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AVERAGE OF CERTAIN MULTIPLICATIVE FUNCTIONS OVER INTEGERS WITHOUT SMALL AND LARGE PRIME FACTORS

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Abstract. Song (2001) described the asymptotic behaviour of sums including certain nonnegative multiplicative functions, where the summation runs over the integers without large prime factors. We generalise the method to restrict the summation to integers without small and large prime factors.

1. Introduction

1.1. Problem statement

Let $1 \le a < b$ be positive integers. Introduce

$$S_{a,\text{-}}(x) := \{n \le x : a < P^{-}(n)\}$$

where $P^{-}(n)$ denotes the smallest prime factor of n, with the convention that $P^{-}(1) = +\infty$; furthermore let

$$S_{\text{-},b}(x) := \{n \le x : P^{+}(n) \le b\}$$

where $P^{+}(n)$ denotes the largest prime factor of n, with the convention that $P^{+}(1) = 1$. These sets form the base of the study in section III.5 and section III.6 respectively of [17]. Denote the intersection of these sets as

$$S_{a,b}(x) := \{n \le x : a < P^{-}(n) \wedge P^{+}(n) \le b\}$$

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and for convenience, introduce $S_{a,b} := \{n \in \mathbb{N} : a < P^-(n) \wedge P^+(n) \leq b\}$ furthermore let $S_{a,b}(x_1, x_2) := S_{a,b}(x_2) \setminus S_{a,b}(x_1)$.

We are going to work with nonnegative multiplicative functions satisfying the two following conditions. Denote such a function as h.

Condition 1.1. *There exists constants* $0 < \delta < 1$ *and* $\kappa > 0$ *such that*

$$\sum_{p \leq y} \frac{h(p)}{p} \ln p = \kappa \ln y + O((\ln y)^{1-\delta})$$

for $y \geq 1$.

Condition 1.2. *There exists a constant* $C > 0$ *such that*

$$\sum_{p, k \geq 2} \frac{h(p^k)}{p^k} \ln p^k \leq C.$$

These conditions pose only a mild average assumption upon the values of h, as it was noted by Tenenbaum [16]. The main theme of this paper is the study of the asymptotic behaviour of the sum

$$m_{a,b}(x) := \sum_{n \in S_{a,b}(x)} \frac{h(n)}{n}$$

where h is a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Analogously we can define $m_{a,\_}(x)$ and $m_{\_,b}(x)$ as $m_{a,b}(x)$, but where the summation runs through $S_{a,\_}(x)$ and $S_{\_,b}(x)$ respectively.

1.2. Past results

To prove our results, we are going to adapt the method of Song, see [14].

There has been a great interest in summations over $S_{\_,b}(x)$, see [4, 5, 8, 9, 18, 19]. Tenenbaum shown a more precise result in a more concise way in his article [16], where he relied on an already established result for the function of Plitz τ_κ. Generalising his method for our case could be a next step in the investigation. (See also [7, 13] concerning sums involving the function of Plitz.)

The extension of the results of Song from his article [15] to handle sums of the form

$$M_{a,b}(x) := \sum_{n \in S_{a,b}(x)} h(n)$$

would be another possible way of investigation.

On the other hand, results concerning summations over $S_{a,\_}(x)$ seem to be scarce. It worth mentioning the results [3, 21] concerning the "complementer" problem, that is where the integers in the sum shouldn't have prime factors from an interval $(a, b]$.

1.3. The differential difference equation

All the following statements are just reiterated from the article of Song [14] for clarity. Define j_κ for a real parameter κ to be the continuous solution of

$$vj_\kappa'(v) = \kappa j_\kappa(v) - \kappa j_\kappa(v-1)$$

for $v > 1$, $j_\kappa(v) = B_\kappa v^\kappa$ for $0 < v \leq 1$, otherwise $j_\kappa(v) = 0$ for $v \leq 0$. Here $B_\kappa := e^{-\gamma\kappa}/\Gamma(\kappa+1)$, with γ being Euler's constant, and Γ being Euler's Gamma function. Note that Dickman's function satisfies $\varrho(v) = e^\gamma j_1'(v)$. For $\kappa \geq 0$ we have that $j_\kappa(v)$ is strictly increasing and positive for all $v > 0$, converges to 1 as $v \to \infty$, furthermore $j_\kappa(\kappa) \geq 1/2$. As

$$(vj_\kappa(v))' = \kappa(j_\kappa(v) - j_\kappa(v-1)) + j_\kappa(v)$$

we have

$$j_\kappa(v) = \frac{\kappa}{v}\int_{v-1}^{v} j_\kappa(t)\,dt + \frac{1}{v}\int_{0}^{v} j_\kappa(t)\,dt = \tag{1.1}$$

$$= \frac{\kappa}{v}\int_{v-1}^{v} j_\kappa(t)\,dt + \frac{1}{v}\int_{1}^{v} j_\kappa(t)\,dt + \frac{B_\kappa}{(\kappa+1)v}$$

for $v \geq 1$. We also define

$$J_\kappa(v) := \int_0^v j_\kappa(t)\,dt.$$

Based on the definition of j_κ we have

$$J_\kappa(v) = \frac{1}{\kappa+1} B_\kappa v^{\kappa+1} \tag{1.2}$$

when $0 < v \leq 1$, and

$$vj_\kappa(v) = vJ_\kappa'(v) = (\kappa+1)J_\kappa(v) - \kappa J_\kappa(v-1) \tag{1.3}$$

when $v > 1$. In particular, we have

$$\frac{J_\kappa(v_2)}{v_2^{\kappa+1}} = \frac{J_\kappa(v_1)}{v_1^{\kappa+1}} - \kappa\int_{v_1}^{v_2} \frac{J_\kappa(t-1)}{t^{\kappa+2}}\,dt. \tag{1.4}$$

1.4. New results

First we are going to prove the following proposition.

