Annales Univ. Sci. Budapest., Sect. Comp. 58 (2025) 47–55

A CONVEXITY-TYPE FUNCTIONAL INEQUALITY WITH INFINITE CONVEX COMBINATIONS

Mátyás Barczy (Szeged, Hungary)

Zsolt Páles (Debrecen, Hungary)

Communicated by László Szili

(Received 8 June 2025; accepted 4 August 2025)

Abstract. Given a function f defined on a nonempty and convex subset of the d-dimensional Euclidean space, we prove that if f is bounded from below and it satisfies a convexity-type functional inequality with infinite convex combinations, then f has to be convex. We also give alternative proofs of a generalization of some known results on convexity with infinite convex combinations due to Daróczy and Páles (1987) and Pavić (2019) using a probabilistic version of Jensen inequality.

1. Introduction

Studying properties of convex functions, in particular inequalities for them, has a long history, and it is still an active area of research, mainly because of their importance in many fields of mathematics. For a recent monograph on some developments in the theory of Jensen-type inequalities (such as Jensen–Steffensen, Jensen–Mercer, Jessen, McShane and Popoviciu inequalities), and for applications in information theory, see Khan et al. [4]. When convexity of a function comes into play, one usually means convexity with finite convex combinations. In this paper, we deal with a convexity-type functional inequality with infinite convex combinations, which, in the language of probability theory,

Key words and phrases: Convex function, Jensen convex function, Jensen-type functional inequality, infinite convex combination.
2010 Mathematics Subject Classification: 26A51, 26B25, 60A05.
Zsolt Páles is supported by the K-134191 NKFIH Grant.
https://doi.org/10.71352/ac.58.040825
First published online: 8 August 2025

can be rephrased as Jensen inequality for discrete random variables. This subfield of the theory of convex functions seems to be less investigated than that of convexity for finite convex combinations.

Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of positive integers, non-negative integers, real numbers and non-negative real numbers, respectively. For a real number $z \in \mathbb{R}$, its positive part $\max(z, 0)$ is denoted by z_+. For a vector $x \in \mathbb{R}^d$, its Euclidean norm is denoted by $\|x\|$. An interval $I \subseteq \mathbb{R}$ will be called nondegenerate if it contains at least two distinct points.

Definition 1.1. Let $d \in \mathbb{N}$ and $D \subseteq \mathbb{R}^d$ be a nonempty convex set.

(i) Given $t \in [0, 1]$, a function $f : D \to \mathbb{R}$ is called t-convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \qquad x, y \in D.$$

If f is $\frac{1}{2}$-convex, then it is called Jensen convex or midpoint convex as well. If f is t-convex for all $t \in (0, 1)$, then it is said to be convex. Note that every function $f : D \to \mathbb{R}$ is automatically 0-convex and 1-convex as well.

(ii) Given $t, s \in [0, 1]$, a function $f : D \to \mathbb{R}$ is called (t, s)-convex if

$$f(tx + (1-t)y) \le sf(x) + (1-s)f(y), \qquad x, y \in D.$$

The following result establishes implications among the convexity notions introduced in part (i) of Definition 1.1.

Theorem 1.1. *Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R}^d$ be a nonempty convex set and $f : D \to \mathbb{R}$. Then the following two assertions hold.*

(i) *If f is t-convex for some $t \in (0, 1)$, then it is Jensen convex.*

(ii) *If D is open as well, f is t-convex for some $t \in (0, 1)$ and f is bounded from above on some nonempty open subset of D, then f is convex and continuous.*

Proof. For the proof of part (i), see Daróczy and Páles [2, Lemma 1] or Kuhn [6]. (For completeness, we note that the compactness assumption of the domain of f in Lemma 1 in Daróczy and Páles [2] is not needed for the validity of their result, so we can indeed apply it.)

Next, we prove part (ii). Using part (i), we have that f is a Jensen convex function defined on the nonempty open and convex set D, and, by the assumption, it is bounded from above on some nonempty open subset of D. By Theorem 6.2.1 in Kuczma [5], it implies that f is locally bounded at every point of D as well. As a consequence, the celebrated Bernstein–Doetsch theorem (see Bernstein and Doetsch [1] or Kuczma [5, Theorems 6.4.2 and 7.1.1]) yields that f is convex and continuous on D, as desired. ∎

Now we recall a result due to Daróczy and Páles [2] about infinite convex combinations. Let us introduce the set

$$\Lambda := \left\{ (\lambda_n)_{n\in\mathbb{N}} : \lambda_n \geq \lambda_{n+1} > 0, n \in \mathbb{N}, \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\}.$$

Theorem 1.2. (Daróczy and Páles [2, Theorem 1].) *Let $(\lambda_n)_{n\in\mathbb{N}} \in \Lambda$, $d \in \mathbb{N}$, $D \subseteq \mathbb{R}^d$ be a compact, convex set, and $f : D \to \mathbb{R}_+$. Then the inequality*

$$f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) \leq \sum_{i=1}^{\infty} \lambda_i f(x_i)$$

holds for all $x_i \in D$, $i \in \mathbb{N}$, if and only if f is convex.

