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NEW FAMILIES OF NORMAL NUMBERS CONSTRUCTED FROM EXISTING ONES

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Abstract. Given an integer $q \geq 2$, let $\mathcal{A}_q := \{0, 1, \ldots, q-1\}$ be the set of base q digits. We say that $\alpha = a_1 \ldots a_k$, where each $a_i \in \mathcal{A}_q$, is a *word* of length $\lambda(\alpha) = k$. Let $H(1), H(2), \ldots$ be a sequence of nonnegative integers and given a sequence of words $\alpha_1, \alpha_2, \ldots$, we examine under which conditions the number $0.\alpha_1^{H(1)}\alpha_2^{H(2)}\ldots$ is a normal number in base q (where $\alpha_j^r = \underbrace{\alpha_j \ldots \alpha_j}_{r \text{ times}}$).

1. Introduction

Fix an integer $q \geq 2$ and let $\mathcal{A}_q := \{0, 1, \ldots, q-1\}$ be the set of base q digits. We say that $\alpha = a_1 \ldots a_k$, where each $a_i \in \mathcal{A}_q$, is a *word* of length $\lambda(\alpha) = k$.

Let $\beta = \alpha_1\alpha_2 \ldots$ be an infinite concatenation of words α_i's, which we write as $\beta = j_1 j_2 \ldots$, where each $j_i \in \mathcal{A}_q$. Then, $\xi = \xi(\beta) := \displaystyle\sum_{\nu=1}^{\infty} \frac{j_\nu}{q^\nu}$ is a *q-ary normal number* if the sequence of fractional parts $\{q^k \xi\}_{k \geq 1}$ is uniformly distributed in $[0, 1]$ (see Theorem 8.1 in Kuipers and Niederreiter [3]).

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Over the past decade, we created several new families of normal numbers [2].

Here, we expose a new approach for creating normal numbers. To do so, we first let $H(1), H(2), H(3), \ldots$ be a sequence of natural numbers and then, given a sequence of words $\alpha_1, \alpha_2, \ldots$, we examine under which conditions the number $0.\alpha_1^{H(1)}\alpha_2^{H(2)}\ldots$ (where $\alpha_j^r = \underbrace{\alpha_j \ldots \alpha_j}_{r \text{ times}}$) is a normal number in base q.

2. Background results

In a 1994 paper, Bassily [1] generalized to polynomials a result which originally applied only to the sum of digits function. To explain his result, we introduce additional notation. Let q and $\mathcal{A}_q$ be as above. Then, every nonnegative integer n can be written as

$$n = \sum_{r=0}^{\infty} a_r(n) q^r, \quad \text{where each } a_r(n) \in \mathcal{A}_q.$$

Clearly, the above sum is finite, since $a_r(n) = 0$ if $r > (\log n)/(\log q)$. Bassily investigated digital functions $\alpha(n)$ which depend on the digital blocks of length k. More precisely, given $k \in \mathbb{N}$ and a function $F_k : \mathcal{A}_q^k \to \mathbb{R}$ which satisfies the condition $F_k(0, \ldots, 0) = 0$, consider the function

$$\alpha(n) := \sum_{j=0}^{\infty} F_k\left(a_j(n), a_{j+1}(n), \ldots, a_{j+k-1}(n)\right),$$

that is, a kind of generalisation of the sum of digits function. Further setting

$$M := \frac{1}{q^k} \sum_{(b_0,\ldots,b_{k-1})\in\mathcal{A}_q^k} F_k(b_0, \ldots, b_{k-1}),$$

Bassily showed [1] that

$$\alpha(n) = (1 + o(1)) M \frac{\log n}{\log q} \qquad (n \to \infty),$$

except perhaps on a set of density 0. Let $\pi(x)$ stand for the number of primes not exceeding x. In his paper, Bassily also proved the following theorem.