Proposition 1.3. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then for $a \geq 1$ we have*

$$m_{a,\_}(x) = C_\kappa(a)(\ln xa^\kappa)^\kappa + O((\ln x)^{\kappa-\delta}), \tag{1.5}$$

where

$$C_\kappa(a) = \frac{1}{\Gamma(\kappa+1)} \lim_{s\to 1+0} \prod_{a<p}\left(1+\sum_{k=1}^{\infty}\frac{h(p^k)}{p^{ks}}\right)\prod_p\left(1-\frac{1}{p^s}\right)^\kappa. \tag{1.6}$$

Then by using Proposition 1.3, we are going to prove the following proposition.

Proposition 1.4. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then for $1 \leq a < b$ with b being sufficiently large we have*

$$m_{a,b}(x) = V_{a,b}\left(j_\kappa(\log_b(xa^\kappa)) + O\left(\frac{\log_b(xa^\kappa)^{2\kappa}\ln(\log_b(xa^\kappa)+1)}{(\ln b)^\delta}\right)\right) \tag{1.7}$$

when $\log_b(xa^\kappa) > 1$, where

$$V_{a,b} := \prod_{a<p\leq b}\left(1+\sum_{k=1}^{\infty}\frac{h(p^k)}{p^k}\right).$$

Finally, we prove the main result by building upon Proposition 1.4.

Theorem 1.5. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then for $1 \leq a < b$ with b being sufficiently large we have*

$$m_{a,b}(x) = V_{a,b}\left(j_\kappa(\log_b(xa^\kappa)) + O\left(\frac{\ln(\log_b(xa^\kappa)+1)}{(\ln b)^\delta}\right)\right)$$

uniformly for

$$1 \leq \log_b(xa^\kappa) \leq \exp\left(\frac{1}{c}(\ln b)^\delta\right) \tag{1.8}$$

for a suitable constant c.

Take note that when we set a as 1, we get back the theorems of [14].

1.5. Application

- One example application can arise during the utilisation of the combinatorial sieve, see section I.4 of [17]. There, one has to compute sums of the form
$$\sum_{\substack{d\le y^u\\ d\mid P(y)}} |R_d|,$$
where $u \ge 1$, R_d is an error term depending upon d, and
$$P(y) := \prod_{\substack{p\in\mathcal{P}\\ p\le y}} p$$
with $\mathcal{P}$ being a set of prime numbers. Now if $\mathcal{P}$ contains all the primes between $a < b$, where $b \le y$, then we can replace this sum with
$$\sum_{d\in S_{a,b}(y^u)} \mu^2(d)|R_d|$$
where μ is the Möbius function. Assuming that we can express $|R_d|$ in a way that we get the sum
$$\sum_{d\in S_{a,b}(y^u)} \frac{\mu^2(d)}{d}$$
we can apply our theorems. Indeed, $h := \mu^2$ is a nonnegative multiplicative function; satisfies Condition 1.1 with $\kappa = 1$ and $\delta \in (0,1)$ based on Mertens' first theorem, see [10]; and also satisfies Condition 1.2 with an arbitrary $C > 0$. Then
$$V_{a,b} = \prod_{a<p\le b}\left(1+\frac{1}{p}\right) \sim \frac{\ln b}{\ln a}$$
and
$$j_1(x) = \int_0^x j_1'(v)\,dv = e^{-\gamma}\int_0^x \varrho(v)\,dv = 1 + O(e^{-x-\gamma}) \tag{1.9}$$
as $x \to \infty$, see articles [6, 20].
- Having an h nonnegative multiplicative function satisfying Conditions 1.1 and 1.2, we can study the growth of the Dirichlet series
$$\sum_{n\ge 1} \frac{h(n)\mathbf{1}_{S_{a,b}}(n)}{n^s}$$

at $s = 1$, where $\mathbf{1}_X$ is the characteristic function of set X. For example when $h := 1$, then Condition 1.1 is satisfied with $\kappa = 1$ and $\delta \in (0, 1)$ as before, and there exists $C > 0$ which satisfies Condition 1.2. Then

$$V_{a,b} = \prod_{a<p\le b} \frac{p}{p-1} \sim \frac{\ln b}{\ln a}$$

based on Theorem 8 and its corollary in article [12], and by using (1.9) and Theorem 1.5 we have

$$\sum_{n\le x} \frac{h(n)\mathbf{1}_{S_{a,b}}(n)}{n} = V_{a,b}\left(1 + O\left(\frac{\ln(\log_b(xa) + 1)}{(\ln b)^\delta}\right)\right)$$

for x satisfying (1.8).

2. Preliminary lemmas

Lemma 2.1. *Let h be a nonnegative multiplicative function satisfying Condition 1.1. Then we have*

$$h(p) \ll \frac{p}{(\ln p)^\delta} \tag{2.1}$$

and

$$\sum_{p\le y} \frac{h(p)}{p}(\ln p)^{1-\delta} \ll (\ln y)^{1-\delta} \tag{2.2}$$

for $y \ge 1$.

Proof. Using Condition 1.1 we have

$$\frac{h(p)}{p}\ln p = \sum_{q\le p} \frac{h(q)}{q}\ln q - \sum_{q\le p-1} \frac{h(q)}{q}\ln q = \kappa \ln \frac{p}{p-1} + O((\ln p)^{1-\delta})$$

from where we get (2.1) by rearrangement.

As $f(t) := (\ln t)^{-\delta}$ has a continuous derivative on $[2, y]$, we can use Abel's identity to get that the sum on the left hand side of (2.2) is equal to

$$\frac{1}{(\ln y)^\delta} \sum_{p\le y} \frac{h(p)}{p} \ln p + \delta \int_2^y \frac{1}{t(\ln t)^{1+\delta}} \sum_{p\le t} \frac{h(p)}{p} \ln p \, dt + O(1)$$

see theorem 4.2 of [2]. By using Condition 1.1, we get that the first sum is in $O((\ln y)^{1-\delta})$, and the integral is

$$\delta\kappa \int_2^y \frac{1}{t(\ln t)^{\delta}}\,dt + O\Big(\int_2^y \frac{1}{t(\ln t)^{2\delta}}\,dt\Big) \ll (\ln y)^{1-\delta}$$