Pavić [10] recently has proven the following similar result.

Theorem 1.3. (Pavić [10, Theorem 2.1].) *Let $f : [a,b] \to \mathbb{R}$ be a convex function, where $a, b \in \mathbb{R}$ with $a < b$. Let $x_i \in [a,b]$, $i \in \mathbb{N}$, and $\lambda_i \in \mathbb{R}_+$, $i \in \mathbb{N}$, be such that $\sum_{i=1}^{\infty} \lambda_i = 1$. Then*

$$f(ta + (1-t)b) = f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) \leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \leq tf(a) + (1-t)f(b),$$

where $t \in [0,1]$ is such that $\sum_{i=1}^{\infty} \lambda_i x_i = ta + (1-t)b$.

As the main result of the paper, given a function f defined on a nonempty and convex subset of $\mathbb{R}^d$, we prove that if f is bounded from below and it satisfies a convexity-type functional inequality with infinite convex combinations, then f has to be convex, see Theorem 3.2. We also give alternative proofs for generalizations of some parts of Theorems 1.2 and 1.3 due to Daróczy and Páles [2] and Pavić [10], respectively, see Proposition 3.1. In the proof of Theorem 3.2, results on (s,t)-convexity, while in the proof of Proposition 3.1, a probabilistic version Jensen inequality (see Section 2) play a crucial role. In Remark 3.1, we give an example which shows that a converse of Theorem 3.2 does not hold in general.

2. A probabilistic version of the Jensen inequality

In this section, we recall a probabilistic version of the Jensen inequality (see, e.g., Dudley [3, 10.2.6, page 348]), which will be used for giving alternative proofs for generalizations of some parts of Theorems 1.2 and 1.3.

Proposition 2.1. (Jensen inequality.) *Let $\xi : \Omega \to \mathbb{R}^d$ be a random vector such that $\mathbb{E}(\|\xi\|) < \infty$, and let $D \subseteq \mathbb{R}^d$ be a Borel measurable, convex set such that $\mathbb{P}(\xi \in D) = 1$. Then the following statements hold.*

(i) *We have* $\mathbb{E}(\xi) \in D$.

(ii) *For all continuous and convex functions* $f : D \to \mathbb{R}$*, we have that* $\mathbb{E}(f(\xi)) \in (-\infty, \infty]$ *and* $f(\mathbb{E}(\xi)) \leq \mathbb{E}(f(\xi))$.

(iii) *Supposing, in addition, that* ξ *is discrete as well, for all convex functions* $f : D \to \mathbb{R}$*, we have that* $\mathbb{E}(f(\xi)) \in (-\infty, \infty]$ *and* $f(\mathbb{E}(\xi)) \leq \mathbb{E}(f(\xi))$.

Proof. We plan to apply Dudley [3, 10.2.6, page 348] to derive the assertions. First note that, for all continuous functions $f : D \to \mathbb{R}$, we have that $f(\xi)$ is a random variable. Further, if ξ is discrete as well, then, for all functions $f : D \to \mathbb{R}$, we have that $f(\xi)$ is a random variable. Indeed, if $\{x_i : i \in \mathbb{N}\}$ denotes the range of ξ, then the range of $f(\xi)$ is $\{f(x_i) : i \in \mathbb{N}\}$ is also countable, and, for each $j \in \mathbb{N}$, we get that

$$\{f(\xi) = f(x_j)\} = \bigcup_{\{i \in \mathbb{N} : f(x_i) = f(x_j)\}} \{\xi = x_i\},$$

which is an event, since $\{\xi = x_i\}$ is an event for each $i \in \mathbb{N}$ and a σ-algebra is closed under countable union. Consequently, if $\xi(\omega) \in D$ for all $\omega \in \Omega$, then (i)–(iii) directly follow from Dudley [3, 10.2.6, page 348].