Theorem A. (Bassily) *Let $q \geq 2$ be a fixed integer. Let $P(x) = c_r x^r + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients taking on positive values for all $x > 0$ and such that $\gcd(c_r, q) = 1$. Then,*

$$\sum_{n \leq x} \left(\alpha(P(n)) - Mr\frac{\log x}{\log q}\right)^2 \ll x \log x,$$

$$\sum_{p \leq x} \left(\alpha(P(p)) - Mr\frac{\log x}{\log q}\right)^2 \ll \pi(x) \log x.$$

Let q and $P(x)$ be as in the statement of Theorem A, and further let $2 = p_1 < p_2 < \cdots$ be the sequence of all primes. Moreover, let η_1 and η_2 be the real numbers whose q-ary expansions are given by

$$\begin{aligned}\eta_1 &= 0.\overline{P(1)}\,\overline{P(2)}\,\overline{P(3)}\,\overline{P(4)}\ldots\overline{P(n)}\ldots,\\ \eta_2 &= 0.\overline{P(2)}\,\overline{P(3)}\,\overline{P(5)}\,\overline{P(7)}\ldots\overline{P(p_n)}\ldots,\end{aligned}$$

where $\overline{n}$ stands for the concatenation of the base q digits of n. Bassily proved that η_1 and η_2 are normal numbers in base q.

Moreover, fix $k \in \mathbb{N}$ and a word $\gamma \in \mathcal{A}_q^k$, that is, a word made of k base q digits. Then, given an arbitrary word β made of base q digits, we define $\omega_\gamma(\beta)$ as the number of occurrences of the subword γ in the word β. Bassily showed that if

$$\Gamma_k := \{\gamma = b_1 b_2 \ldots b_k : b_i \in \mathcal{A}_q \text{ for } i = 1, \ldots, k\}, \tag{2.1}$$

then,

$$\sum_{n \le x} \max_{\gamma \in \Gamma_k} \left(\omega_\gamma(P(n)) - \frac{r}{q^k}\frac{\log x}{\log q}\right)^2 \ll x \log x, \tag{2.2}$$

$$\sum_{p \le x} \max_{\gamma \in \Gamma_k} \left(\omega_\gamma(P(p)) - \frac{r}{q^k}\frac{\log x}{\log q}\right)^2 \ll \pi(x) \log x. \tag{2.3}$$

Now, before we state our main results, we introduce two new functions as follows. Given an arithmetic function $H : \mathbb{N} \to \mathbb{N} \cup \{0\}$, consider the two functions

$$s_H(x) := \sum_{n \le x} H^2(n) \Big/ \Big(\sum_{n \le x} H(n)\Big)^2 \quad \text{and}$$

$$t_H(x) := \sum_{p \le x} H^2(p) \Big/ \Big(\sum_{p \le x} H(p)\Big)^2.$$

3. The main results

Theorem 1. *Let $q \ge 2$ be a fixed integer. Let $P(x) = c_r x^r + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients taking on nonnegative values for all $x > 0$ and such that $gcd(c_r, q) = 1$. Let $H(1), H(2), H(3), \ldots$ be a sequence of natural numbers satisfying the condition*

$$\lim_{x \to \infty} s_H(x) \cdot \frac{x}{\log x} = 0. \tag{3.1}$$

Then, the number

$$\eta_1 := 0.\overline{P(1)}^{H(1)}\,\overline{P(2)}^{H(2)}\,\overline{P(3)}^{H(3)}\,\overline{P(4)}^{H(4)}\ldots \tag{3.2}$$

is a q-normal number.

Theorem 2. *Let q, $P(x)$ and $H(n)$ be as in Theorem 1. Assuming that*

$$\lim_{x\to\infty} t_H(x)\cdot\frac{x}{\log^2 x} = 0, \tag{3.3}$$

then,

$$\eta_2 := 0.\overline{P(2)}^{H(2)}\,\overline{P(3)}^{H(3)}\,\overline{P(5)}^{H(5)}\,\overline{P(7)}^{H(7)}\ldots \tag{3.4}$$

is a q-normal number.

4. Some preliminary results

We start with a classic result in analytic number theory. As is common, $\zeta(s)$ stands for the Riemann Zeta Function.

Proposition 1. *Let f be an arithmetic function and let t be a positive real number. Assume that, for every real $s>1$,*

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \zeta^t(s)\sum_{n=1}^{\infty}\frac{g(n)}{n^s},$$

where g is such that $\displaystyle\sum_{n=1}^{\infty}\frac{g(n)}{n}$ converges absolutely. Then,

$$\sum_{n\le x} f(n) = (c+o(1))\,x\,\log^{t-1}x \qquad (x\to\infty),$$

where $c = \dfrac{1}{\Gamma(t)}\displaystyle\sum_{n-1}^{\infty}\frac{g(n)}{n}$.