from where we get the right hand side of (2.2). ∎

Lemma 2.2. *Let h be a nonnegative multiplicative function. Then we have*

$$m_{a,b}(x)\ln x = T_{a,b}(x) + \sum_{\substack{mp^k\in S_{a,b}(x)\\ p\nmid m}} \frac{h(m)}{m}\frac{h(p^k)}{p^k}\ln p^k \tag{2.3}$$

for $1 \le a < b \le x$, where

$$T_{a,b}(x) := \int_1^x \frac{m_{a,b}(t)}{t}\,dt.$$

Proof. As $f(t) := \ln(x/t)$ has a continuous derivative on $[1,x]$, we can use Abel's identity to get

$$\sum_{1<n\le x}\frac{h(n)}{n}\mathbf{1}_{S_{a,b}(x)}(n)\ln\frac{x}{n} = -(\ln x)h(1) + \int_1^x \frac{1}{t}\sum_{n\le t}\frac{h(n)}{n}\mathbf{1}_{S_{a,b}(x)}(n)\,dt$$

see theorem 4.2 of [2], which can be rewritten as

$$\sum_{n\in S_{a,b}(x)}\frac{h(n)}{n}\ln\frac{x}{n} = \int_1^x \frac{m_{a,b}(t)}{t}\,dt.$$

We can reorder this as

$$m_{a,b}(x)\ln x = T_{a,b}(x) + \sum_{n\in S_{a,b}(x)}\frac{h(n)}{n}\ln n$$

due to the finiteness of the sum. The sum on the right hand side can be written as

$$\sum_{n\in S_{a,b}(x)}\frac{h(n)}{n}\sum_{p^k|n}\ln p = \sum_{p^k\le x}\ln p\sum_{\substack{n\in S_{a,b}(x)\\ p^k|n}}\frac{h(n)}{n} = \sum_{p^k\le x}\ln p\sum_{\substack{mp^k\in S_{a,b}(x)\\ p\nmid m}}\frac{h(mp^k)}{mp^k}$$

from where we get our statement due to the multiplicative nature of h. ∎

Lemma 2.3. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then we have*

$$m_{a,b}(x)\ln x = T_{a,b}(x) + \sum_{m\in S_{a,b}(x)} \frac{h(m)}{m} \sum_{a<p\le \min(x/m,b)} \frac{h(p)}{p}\ln p + \qquad (2.4)$$
$$+ O(m_{a,b}(x)(\ln b)^{1-\delta})$$

for $1 \le a < b \le x$.

Proof. We are going to derive (2.4) by examining the sum in (2.3) for different values of k.

- First we look at the case when $k = 1$. We can split the sum as

$$\sum_{mp\in S_{a,b}(x)} \frac{h(m)}{m}\frac{h(p)}{p}\ln p - \sum_{\substack{mp\in S_{a,b}(x)\\ p|m}} \frac{h(m)}{m}\frac{h(p)}{p}\ln p$$

which can be rewritten as

$$\sum_{m\in S_{a,b}(x)} \frac{h(m)}{m} \sum_{a<p\le \min(x/m,b)} \frac{h(p)}{p}\ln p - \sum_{\substack{mp^{l+1}\in S_{a,b}(x)\\ p\nmid m\\ l\ge 1}} \frac{h(mp^l)}{mp^l}\frac{h(p)}{p}\ln p.$$

The sum on the right hand side is less than or equal to

$$\sum_{\substack{p^{l+1}\in S_{a,b}(x)\\ l\ge 1}} m_{a,\_}(x/p^{l+1})\frac{h(p^l)}{p^l}\frac{h(p)}{p}\ln p$$

which is less than or equal to

$$m_{a,b}(x)\sum_{\substack{p^{l+1}\in S_{a,b}(x)\\ l\ge 1}} \frac{h(p^l)}{p^l}\frac{h(p)}{p}\ln p \ll m_{a,b}(x)\sum_{\substack{p^{l+1}\in S_{a,b}(x)\\ l\ge 1}} \frac{h(p^l)}{p^l}(\ln p)^{1-\delta}$$

using (2.1).

  – When $l = 1$ in the sum on the right hand side, then we have

$$\sum_{p^2\in S_{a,b}(x)} \frac{h(p)}{p}(\ln p)^{1-\delta} = \sum_{a<p\le b} \frac{h(p)}{p}(\ln p)^{1-\delta} \ll (\ln b)^{1-\delta}$$

  by (2.2).

– When $l > 1$, then the sum can be bounded by

$$\sum_{\substack{p,l\geq 2\\ a<p}} \frac{h(p^l)}{p^l}(\ln p)^{1-\delta} \leq C$$

using Condition 1.2.

- For the case when $k > 1$, the sum can be bounded by

$$\sum_{p,k\geq 2} m_{a,b}(x/p^k)\frac{h(p^k)}{p^k}\ln p^k \leq m_{a,b}(x)\sum_{p,k\geq 2}\frac{h(p^k)}{p^k}\ln p^k \leq Cm_{a,b}(x)$$

using Condition 1.2. ■

Lemma 2.4. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then we have*

$$m_{a,b}(x)\ln xa^{\kappa} = (\kappa+1)T_{a,b}(x) - \kappa T_{a,b}(x/b) + O(V_{a,b}(\ln b)^{1-\delta}) \tag{2.5}$$

for $1 \leq a < b \leq x$.