In the more general case when $\mathbb{P}(\xi \in D) = 1$ holds, let us introduce the mapping $\widetilde{\xi} : \Omega \to \mathbb{R}^d$,

$$\widetilde{\xi}(\omega) := \big(\xi \mathbf{1}_{\{\xi \in D\}} + d_0 \mathbf{1}_{\{\xi \notin D\}}\big)(\omega) = \begin{cases} \xi(\omega) & \text{if } \omega \in \Omega \text{ is such that } \xi(\omega) \in D, \\ d_0 & \text{otherwise,} \end{cases}$$

where d_0 is arbitrarily fixed element of D. Then $\widetilde{\xi}(\omega) \in D$ for all $\omega \in \Omega$, and, since $\mathbb{P}(\xi \notin D) = 0$, we have

$$\begin{aligned} \mathbb{E}(\|\widetilde{\xi}\|) &= \mathbb{E}\big(\|\widetilde{\xi}\| \mathbf{1}_{\{\xi \in D\}}\big) + \mathbb{E}\big(\|\widetilde{\xi}\| \mathbf{1}_{\{\xi \notin D\}}\big) = \\ &= \mathbb{E}\big(\|\xi\| \mathbf{1}_{\{\xi \in D\}}\big) + \mathbb{E}\big(\|d_0\| \mathbf{1}_{\{\xi \notin D\}}\big) = \\ &= \mathbb{E}(\|\xi\|) - \mathbb{E}\big(\|\xi\| \mathbf{1}_{\{\xi \notin D\}}\big) + \|d_0\| \mathbb{P}(\xi \notin D) = \mathbb{E}(\|\xi\|) < \infty. \end{aligned}$$

Using again Dudley [3, 10.2.6, page 348], we obtain that $\mathbb{E}(\widetilde{\xi}) \in D$, $\mathbb{E}\big(f(\widetilde{\xi})\big) \in$ $\in (-\infty, \infty]$ and $f\big(\mathbb{E}(\widetilde{\xi})\big) \leq \mathbb{E}\big(f(\widetilde{\xi})\big)$. Since $\mathbb{P}(\xi \notin D) = 0$, we get that

$$\begin{aligned} \mathbb{E}(\widetilde{\xi}) &= \mathbb{E}\big(\widetilde{\xi} \mathbf{1}_{\{\xi \in D\}}\big) + \mathbb{E}\big(\widetilde{\xi} \mathbf{1}_{\{\xi \notin D\}}\big) = \mathbb{E}\big(\xi \mathbf{1}_{\{\xi \in D\}}\big) + \mathbb{E}\big(d_0 \mathbf{1}_{\{\xi \notin D\}}\big) = \\ &= \mathbb{E}(\xi) - \mathbb{E}\big(\xi \mathbf{1}_{\{\xi \notin D\}}\big) + d_0 \mathbb{P}(\xi \notin D) = \mathbb{E}(\xi), \end{aligned}$$

and therefore $\mathbb{E}(\xi) \in D$ holds as well. Similarly, one can check that $\mathbb{E}\big(f(\widetilde{\xi})\big) =$ $= \mathbb{E}\big(f(\xi)\big)$, and consequently (i)–(iii) also follow in the case $\mathbb{P}(\xi \in D) = 1$. ■

Note that if $d > 1$ and $D \subseteq \mathbb{R}^d$ is a convex set, then D may be not Borel measurable. Therefore, the assumption about the Borel measurability of D in the above proposition is indispensable. Namely, the union of an open ball in $\mathbb{R}^d$ with any non Borel measurable subset of its boundary provides an example of a non Borel measurable convex set of $\mathbb{R}^d$, cf. Lang [8].

3. Main results

First, we give alternative proofs for a generalized form of the ‘only if’ part of Theorem 1.2 and the first inequality of Theorem 1.3 using a probabilistic version of Jensen inequality (see part (iii) of Proposition 2.1).