Proof. This is a particular case of Theorem 2 in the 1954 paper of Atle Selberg [5]. ■

Let ϕ stand for the Euler totient function. The following is often called the *prime number theorem for arithmetic progressions.*

Proposition 2. *Given two coprime integers $k\ge 1$ and ℓ, let $\pi(x;k,\ell) :=$ $:= \#\{p\le x : p\equiv \ell \pmod k\}$. Then,*

$$\pi(x;k,\ell) = (1+o(1))\frac{1}{\phi(k)}\frac{x}{\log x} \qquad (x\to\infty).$$

Proof. See Theorems 5.11 and 5.14 in the book of Narkiewicz [4]. ■

We now recall a 1961 result of Wirsing [6] which we state as a proposition.

Proposition 3. *Let $f(n)$ be a real valued non negative multiplicative function satisfying $f(p^\nu) \le c_1 c_2^\nu$ for all prime powers p^ν, $\nu \ge 2$, for some positive constant c_1 and c_2, with $c_2 < 2$. Assuming that there exists a positive constant τ for which*

$$\sum_{p\le x} f(p) = (\tau + o(1))\frac{x}{\log x} \qquad (x\to\infty),$$

then, as $x\to\infty$,

$$\sum_{n\le x} f(n) = \left(\frac{e^{\gamma r}}{\Gamma(r)} + o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right),$$

where γ stands for the Euler-Mascheroni constant and $\Gamma(r)$ is the Gamma function.

The next proposition is an immediate application of Proposition 3.

Proposition 4. *Let f be a completely multiplicative function such that $f(p) \in$ $\in \{0,1\}$ for all primes p. Assume that there exists a real number $\delta \in (0,1)$ such that*

$$\sum_{p\le x} f(p) = (\delta + o(1))\frac{x}{\log x} \qquad (x\to\infty).$$

Then, as $x\to\infty$,

$$\sum_{n\le x} f(n) = \left(\frac{e^{\gamma r}}{\Gamma(r)} + o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1-\frac{f(p)}{p}\right)^{-1}.$$

5. Proof of Theorem 1

Letting Γ_k be the set defined in (2.1), in order to prove that η_1 is a q-normal number, it is sufficient to prove that

$$\sum_{n\le x} H(n)\cdot\max_{\gamma\in\Gamma_k}\left|\omega_\gamma(P(n)) - \frac{r}{q^k}\frac{\log x}{\log q}\right| = = o\Big((\log x)\cdot\sum_{n\le x} H(n)\Big) \quad (x\to\infty). \tag{5.1}$$

Indeed, we need to show that for most integers $n\le x$, the number of occurrences of $P(n)$ in the words $\gamma = b_1\dots b_k\in\mathcal{A}_q^k$ is asymptotic to $\dfrac{r}{q^k}\left\lfloor\dfrac{\log n}{\log q}\right\rfloor$ (which for large $n \le x$ is asymptotic to $\frac{r}{q^k}\frac{\log x}{\log q}$). By proving (5.1), we will have shown that the largest possible difference between $\omega_\gamma(P(n))$ and $\frac{r}{q^k}\frac{\log x}{\log q}$ is on average $o(\log x)$, which is sufficient to show that η_1 is a q-normal number.

Now, using the Cauchy–Schwarz inequality, the Bassily result (2.2) and finally the condition (3.1), we obtain that

$$\begin{aligned}
\sum_{n\le x} H(n) \quad \cdot \quad & \max_{\gamma\in\Gamma_k}\left|\omega_\gamma(P(n)) - \frac{r}{q^k}\frac{\log x}{\log q}\right| \le \sqrt{\sum_{n\le x} H^2(n)\cdot x\log x} \le \\
\le \quad & \sqrt{x\log x\cdot s_H(x)\cdot\Big(\sum_{n\le x} H(n)\Big)^2} = \\
= \quad & \sqrt{x\log x}\cdot\sqrt{s_H(x)}\cdot\sum_{n\le x} H(n) = \\
= \quad & o\Big((\log x)\cdot\sum_{n\le x} H(n)\Big) \quad (x\to\infty),
\end{aligned}$$

thus establishing (5.1) and completing the proof of Theorem 1. ■

6. Proof of Theorem 2

Theorem 2 will follow if we can prove that

$$\sum_{p\le x} H(p)\max_{j\in\Gamma_k}\left|\omega_\gamma(P(p)) - \frac{r}{q^k}\frac{\log x}{\log q}\right| = o\Big(\log x\cdot\sum_{p\le x} H(p)\Big) \quad (x\to\infty). \tag{6.1}$$

Proceeding esentially as in the proof of Theorem 1, one can see that (6.1) is a consequence of (3.3), from which the proof of Theorem 2 follows. ■

7. Applications

We now identify some particular types of functions $H(n)$ for which Theorems 1 and 2 apply.