Proof. The double sum in (2.4) can be split as

$$\sum_{m\in S_{a,b}(x/b)}\frac{h(m)}{m}\sum_{a<p\leq b}\frac{h(p)}{p}\ln p + \sum_{m\in S_{a,b}(x/b,x)}\frac{h(m)}{m}\sum_{a<p\leq x/m}\frac{h(p)}{p}\ln p. \tag{2.6}$$

Based on Condition 1.1, we have

$$\sum_{a<p\leq b}\frac{h(p)}{p}\ln p = \kappa\ln b - \kappa\ln a + O((\ln b)^{1-\delta})$$

and

$$\sum_{a<p\leq x/m}\frac{h(p)}{p}\ln p = \kappa\ln\frac{x}{m} - \kappa\ln a + O((\ln b)^{1-\delta}) \tag{2.7}$$

as $m \geq x/b$. Also, we have

$$\sum_{m\in S_{a,b}(x/b,x)}\frac{h(m)}{m}\ln\frac{x}{m} = -(\ln b)m_{a,b}(x/b) + \int_{x/b}^{x}\frac{m_{a,b}(t)}{t}\,dt$$

so (2.6) can be written as

$$\kappa T_{a,b}(x) - \kappa T_{a,b}(x/b) - \kappa(\ln a)m_{a,b}(x) + O(m_{a,b}(x)(\ln b)^{1-\delta})$$

and using this we get our result from (2.4). ■

3. Proof of Proposition 1.3

Fix an $a \geq 1$, and let $a < x \leq b$. Using (2.7), expression (2.4) can be written as

$$m_{a,\text{-}}(x)\ln x = (\kappa+1)T_{a,\text{-}}(x) - \kappa m_{a,\text{-}}(x)\ln a + O(m_{a,\text{-}}(x)(\ln x)^{1-\delta}),$$

where

$$T_{a,\text{-}}(x) := \int_1^x \frac{m_{a,\text{-}}(t)}{t}\,dt.$$

Moving the items which contain $m_{a,\text{-}}(x)$ on the left hand side we get

$$m_{a,\text{-}}(x)\left(1+\kappa\frac{\ln a}{\ln x} + O((\ln x)^{-\delta})\right) = \frac{\kappa+1}{\ln x}T_{a,\text{-}}(x)$$

which can be written as

$$m_{a,\text{-}}(x) = \frac{1}{1-\varepsilon_a(x)}\frac{\kappa+1}{\ln x}T_{a,\text{-}}(x), \tag{3.1}$$

where

$$\varepsilon_a(x) := -\kappa\frac{\ln a}{\ln x} + O((\ln x)^{-\delta}).$$

Note that there exists x_0 such that $|\varepsilon_a(x)| < 1$ when $x \geq x_0$. Define

$$E_a(t) := \ln\left(\frac{\kappa+1}{(\ln t)^{\kappa+1}}T_{a,\text{-}}(t)\right).$$

Using (3.1) we get that

$$E'_a(t) = \frac{\kappa+1}{t\ln t}\frac{\varepsilon_a(t)}{1-\varepsilon_a(t)} \tag{3.2}$$

holds. We are going to prove the proposition separately in the case when

$$\exp((\ln a)^{\frac{1}{1-\delta}}) < x \tag{3.3}$$

and in the case when

$$\max\{x_0, \exp((\ln a)^{\frac{1}{1-\delta/(\nu+1)}})\} < x \leq \exp((\ln a)^{\frac{1}{1-\delta/(\nu+\eta)}}) \tag{3.4}$$

where $\nu \geq 1$ is an integer and $\eta \in (0,1)$.

- First we look at the case when (3.3) holds. When $\ln a < (\ln t)^{1-\delta}$, then $\varepsilon_a(t) \ll (\ln t)^{-\delta}$ and we have that

$$E'_a(t) \ll \frac{1}{t(\ln t)^{1+\delta}} \tag{3.5}$$

holds. By this, we have that

$$E_0 := \int_1^\infty E'_a(t)\,dt$$

converges absolutely, and therefore

$$E_a(x) = E_0 - \int_x^\infty E'_a(t)\,dt.$$

Let the inequality (3.3) hold for x. Exponentiating $E_a(x)$, we get

$$\frac{\kappa+1}{(\ln x)^{\kappa+1}} T_{a,-}(x) = \exp(E_a(x)) = C_\kappa(a) \exp\Big(- \int_x^\infty E'_a(t)\,dt \Big),$$

where $C_\kappa(a) := \exp(E_0)$. Here, by using (3.5) we have

$$- \int_x^\infty E'_a(t)\,dt \ll \int_x^\infty \frac{1}{t(\ln t)^{1+\delta}}\,dt = (\ln x)^{-\delta} \tag{3.6}$$

so

$$\frac{\kappa+1}{\ln x} T_{a,-}(x) = C_\kappa(a)(\ln x)^\kappa (1 + O((\ln x)^{-\delta}))$$

by the power series representation of the exponential function. Using this in (3.1) we get (1.5), as we can rewrite

$$(\ln xa^\kappa)^\kappa = (\ln x)^\kappa \left(1 + \kappa \frac{\ln a}{\ln x}\right)^\kappa = (\ln x)^\kappa (1 + O((\ln x)^{-\delta}))$$

in this case.

As $f(t) := 1/t^{s-1}$ has a continuous derivative on $[1, \infty)$, we can use Abel's identity, and rely on $P^-(1) = +\infty$ to get

$$\prod_{a<p} \left(1 + \sum_{k=1}^\infty \frac{h(p^k)}{p^{ks}}\right) = \sum_{\substack{n=1 \\ a<P^-(n)}}^\infty \frac{h(n)}{n^s} = (s-1) \int_1^\infty \frac{m_{a,-}(t)}{t^s}\,dt, \tag{3.7}$$

where we split the integral as

$$(3.8)\qquad (s-1)\int_1^x \frac{m_{a,-}(t)}{t^s}\,dt + (s-1)\int_x^\infty \frac{m_{a,-}(t)}{t^s}\,dt.$$

Using (1.5) in the second integral, we get

$$C_\kappa(a)(s-1)\int_x^\infty \frac{(\ln t)^\kappa + O((\ln t)^{\kappa-\delta})}{t^s}\,dt$$

which is equal to

$$(3.9)\qquad C_\kappa(a)\frac{\Gamma(\kappa+1,(s-1)\ln x)}{(s-1)^\kappa} + O\left(\frac{\Gamma(\kappa+1,(s-1)\ln x)}{(s-1)^{\kappa-\delta}}\right)$$

because

$$\int_x^\infty \frac{(\ln t)^\alpha}{t^s}\,dt = -\left[\frac{\Gamma(\alpha+1,(s-1)\ln t)}{(s-1)^{\alpha+1}}\right]_x^\infty = \frac{\Gamma(\alpha+1,(s-1)\ln x)}{(s-1)^{\alpha+1}}$$

by the definition of the upper incomplete gamma function. Substituting (3.9) into (3.8), and the resulting expression into (3.7) we get

$$C_\kappa(a)\frac{\Gamma(\kappa+1,(s-1)\ln x)}{(s-1)^\kappa} = \prod_{a<p}\left(1+\sum_{k=1}^\infty \frac{h(p^k)}{p^{ks}}\right) -$$
$$-(s-1)\int_1^x \frac{m_{a,-}(t)}{t^s}\,dt + O\left(\frac{\Gamma(\kappa+1,(s-1)\ln x)}{(s-1)^{\kappa-\delta}}\right).$$

