mathcal{X}$, the Bounded Perturbation Theorem implies that the operator $\mathbf{D} + A_\mu$ generates a C_0-semigroup. ∎

Thus, the robustness of the asymptotic stability of the stationary solution u^* (with fixed diffusion coefficients) can be examined similarly as in the case of the Dirichlet boundary condition.

Theorem 3.6. *Suppose that with a certain value μ^* of the parameter the stationary solution $u^*(\mu^*)$ is an asymptotically stable solution of the nonlinear reaction-diffusion system (3.1), such that the zero solution of the linearized system (3.2) is asymptotically stable. Assume furthermore that the map*

$$A : \Pi \to \mathcal{L}(\mathcal{X}), \qquad A(\mu) := A_\mu$$

is Lipschitz continuous with constant $L > 0$*. Then, there exists an* $R > 0$ *such that for all parameter values* $\widetilde{\mu}$ *satisfying*

$$\|\widetilde{\mu} - \mu^*\|_{\mathbb{R}^p} < R, \tag{3.12}$$

the equilibrium solution $u^*(\widetilde{\mu})$ *is also an asymptotically stable solution of system* (3.1)*.*

Proof. The asymptotic stability of the zero solution of the linearized system (3.2) implies the exponential stability of the generated strongly continuous semigroup $\mathcal{T}_{\mu^*}$, i.e. there exist $w > 0$ and $M \geq 1$ constants, such that $\|T(t)x\| \leq Me^{-wt}\|x\|$ holds for all $x \in \mathcal{X}$. Let us define

$$R := \frac{w}{L \cdot M}.$$

Then, similarly as in the case of Thm. 3.3 we write the operator

$$\mathcal{A}_{\widetilde{\mu}} = D \cdot \mathbf{\Delta}_x + A(\widetilde{\mu})$$

in form of

$$D \cdot \mathbf{\Delta}_x + A(\widetilde{\mu}) = (D \cdot \mathbf{\Delta}_x + A(\mu^*)) + (A(\widetilde{\mu}) - A(\mu^*)),$$

then condition (3.12) (with the above defined R) implies that the norm of the operator $B := A(\widetilde{\mu}) - A(\mu^*) \in \mathcal{L}(\mathcal{X})$ can be estimated as

$$\|B\|_{\mathcal{L}(\mathcal{X})} = \|A(\widetilde{\mu}) - A(\mu^*)\|_{\mathcal{L}(\mathcal{X})} \leq L \cdot \|\widetilde{\mu} - \mu^*\|_{\mathbb{R}^p} < \frac{w}{L \cdot M}.$$

Thus, $(T_{\widetilde{\mu}}(t))_{t\geq 0}$ is exponentially stable, which has a consequence that the zero solution of the linearized system is asymptotically stable with parameter value $\widetilde{\mu}$ (cf. Thm. 3.1), which implies the asymptotic stability of $u^*(\widetilde{\mu})$. ■

4. Discussion

Summarizing, we have worked on two topics in this paper. On the one hand, by giving a bound estimation for the perturbed operator we dealt with the robustness of exponential stability of strongly continuous semigroups. On the other hand, we showed how much parameter change in the kinetic term of a reaction-diffusion system with homogeneous Dirichlet, resp. Neumann boundary conditions doesn't cause loss of exponential stability of its (spatially constant) stationary solution. Thus, in the last section, we have shown that the exponential stability of stationary solutions is a robust property for reaction-diffusion systems.

References

[1] **Arendt, W., A. Grabosch, G. Greiner, U. Moustakas, R. Nagel, U. Schlotterbeck, U. Groh, H.P. Lotz and F. Neubrander,** *One-parameter Semigroups of Positive Operators,* Springer-Verlag, Berlin, 1986.
https://doi.org/10.1007/BFb0074922

[2] **Bátkai, A. and S. Piazzera,** *Semigroups for Delay Equations,* AK Peters, New York, 2005.
https://doi.org/10.1201/9781439865682

[3] **Britton, N.F.,** *Reaction-diffusion equations and their applications to biology,* Academic Press, Inc., London, 1986.

[4] **Casten, R. and Ch. Holland,** Stability propertiesof solutions to system of reaction-diffusion equations, *SIAM J. Appl. Math.*, **33** (1977), 353–364.
https://doi.org/10.1137/0133023

[5] **Chicone, C. and Y. Latushkin,** *Evolution Semigroups in Dynamical Systems and Differential Equations*, American Mathematical Society, Providence, 1999.
https://doi.org/10.1090/surv/070

[6] **Engel, KJ. and R. Nagel,** *One-parameter semigroups for linear evolution equations*, *Semigroup Forum*, **63** (2001), 278–280.
https://doi.org/10.1007/s002330010042

[7] **Henry, D.,** *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, Heidelberg, 1981.
https://doi.org/10.1007/BFb0089647

[8] **Kovács, S. and Sz. György,** On the qualitative behaviour of parameter dependent systems, *Miskolc Mathematical Notes*, **20** (2019), 353–364.
https://doi.org/10.18514/MMN.2019.2736

[9] **Leiva, H.,** Stability of a periodic solution for a system of parabolic equations, *Applicable Analysis*, **60(3-4)** (1996), 277–300.
https://doi.org/10.1080/00036819608840433

[10] **Lorenzi, L., A. Lunardi, G. Metafune and D. Pallara,** Analytic semigroups and reaction-diffusion problems, Internet seminar (2004-2005), 2005.
https://www.math.tecnico.ulisboa.pt/~czaja/ISEM/08internetseminar200405.pdf

[11] **Pazy, A.,** *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
https://doi.org/10.1007/978-1-4612-5561-1

[12] **Phóng, V.Q.,** On the exponential stability and dichotomy of $\mathcal{C}_0$-semigropus, *Studia Math.*, **132** (1999), 141–149.
http://doi.org/10.4064/SM-132-2-141-149

[13] **Sasu, A.L.,** Exponential dichotomy for evolution families on the real line, *Abstr. Appl. Anal.* Art. ID 31641 (2006), 1–16.

[14] **Sasu, B.,** On exponential dichotomy of semigroups, *Acta Math. Univ. Comenian.*, **75** (2006), 55–61.

[15] **Nguyen Van Minh, N., F. Räbiger and R. Schnaubelt,** Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations and Operator Theory*, **32** (1998), 332–353.
https://doi.org/10.1007/BF01203774

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