7.1. Choosing $H(n)$ to be a polynomial

Let $Q(x) \in \mathbb{R}[x]$ be a polynomial of degree $d \ge 1$, and set $H(n) := \lfloor |Q(n)| \rfloor$ for each $n \in \mathbb{N}$. In this case,

$$\sum_{n\le N} H(n) \approx N^{d+1} \quad \text{and} \quad \sum_{n\le x} H^2(n) \approx \sum_{n\le x} n^{2d} \sim x^{2d+1}.$$

Hence,

$$\left(\sum_{n\le x} H(n)\right)^2 \approx \left(\sum_{n\le x} n^d\right)^2 \approx (x^{d+1})^2 = x^{2d+2},$$

from which it follows that condition (3.1) is satisfied, implying that the corresponding real number defined by (3.2) is indeed a normal number in base q.

Following essentially the same kind of reasoning, one easily sees that the conditions of Theorem 2 are also satisfied and therefore that the corresponding real number defined by (3.4) is a normal number in base q.

7.2. Choosing $H(n)$ to be a multiplicative function

Let H_0 be an arithmetic multiplicative function such that $H_0(p) = w$ for all primes p, where $1 < w < 2$, and also such that $H_0(p^r) < r + 1$ for each prime power p^r. Then, set $H(n) := \lfloor H_0(n) \rfloor$ for every $n \in \mathbb{N}$. In order to show that condition (3.1) holds, we will prove that

$$\lim_{x\to\infty} S_{H_0}(x) \cdot \frac{x}{\log x} = 0. \tag{7.1}$$

and

$$\lim_{x\to\infty} \frac{s_{H_0}(x)}{s_H(x)} = 1. \tag{7.2}$$

First observe that for $\Re s > 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_0(n)}{n^s} &= \prod_p \left(1 + \frac{w}{p^s} + \sum_{r\ge 2} \frac{H_0(p^r)}{p^{rs}}\right) = \zeta(s)^w A_1(s),\\ \sum_{n=1}^{\infty} \frac{H_0^2(n)}{n^s} &= \zeta(s)^{w^2} A_2(s),\end{aligned}$$

where $A_1(s)$ and $A_2(s)$ are two holomorphic functions bounded in the half plane $\Re s > 1 - \delta$, where $\delta > 0$ is a suitable small number, with $A_1(1) \neq 0$ and $A_2(1) \neq 0$. Then, it follows from Proposition 1 that there exist positive constants c_1 and c_2 such that, as $x \to \infty$,

$$\begin{aligned}\sum_{n\le x} H_0(n) &= (1 + o(1)) c_1 x \log^{w-1} x,\\ \sum_{n\le x} H_0^2(n) &= (1 + o(1)) c_2 x \log^{w^2-1} x,\end{aligned}$$

from which it follows that

$$\begin{aligned}s_{H_0}(x) &= (1 + o(1)) \frac{c_2 x \log^{w^2-1} x}{c_1^2 x^2 \log^{2(w-1)} x} =\\ &= (1 + o(1)) \frac{c_2}{c_1^2} \frac{\log^{(w-1)^2} x}{x} \qquad (x \to \infty).\end{aligned} \tag{7.3}$$

Since $(w-1)^2 < 1$, it is clear that (7.3) implies condition (7.1). On the other hand, since $H(n) = H_0(n) + O(1)$, it follows that

$$H^2(n) = H_0^2(n) + O(H_0(n)) + O(1),$$

which combined with (7.3) implies (7.2). From this, condition (3.1) is proved.

Hence, it follows that the conditions of Theorem 1 is satisfied, so that the corresponding real number defined by (3.2) is a normal number in base q.

Similarly, one can show that the conditions of Theorem 2 are satisfied and therefore that the corresponding real number defined by (3.4) is also a normal number in base q.