By dividing with the quotient after $C_\kappa(a)$ we obtain

$$C_\kappa(a) = \frac{(s-1)^\kappa}{\Gamma(\kappa+1,(s-1)\ln x)}\prod_{a<p}\left(1+\sum_{k=1}^\infty \frac{h(p^k)}{p^{ks}}\right)+$$
$$+O((s-1)^\delta) - \frac{(s-1)^{\kappa+1}}{\Gamma(\kappa+1,(s-1)\ln x)}\int_1^x \frac{m_{a,-}(t)}{t^s}\,dt\ ,$$

where by keeping with s to 1 from the right, we get

$$C_\kappa(a) = \frac{1}{\Gamma(\kappa+1)}\lim_{s\to 1+0}\prod_{a<p}\left(1+\sum_{k=1}^\infty \frac{h(p^k)}{p^{ks}}\right)(s-1)^\kappa$$

as the integral keeps to $T_{a,\_}(x)$. Also, we have

$$\lim_{s\to 1+0}(s-1)\zeta(s)=1,$$

where

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}$$

when $s>1$. From these we get (1.6).

- Now we look at the case when (3.4) holds. If $t\geq x_0$, then $|\varepsilon_a(t)|<1$, and by the geometric summation we have that (3.2) is equal to

$$\frac{\kappa+1}{t\ln t}\Big(\varepsilon_a(t)+\varepsilon_a(t)^2+\ldots+\varepsilon_a(t)^\nu+O(\varepsilon_a(t)^{\nu+1})\Big). \tag{3.10}$$

Fix a $\nu\geq 1$, and assume that $(\ln t)^{1-\delta/(\nu+\eta)}\leq \ln a<(\ln t)^{1-\delta/(\nu+1)}$ holds with $\eta\in(0,1)$. Using the binomial theorem to compute $\varepsilon_a(t)^m$, where $1<m\leq\nu+1$, we get that for its main term we have

$$\left(-\kappa\frac{\ln a}{\ln t}\right)^m\ll\frac{1}{(\ln t)^{m\delta/(\nu+1)}}$$

which can be bounded by $(\ln t)^{-\delta}$ when $m=\nu+1$; and all the remaining terms $(l=0,\ldots,m-1)$ can be bounded by

$$\frac{(\ln a)^l}{(\ln t)^{l+(m-l)\delta}}\ll\frac{1}{(\ln t)^{(m-l)\delta+l\delta/(\nu+1)}}\ll(\ln t)^{-\delta}$$

using our assumption. So we have that the expression inside the parentheses of (3.10) is

$$-\kappa\frac{\ln a}{\ln t}+\left(-\kappa\frac{\ln a}{\ln t}\right)^2+\cdots+\left(-\kappa\frac{\ln a}{\ln t}\right)^\nu+O((\ln t)^{-\delta})$$

from where we get

$$E_a'(t)=\frac{\kappa+1}{t\ln t}\left(\sum_{i=1}^{\nu}\left(-\kappa\frac{\ln a}{\ln t}\right)^i+O((\ln t)^{-\delta})\right)\ll \\ \ll\frac{1}{t(\ln t)^{1+\delta/(\nu+1)}}. \tag{3.11}$$

Let the inequalities (3.4) hold for x. By using (3.11), from (3.6) we get

$$-\int_x^\infty E_a'(t)\,dt=-\int_x^\infty\frac{\kappa+1}{t\ln t}\sum_{i=1}^{\nu}\left(-\kappa\frac{\ln a}{\ln t}\right)^i dt+O((\ln x)^{-\delta})$$

which is equal to

$$-(\kappa+1)\sum_{i=1}^{\nu}\frac{\lambda^i}{i}+O((\ln x)^{-\delta}) \tag{3.12}$$

with $\lambda := -\kappa(\ln a)/(\ln x)$. For $x \geq x_0$ we have $|\lambda| < 1$, so we can use equation 8. from section 4.1.7. of [11] to get that the sum is equal to

$$-\ln(1-\lambda)-\lambda^{\nu+1}\int_0^1 \frac{t^\nu}{(1-\lambda+\lambda t)^{\nu+1}}\,dt. \tag{3.13}$$

Here, the integral is equal to

$$\left[\left(\frac{t}{1-\lambda}\right)^{\nu+1}\frac{{}_2F_1(\nu+1,\nu+1;\nu+2;t\lambda/(\lambda-1))}{\nu+1}\right]_0^1, \tag{3.14}$$

where we can write

$${}_2F_1(\nu+1,\nu+1;\nu+2;\mu)=(1-\mu)^{-\nu}{}_2F_1(1,1;\nu+2;\mu)$$

using the formula 15.3.3 from [1], with $\mu := \lambda/(\lambda-1)$. As $0 < \mu < 1$, based on 15.1.1 of [1] we can write ${}_2F_1(1,1;\nu+2;\mu)$ as

$$\frac{\Gamma(\nu+2)}{\Gamma(1)\Gamma(1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(n+\nu+2)}\frac{\mu^n}{n!} \leq (\nu+1)!\sum_{n=0}^{\infty}\mu^n.$$

So expression (3.14) can be bounded by

$$\left(\frac{1}{1-\lambda}\right)^{\nu+1}\frac{\nu!}{(1-\mu)^{\nu+1}}=\nu!$$

and by using this in expression (3.13), we get that (3.12) can be written as

$$(\kappa+1)\ln\left(1+\kappa\frac{\ln a}{\ln x}\right)+O((\ln x)^{-\delta}).$$

With this

$$\frac{\kappa+1}{\ln x}T_{a,\text{-}}(x)=C_\kappa(a)(\ln x)^\kappa\left(1+\kappa\frac{\ln a}{\ln x}\right)^{\kappa+1}(1+O((\ln x)^{-\delta}))$$

again by the power series representation of the exponential function. We have that in (3.1)

$$\frac{1}{1-\varepsilon_a(x)}=1-\kappa\frac{\ln a}{\ln x}+\cdots+\left(-\kappa\frac{\ln a}{\ln x}\right)^\nu+O((\ln x)^{-\delta})$$

similarly as in the case of expression (3.10). By using the sum formula of geometric progressions, we get (1.5).