Proposition 3.1. *Let $\lambda_i \in \mathbb{R}_+$, $i \in \mathbb{N}$, be such that $\sum_{i=1}^\infty \lambda_i = 1$. Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R}^d$ be a convex set, and let $x_i \in D$, $i \in \mathbb{N}$, be such that $\sum_{i=1}^\infty \lambda_i \|x_i\| < \infty$. Then, for any convex function $f : D \to \mathbb{R}$, we have that*

$$f\left(\sum_{i=1}^\infty \lambda_i x_i\right) \leq \sum_{i=1}^\infty \lambda_i f(x_i).$$

Proof. Let ξ be a discrete random variable with range $\{x_i : i \in \mathbb{N}\}$ and distribution $\mathbb{P}(\xi = x_i) = \lambda_i$, $i \in \mathbb{N}$. Since $\lambda_i \in \mathbb{R}_+$, $i \in \mathbb{N}$, and $\sum_{i=1}^\infty \lambda_i = 1$, the distribution of ξ is indeed well-defined. Then, by the assumption, we have that

$$\mathbb{E}(\|\xi\|) = \sum_{i=1}^\infty \lambda_i \|x_i\| < \infty.$$

Consequently, by part (i) of Proposition 2.1, we get $\mathbb{E}(\xi) \in D$, i.e.,

$$\sum_{i=1}^\infty \lambda_i x_i = \sum_{i=1}^\infty x_i \mathbb{P}(\xi = x_i) = \mathbb{E}(\xi) \in D.$$

Furthermore, by part (iii) of Proposition 2.1, for any convex function $f : D \to \mathbb{R}$, we have that $f(\mathbb{E}(\xi)) \leq \mathbb{E}(f(\xi))$, i.e.,

$$f\left(\sum_{i=1}^\infty \lambda_i x_i\right) = f(\mathbb{E}(\xi)) \leq \mathbb{E}(f(\xi)) = \sum_{i=1}^\infty f(x_i)\mathbb{P}(\xi = x_i) = \sum_{i=1}^\infty \lambda_i f(x_i).$$

This completes the proof of the statement. ■

Next, we present a generalization of the ‘if part‘ of Theorem 1.2.

Theorem 3.2. *Let $(\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}} \in \Lambda$, let $d \in \mathbb{N}$, $D \subseteq \mathbb{R}^d$ be nonempty and convex, and let $f : D \to \mathbb{R}$ be bounded from below. If the inequality*

$$f\left(\sum_{i=1}^\infty \lambda_i x_i\right) \leq \sum_{i=1}^\infty \mu_i f(x_i) \tag{3.1}$$

holds for all bounded sequences $x_i \in D$, $i \in \mathbb{N}$, then f is convex. Here the right hand side of (3.1) is either convergent, or else, it diverges to the extended real number $+\infty$.

The inequality (3.1) can be reformulated as

$$f(\mathbb{E}(\xi)) \leq \mathbb{E}(f(\eta)),$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, $\xi : \Omega \to \mathbb{R}^d$ and $\eta : \Omega \to \mathbb{R}^d$ are random vectors with common ranges $\{x_i : i \in \mathbb{N}\}$ and with possibly different distributions $\mathbb{P}(\xi = x_i) = \lambda_i$, $i \in \mathbb{N}$, and $\mathbb{P}(\eta = x_i) = \mu_i$, $i \in \mathbb{N}$, respectively.

Proof of Theorem 3.2. First, we check that the left hand side of (3.1) is well-defined for all bounded sequences $x_i \in D$, $i \in \mathbb{N}$, that is, $\sum_{i=1}^{\infty} \lambda_i x_i$ belongs to D. Indeed, one can interpret the sum $\sum_{i=1}^{\infty} \lambda_i x_i$ as the expectation of a discrete random vector ξ with range $\{x_i : i \in \mathbb{N}\}$ and with distribution $\mathbb{P}(\xi = x_i) = \lambda_i$, $i \in \mathbb{N}$. Since $(x_i)_{i\in\mathbb{N}}$ is bounded, there exists $K > 0$ such that $\|x_i\| \leq K$, $i \in \mathbb{N}$, yielding that

$$\sum_{i=1}^{\infty} \|\lambda_i x_i\| = \sum_{i=1}^{\infty} \lambda_i \|x_i\| \leq K \sum_{i=1}^{\infty} \lambda_i = K < \infty.$$

As a consequence, we have that ξ is integrable, and, by part (i) of Proposition 2.1, we get that $\sum_{i=1}^{\infty} \lambda_i x_i = \mathbb{E}(\xi) \in D$.

Next, we show that the right hand side of (3.1) is either convergent, or else, it diverges to the extended real number $+\infty$. Assume that B is a lower bound for f. Then $f - B \geq 0$, and hence the series $\sum_{i=1}^{\infty} \mu_i (f(x_i) - B)$ is either convergent, or else, it diverges to the extended real number $+\infty$. Therefore, observing that

$$\sum_{i=1}^{\infty} \mu_i f(x_i) = \sum_{i=1}^{\infty} \mu_i (f(x_i) - B) + \sum_{i=1}^{\infty} (\mu_i B) = \sum_{i=1}^{\infty} \mu_i (f(x_i) - B) + B,$$

we can conclude that the series $\sum_{i=1}^{\infty} \mu_i f(x_i)$ is also either convergent, or else, it diverges to the extended real number $+\infty$.