7.3. Choosing $H(n)$ as completely multiplicative functions

7.3.1. Considering sets of integers whose prime factors are all congruent to 1 mod D

Fix an integer $D \geq 3$ and consider the completely multiplicative function $H(n)$ defined on primes p by

$$H(p) := \begin{cases} 1 & \text{if } p \equiv 1 \pmod{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Proposition 4 with $f(n) = H(n)$, we find that, as $x \to \infty$,

$$\sum_{n \leq x} H(n) = \left(\frac{e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left(1 - \frac{f(p)}{p} \right)^{-1}. \tag{7.4}$$

Now, setting $\wp_0 := \{p \in \wp : f(p) = H(p) = 1\}$, we have that, in light of Proposition 2,

$$\sum_{\substack{p \leq x \\ p \in \wp_0}} 1 = \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{D}}} 1 = (1 + o(1)) \frac{1}{\phi(D)} \frac{x}{\log x}, \tag{7.5}$$

so that, as $x \to \infty$,

$$\sum_{\substack{p \leq x \\ p \in \wp_0}} \frac{1}{p} = \frac{1}{\phi(D)} \log\log x + C_1 + o(1) \tag{7.6}$$

for some constant C_1. Hence, using (7.6), we have, as $x \to \infty$,

$$\prod_{p \leq x} \left(1 - \frac{f(p)}{p} \right)^{-1} = \prod_{\substack{p \leq x \\ p \in \wp_0}} \left(1 - \frac{1}{p} \right)^{-1} = \exp\left\{ - \sum_{\substack{p \leq x \\ p \in \wp_0}} \log \left(1 - \frac{1}{p} \right) \right\} =$$

$$
\begin{aligned}
&= \exp\left\{ \sum_{\substack{p\le x\\ p\in\wp_0}} \frac{1}{p} + C_2 + o(1) \right\} = \\
&= \exp\left\{ \frac{1}{\phi(D)} \log\log x + C_3 + o(1) \right\} = \\
&= (C_4 + o(1))(\log x)^{1/\phi(D)} \qquad (7.7)
\end{aligned}
$$

for some constants C_2, C_3 and $C_4 > 0$.

Using (7.7) in (7.4), we obtain that

$$\sum_{n\le x} H(n) = \left(\frac{C_4 e^{\gamma r}}{\Gamma(r)} + o(1) \right) \frac{x}{(\log x)^{1-1/\phi(D)}} \qquad (x\to\infty),$$

from which it follows, setting $C = \dfrac{C_4 e^{\gamma r}}{\Gamma(r)}$ and using the trivial fact that in this case $H(n)^2 = H(n)$, that

$$s_H(x) = \frac{\sum_{n\le x} H^2(n)}{\left(\sum_{n\le x} H(n)\right)^2} = (1+o(1))\frac{(\log x)^{1-1/\phi(D)}}{Cx} = o\left(\frac{\log x}{x}\right) \qquad (x\to\infty),$$

implying that condition (3.1) of Theorem 1 is satisfied. On the other hand, to see that the conditions of Theorem 2 are satisfied, we first observe that $H(p)^2 = H(p)$ for all primes p and that it follows from (7.5) that

$$\sum_{p\le x} H(p) = (1+o(1))\frac{1}{\phi(D)}\frac{x}{\log x} \qquad (x\to\infty),$$

from which we have that

$$t_H(x) = \frac{1}{\sum_{p\le x} H(p)} = (1+o(1))\frac{\phi(D)\log x}{x} = o\left(\frac{\log^2 x}{x}\right) \qquad (x\to\infty),$$

as required. It is then easy to conclude that the conditions of Theorem 2 are satisfied.

The conditions of both Theorems 1 and 2 having been verified for this particular function H, we may therefore conclude that the corresponding real numbers defined by (3.2) and (3.4) are normal numbers in base q.

7.3.2. Considering sets of integers generated by particular subsets of primes

Let $\wp$ be a set of primes for which there exists a positive constant $\delta < 1$ satisfying

$$\#\{p\le x : p\in\wp\} = (\delta + o(1))\frac{x}{\log x} \qquad (x\to\infty).$$

Then, consider the completely multiplicative function $H(n)$ defined on primes p by

$$H(p) := \begin{cases} 1 & \text{if } p \in \wp, \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding in a manner similar to the one used in the preceding subsection, we obtain that

$$\sum_{n \le x} H(n) = (A + o(1)) \frac{x}{(\log x)^{1-\delta}} \qquad (x \to \infty),$$

where A is a computable constant which depends on δ.

Again, proceeding as in preceding subsection, one will reach the conclusion that with this particular function $H(n)$, the corresponding real numbers defined by (3.2) and (3.4) are normal numbers in base q.

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