The second integral in (3.8) can be written as

$$C_\kappa(a)(s-1)\int_x^\infty \frac{(\ln ta^\kappa)^\kappa}{t^s}\,dt + O\left(\frac{\Gamma(\kappa+1,(s-1)\ln x)}{(s-1)^{\kappa-\delta}}\right),$$

where the integral is

$$a^{(s-1)\kappa}\int_{xa^\kappa}^\infty \frac{(\ln v)^\kappa}{v^s}\,dv = -a^{(s-1)\kappa}\left[\frac{\Gamma(\kappa+1,(s-1)\ln v)}{(s-1)^{\kappa+1}}\right]_{xa^\kappa}^\infty$$

with the substitution $v = ta^\kappa$. By gathering the terms, and keeping with s to 1 from the right we get (1.6). ■

4. Proof of Proposition 1.4

For h being a nonnegative multiplicative function satisfying conditions 1.1 and 1.2 we have

$$\prod_{b<p\le c}\left(1+\sum_{k=1}^\infty \frac{h(p^k)}{p^{ks}}\right)\left(1-\frac{1}{p^s}\right)^\kappa = 1 + O((\ln b)^{-\delta})$$

uniformly in $s \ge 1$ and $c \ge b$, according [14], see the proofs of Theorems A and B. Using this in (1.6) we get

$$C_\kappa(a) = \frac{V_{a,b}}{\Gamma(\kappa+1)}\prod_{p\le b}\left(1-\frac{1}{p}\right)^\kappa (1+O((\ln b)^{-\delta})).$$

Based on Mertens' theorem, we get

$$V_{a,b}\frac{e^{-\gamma\kappa}}{\Gamma(\kappa+1)}\frac{1}{(\ln b)^\kappa}(1+O((\ln b)^{-1}))(1+O((\ln b)^{-\delta}))$$

on the right hand side. Using the definition of B_κ from section 1.3, and moving everything except $V_{a,b}$ to the left hand side we get

$$V_{a,b} = \frac{C_\kappa(a)}{B_\kappa}(\ln b)^\kappa(1+O((\ln b)^{-\delta})). \tag{4.1}$$

When $x \leq b$, both being sufficiently large, we can apply Proposition 1.3 to get

$$\begin{aligned} m_{a,b}(x) = m_{a,\text{-}}(x) &= C_\kappa(a)(\ln x + \ln a^\kappa)^\kappa + O((\ln x)^{\kappa-\delta}) = \\ &= C_\kappa(a)(\log_b(x)\ln b + \ln a^\kappa)^\kappa + O((\log_b(x)\ln b)^{\kappa-\delta}) = \\ &= C_\kappa(a)(\ln b)^\kappa\left(\log_b(xa^\kappa)^\kappa + O\left(\frac{\log_b(x)^{\kappa-\delta}}{(\ln b)^\delta}\right)\right) = \\ &= V_{a,b}B_\kappa\left(\log_b(xa^\kappa)^\kappa + O\left(\frac{\log_b(x)^{\kappa-\delta}}{(\ln b)^\delta}\right)\right). \end{aligned}$$

Thus when $x \leq b/a^\kappa$, that is when $\log_b(xa^\kappa) \leq 1$, we have

$$m_{a,b}(x) = V_{a,b}\left(j_\kappa(\log_b(xa^\kappa)) + O\left(\frac{\log_b(x)^{\kappa-\delta}}{(\ln b)^\delta}\right)\right). \tag{4.2}$$

Assume that $b/a^\kappa < x$, that is $\log_b(xa^\kappa) > 1$. Based on (2.5) we have

$$\begin{aligned} \frac{d}{dx}\frac{T_{a,b}(x)}{(\ln xa^\kappa)^{\kappa+1}} &= \frac{m_{a,b}(x)\ln xa^\kappa - (\kappa+1)T_{a,b}(x)}{x(\ln xa^\kappa)^{\kappa+2}} = \\ &= -\frac{\kappa T_{a,b}(x/b)}{x(\ln xa^\kappa)^{\kappa+2}} + O\left(\frac{V_{a,b}(\ln b)^{1-\delta}}{x(\ln xa^\kappa)^{\kappa+2}}\right). \end{aligned}$$

Integrating this in variable x from ξ_1 to ξ_2, where $b \leq \xi_1 \leq \xi_2$ we get

$$\begin{aligned} \frac{T_{a,b}(\xi_2)}{(\ln \xi_2 a^\kappa)^{\kappa+1}} = \frac{T_{a,b}(\xi_1)}{(\ln \xi_1 a^\kappa)^{\kappa+1}} - \kappa\int_{\xi_1}^{\xi_2}\frac{T_{a,b}(x/b)}{x(\ln xa^\kappa)^{\kappa+2}}\,dx + \\ + O\left(\frac{V_{a,b}(\ln b)^{1-\delta}}{(\ln \xi_1 a^\kappa)^{\kappa+1}}\right). \end{aligned} \tag{4.3}$$

Assume also that $T_{a,b}(\xi)$ can be written as

$$T_{a,b}(\xi) = (J_\kappa(\log_b(\xi a^\kappa)) - J_\kappa(\log_b(a^\kappa)))V_{a,b}\ln b + R_{a,b}(\xi). \tag{4.4}$$

After substituting this in (4.3), and canceling the terms containing J_κ or melding them into the ordo term, we get

$$\begin{aligned} \frac{R_{a,b}(\xi_2)}{(\ln \xi_2 a^\kappa)^{\kappa+1}} = \frac{R_{a,b}(\xi_1)}{(\ln \xi_1 a^\kappa)^{\kappa+1}} - \kappa\int_{\xi_1}^{\xi_2}\frac{R_{a,b}(x/b)}{x(\ln xa^\kappa)^{\kappa+2}}\,dx + \\ + O\left(\frac{V_{a,b}(\ln b)^{1-\delta}}{(\ln \xi_1 a^\kappa)^{\kappa+1}}\right) \end{aligned} \tag{4.5}$$

using (1.4).