Now, suppose that (3.1) holds for all bounded sequences $x_i \in D$, $i \in \mathbb{N}$. Let $x, y \in D$ be fixed. By choosing $x_1 := x$ and $x_i := y$, $i \in \mathbb{N} \setminus \{1\}$, in (3.1), we have that

$$\begin{aligned} f(\lambda_1 x + (1 - \lambda_1) y) &= f\left(\lambda_1 x + \left(\sum_{i=2}^{\infty} \lambda_i\right) y\right) \leq \\ &\leq \mu_1 f(x) + \left(\sum_{i=2}^{\infty} \mu_i\right) f(y) = \\ &= \mu_1 f(x) + (1 - \mu_1) f(y). \end{aligned}$$

Since $\lambda_i > 0$ and $\mu_i > 0$ for each $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \mu_i = 1$, we have that $\lambda_1, \mu_1 \in (0, 1)$. Consequently, we get that f is (λ_1, μ_1)-convex on D. By Fact on page 135 in Matkowski and Pycia [9], we get that f is Jensen convex on D. Here, for historical fidelity, we mention that this result is due to Kuhn [7] and, then, using the idea in the proof of Lemma 1 in Daróczy and Páles [2], later Matkowski and Pycia [9] gave an elementary proof of the result of Kuhn [7]. This together with Theorem 5.3.5 in Kuczma [5] imply that f is q-convex for all $q \in \mathbb{Q} \cap [0, 1]$. (For completeness, we note that the openness of D in Theorem 5.3.5 in Kuczma [5] is assumed, but, in fact, it is not needed for the validity of the result.)

By Lemma 2 in Daróczy and Páles [2], for any $t \in [0, 1]$, there exist $q_i \in \mathbb{Q} \cap [0, 1]$, $i \in \mathbb{N}$, such that $t = \sum_{i=1}^{\infty} \lambda_i q_i$, and hence

$$1 - t = \sum_{i=1}^{\infty} \lambda_i - \sum_{i=1}^{\infty} \lambda_i q_i = \sum_{i=1}^{\infty} \lambda_i (1 - q_i).$$

Therefore, for any $x, y \in D$ and $t \in [0, 1]$, we get that

$$\begin{aligned} f(tx + (1-t)y) &= f\left(\left(\sum_{i=1}^{\infty} \lambda_i q_i \right) x + \left(\sum_{i=1}^{\infty} \lambda_i (1 - q_i) \right) y \right) = \\ &= f\left(\sum_{i=1}^{\infty} \lambda_i (q_i x + (1 - q_i) y) \right). \end{aligned}$$

Using (3.1) for the bounded sequence $q_i x + (1 - q_i) y \in D$, $i \in \mathbb{N}$, and that f is q_i-convex for each $i \in \mathbb{N}$, we have that

$$\begin{aligned} f(tx + (1-t)y) &\le \sum_{i=1}^{\infty} \mu_i f(q_i x + (1 - q_i) y) \le \\ &\le \sum_{i=1}^{\infty} \mu_i (q_i f(x) + (1 - q_i) f(y)) = \\ &= \left(\sum_{i=1}^{\infty} \mu_i q_i \right) f(x) + \left(\sum_{i=1}^{\infty} \mu_i (1 - q_i) \right) f(y) \le \\ &\le \left(\sum_{i=1}^{\infty} \mu_i q_i \right) (f(x))_+ + \left(\sum_{i=1}^{\infty} \mu_i (1 - q_i) \right) (f(y))_+ \le \\ &\le \left(\sum_{i=1}^{\infty} \mu_i \right) (f(x))_+ + \left(\sum_{i=1}^{\infty} \mu_i \right) (f(y))_+ = \\ &= (f(x))_+ + (f(y))_+. \end{aligned}$$

This proves that f is bounded from above on the segment $[x, y] := \{rx + +(1 - r)y : r \in [0, 1]\}$ connecting x and y. All in all, $f : D \to \mathbb{R}$ is a Jensen convex function defined on a non-empty and convex set $D \subseteq \mathbb{R}^d$ such that, for all $x, y \in D$, it is bounded from above on the segment $[x, y]$. To finish the proof, for any $x, y \in D$ with $x \neq y$, let us apply the Bernstein–Doetsch theorem (see