We are going to bound the error term $R_{a,b}(\xi)$ by induction. Let $\xi \le b/a^\kappa$. We can use (4.2) to write $T_{a,b}(\xi)$ as

$$\int_1^\xi \frac{m_{a,b}(t)}{t}\,dt = V_{a,b}(\ln b)\left(J_\kappa(\log_b(\xi a^\kappa)) - J_\kappa(\log_b(a^\kappa)) + O\left(\frac{\log_b(\xi)^{\kappa+1-\delta}}{(\ln b)^\delta}\right)\right)$$

which agrees with (4.4). So in this case we have

$$|R_{a,b}(\xi)| \le D_1 \frac{V_{a,b}}{(\ln b)^{\delta-1}} \tag{4.6}$$

with the constant D_1 at least as large as the one in (4.2).

Let $\xi_1 = b/a^\kappa$, and $\xi_2 = \xi$ with $b/a^\kappa < \xi \le b^2/a^\kappa$. With these choices we can bound every $R_{a,b}(\cdot)$ using (4.6) on the right hand side of (4.5), and we get

$$\frac{|R_{a,b}(\xi)|}{(\ln \xi a^\kappa)^{\kappa+1}} \le D_1 \frac{V_{a,b}}{(\ln b)^{\delta-1}}\left(\kappa \int_{b/a^\kappa}^{b^2/a^\kappa} \frac{dx}{x(\ln x a^\kappa)^{\kappa+2}} + \frac{1 + D_0/D_1}{(\ln b)^{\kappa+1}}\right)$$

with D_0 being the constant in the O term of (4.5). This is less than or equal to

$$D_1 \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}}\left(\frac{\kappa}{\kappa+1} + 1 + \frac{D_0}{D_1}\right) \le 2D_1 \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}}$$

given that $D_1 \ge (\kappa+1)D_0$.

Suppose that for $\nu = 1, 2, \ldots$ there exists a constant D^* independent of ν such that

$$\frac{|R_{a,b}(\xi)|}{(\ln \xi a^\kappa)^{\kappa+1}} \le \eta(\nu) D^* \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}} \tag{4.7}$$

for $b^\nu/a^\kappa < \xi \le b^{\nu+1}/a^\kappa$, where $\eta(\nu) := \nu^\kappa \ln(\nu+1)$. We have that (4.7) is true for $\nu = 1$ given that $D^* \ge 2D_1/\ln 2$. So we assume that $\nu > 1$.

Let $\xi_1 = b^\nu/a^\kappa$, and $\xi_2 = \xi$ with $b^\nu/a^\kappa < \xi \le b^{\nu+1}/a^\kappa$. We can bound every $R_{a,b}(\cdot)$ using (4.7) on the right hand side of (4.5), and the bound this time is

$$D^* \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}}\left(\eta(\nu-1) + \kappa\eta(\nu-1) \int_{b^\nu/a^\kappa}^{b^{\nu+1}/a^\kappa} \frac{(\ln x a^\kappa/b)^{\kappa+1}}{x(\ln x a^\kappa)^{\kappa+2}}\,dx + \frac{D_0}{D^*\nu^{\kappa+1}}\right),$$

where the integral term can be handled the same way as in [14], and we get

$$D^* \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}}\left(\frac{\nu+\kappa}{\nu}\eta(\nu-1) + \frac{D_0}{D^*\nu}\right) \le \eta(\nu) D^* \frac{V_{a,b}}{(\ln b)^{\kappa+\delta}}$$

which is true if $D^* \geq 4D_0/3$. This proves (4.7), and we have

$$|R_{a,b}(\xi)| \leq \log_b(\xi a^\kappa)^{2\kappa+1} \ln(\log_b(\xi a^\kappa)+1) D^* \frac{V_{a,b}}{(\ln b)^{\delta-1}}$$

when $b/a^\kappa < \xi$.

Using this in (4.4) we can write $T_{a,b}(\xi)$ as

$$\left(J_\kappa(\log_b(\xi a^\kappa)) - J_\kappa(\log_b(a^\kappa)) + O\left(\frac{\log_b(\xi a^\kappa)^{2\kappa+1}\ln(\log_b(\xi a^\kappa)+1)}{(\ln b)^\delta}\right)\right) V_{a,b} \ln b.$$

Substituting this in (2.5), dividing with $\log_b(xa^\kappa)\ln b$, and using (1.3) we get

$$V_{a,b}\left(j_\kappa(\log_b(xa^\kappa)) - \frac{J_\kappa(\log_b(a^\kappa))}{\log_b(xa^\kappa)} + O\left(\frac{\log_b(xa^\kappa)^{2\kappa}\ln(\log_b(xa^\kappa)+1)}{(\ln b)^\delta}\right)\right)$$

for $m_{a,b}(x)$ when $b/a^\kappa < x$. If b is large enough, so that

$$a^\kappa \leq \exp((\ln b)^{1-\delta/(\kappa+1)})$$

holds, then $\log_b(a^\kappa) \leq 1$, and we can use (1.2) to get that

$$J_\kappa(\log_b(a^\kappa)) = \frac{B_\kappa}{\kappa+1}\log_b(a^\kappa)^{\kappa+1} \ll (\ln b)^{-\delta}.$$

Thus we can meld $J_\kappa(\log_b(a^\kappa))/\log_b(xa^\kappa)$ into the ordo, and we get (1.7). ■