Bernstein and Doetsch [1] or Kuczma [5, Theorem 6.4.2]) to the convex and open set $(x, y) := \{rx + (1-r)y : r \in (0,1)\}$ and the Jensen convex function $f_{x,y} : (x, y) \to \mathbb{R}$, $f_{x,y}(v) := f(v)$, $v \in (x, y)$, which is bounded from above. Then, for all $x, y \in D$ with $x \neq y$, we have that $f_{x,y}$ is continuous and convex on (x, y). Since $f_{x,y}$ is a restriction of f onto (x, y), it readily yields that f is convex on D, as desired. ■

In the next remark, we provide an example, which shows that a converse of Theorem 3.2 does not hold in general.

Remark 3.1. Let $(\lambda_n)_{n\in\mathbb{N}} \in \Lambda$ and $(\mu_n)_{n\in\mathbb{N}} \in \Lambda$ be such that $(\mathrm{e}^{\lambda_1} - 1)/ /(\mathrm{e} - 1) > \mu_1$. Let $d := 1$, $D := \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \mathrm{e}^x$, $x \in \mathbb{R}$. Then f is convex, and, with $x_1 := 1$ and $x_i := 0$, $i \in \mathbb{N} \setminus \{1\}$, we get that

$$f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) = f(\lambda_1) = \mathrm{e}^{\lambda_1},$$

and

$$\sum_{i=1}^{\infty} \mu_i f(x_i) = \mu_1 f(1) + (1-\mu_1) f(0) = \mu_1 \mathrm{e} + 1 - \mu_1 = 1 + \mu_1(\mathrm{e} - 1).$$

Since $(\mathrm{e}^{\lambda_1} - 1)/(\mathrm{e} - 1) > \mu_1$, we have that the inequality (3.1) does not hold.

References

[1] **Bernstein, F. and G. Doetsch,** Zur Theorie der konvexen Funktionen, *Math. Ann.*, **76(4)** (1915), 514–526.
https://doi.org/10.1007/BF01458222

[2] **Daróczy, Z. and Zs. Páles,** Convexity with given infinite weight sequences, *Stochastica*, **11(1)** (1987), 5–12.

[3] **Dudley, R.M.,** *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics **74**, Cambridge University Press, Cambridge, 2002.
https://doi.org/10.1017/CBO9780511755347

[4] **Khan, M.A., K.A. Khan, D. Pečarić and J. Pečarić,** *Some new improvements of Jensen's inequality—Jensen's type inequalities in information theory*, *Monographs in Inequalities* **19**, ELEMENT, Zagreb, 2020.

[5] **Kuczma, M.,** *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's equation and Jensen's inequality*, Birkhäuser Verlag, Basel, 2nd edition, 2009.
https://doi.org/10.1007/978-3-7643-8749-5

[6] **Kuhn, N.,** A note on t-convex functions, in: W. Walter (Ed.) *General Inequalities, 4* (Oberwolfach, 1983), International Series of Numerical Mathematics **71**, Birkhäuser Verlag, Basel, 1984, 269–276.
https://doi.org/10.1007/978-3-0348-6259-2_25

[7] **Kuhn, N.,** On the structure of (s, t)-convex functions, in: W. Walter (Ed.) *General Inequalities, 5* (Oberwolfach, 1986), International Series of Numerical Mathematics **80**, Birkhäuser Verlag, Basel, 1987, 161–174.
https://doi.org/10.1007/978-3-0348-7192-1_13

[8] **Lang, R.,** A note on the measurability of convex sets. *Arch. Math.* (*Basel*), **47(1)** (1986), 90–92.
https://doi.org/10.1007/BF01202504

[9] **Matkowski, J. and M. Pycia,** On (α, a)-convex functions, *Arch. Math.* (*Basel*), **64(2)** (1995), 132–138.
https://doi.org/10.1007/BF01196632

[10] **Pavić, Z.,** Inequalities with infinite convex combinations, *Turkish J. Ineq.*, **3(2)** (2019), 53–61.
https://www.tjinequality.com/articles/03-02-006.pdf

Mátyás Barczy
https://orcid.org/0000-0003-3119-7953
HUN-REN–SZTE Analysis and Applications Research Group
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
H–6720 Szeged
Hungary
barczy@math.u-szeged.hu

Zsolt Páles
https://orcid.org/0000-0003-2382-6035
Institute of Mathematics
University of Debrecen
Egyetem tér 1
H–4032 Debrecen
Hungary
pales@science.unideb.hu