5. Auxiliary lemmas for the proof of Theorem 1.5

Lemma 5.1. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. Then there exists positive x_0 such that*

$$\int_1^{b/a^\kappa} \frac{m_{a,\text{-}}(t)}{t}\,dt = \frac{B_\kappa}{\kappa+1} V_{a,b}(\ln b)(1+O((\ln b)^{-\delta})) \tag{5.1}$$

with $1 \leq a < b$ and $b_0 \leq b$, where b_0 is the least value of b satisfying

$$(\ln b)^{\kappa+1-\delta} \geq (\ln x_0)^{\kappa+1}. \tag{5.2}$$

Proof. As h satisfies the requirements of Proposition 1.3, we have that there exist constants C' and x_0 such that

$$|m_{a,\text{-}}(x) - C_\kappa(a)(\ln xa^\kappa)^\kappa| \leq C'(\ln x)^{\kappa-\delta} \tag{5.3}$$

for all $x \geq x_0$. We can split the integral on the left hand side of (5.1) as

$$\int_1^{b/a^\kappa} \frac{m_{a,\text{-}}(t)}{t}\,dt = \int_1^{x_0/a^\kappa} \frac{m_{a,\text{-}}(t)}{t}\,dt + \int_{x_0/a^\kappa}^{b/a^\kappa} \frac{m_{a,\text{-}}(t)}{t}\,dt$$

due to our requirements for b. As $m_{a,\text{-}}(t) \leq V_{a,x_0/a^\kappa}$ when $1 \leq t \leq x_0/a^\kappa \leq b$, we can bound the first integral on the right hand side with $V_{a,x_0/a^\kappa} \ln x_0/a^\kappa$. Concerning the second integral, we can subtract and add $C_\kappa(a)(\ln ta^\kappa)^\kappa/t$ to get

$$C' \int_{x_0/a^\kappa}^{b/a^\kappa} \frac{(\ln t)^{\kappa-\delta}}{t}\,dt + C_\kappa(a) \int_{x_0/a^\kappa}^{b/a^\kappa} \frac{(\ln ta^\kappa)^\kappa}{t}\,dt$$

based on (5.3). By summing everything we get, computing the integrals, and extracting $B_\kappa V_{a,b}(\ln b)/(\kappa+1)$, we get our result based on (4.1) and (5.2). ■

Lemma 5.2. *Let h be a nonnegative multiplicative function satisfying Conditions 1.1 and 1.2. With $1/2 \leq \theta \leq 1$ we have*

(5.4)
$$\sum_{a<p\leq b^\theta} j_\kappa(\log_b(xa^\kappa) - \log_b(p)) \frac{h(p)}{p} \ln p =$$
$$= \kappa(\ln b) \int_{\log_b(a)}^{\theta} j_\kappa(\log_b(xa^\kappa) - v)\,dv + O((\ln b)^{1-\delta})$$

for $1 \leq a < b$, and $\log_b(xa^\kappa) \geq 2$.

Proof. Let

$$s_a(y) := \sum_{a<p\leq y} \frac{h(p)}{p} \ln p.$$

Using Condition 1.1 we have $s_a(y) - \kappa \ln y/a = r(y)$, where $|r(y)| \ll (\ln y)^{1-\delta}$. We can write the sum on the left hand side of (5.4) as

(5.5)
$$\int_a^{b^\theta} j_\kappa(\log_b(xa^\kappa) - \log_b(t))\,d\{s_a(t) - \kappa \ln t/a\} +$$
$$+\kappa \int_a^{b^\theta} \frac{j_\kappa(\log_b(xa^\kappa) - \log_b(t))}{t}\,dt.$$

The first integral can be written as

$$(5.6)\qquad j_\kappa(\log_b(xa^\kappa)-\theta)|r(b^\theta)| - j_\kappa(\log_b(xa^\kappa)-\log_b(a))|r(a)| - \\ -\int_a^{b^\theta} r(t)\frac{d}{dt}j_\kappa(\log_b(xa^\kappa)-\log_b(t))\,dt,$$

where the integral is in

$$O\Bigg((\ln b)^{1-\delta}\int_a^{b^\theta} j_\kappa'(\log_b(xa^\kappa)-\log_b(t))\,dt\Bigg)$$

which means that the first integral of (5.5) is in $O((\ln b)^{1-\delta})$. We get the integral on the right hand side of (5.4) by the change of variable $v=\log_b(t)$ in the second integral. ■

6. Proof of Theorem 1.5

The proposition is already proved for $\log_b(xa^\kappa)$ in some bounded range because of Proposition 1.4. Say we have

$$(6.1)\qquad m_{a,b}(x) = V_{a,b}(j_\kappa(\log_b(xa^\kappa)) + R_{a,b}(\log_b(xa^\kappa)))$$

for $b\ge b_0$, where

$$|R_{a,b}(\log_b(xa^\kappa))| \ll (\ln b)^{-\delta}$$

if $1<\log_b(xa^\kappa)\le\kappa+5/2$. If $\log_b(xa^\kappa)>\kappa$, then

$$(6.2)\qquad m_{a,b}(x) = V_{a,b}j_\kappa(\log_b(xa^\kappa))(1+\Delta_{a,b}(\log_b(xa^\kappa)))$$

where $\Delta_{a,b}(\log_b(xa^\kappa))\le 2R_{a,b}(\log_b(xa^\kappa))$ as $j_\kappa(\kappa)\ge 1/2$. Let

$$\Delta^*_{a,b}(\log_b(xa^\kappa)) := \sup_{\kappa\le v\le\log_b(xa^\kappa)}|\Delta_{a,b}(v)|.$$

Then we have

$$(6.3)\qquad \Delta^*_{a,b}(\log_b(xa^\kappa)) \ll (\ln b)^{-\delta}$$

for $\kappa<\log_b(xa^\kappa)\le\kappa+5/2$. What remains is to show that

$$(6.4)\qquad \Delta^*_{a,b}(\log_b(xa^\kappa)) \ll \frac{\ln\log_b(xa^\kappa)}{(\ln b)^\delta}$$

uniformly for $\log_b(xa^\kappa) \geq \kappa + 5/2$ and for all sufficiently large b. So let $\log_b(xa^\kappa) \geq \kappa + 5/2$ from here onward.

By changing the order of summation in (2.4) we get

$$m_{a,b}(x)\ln x = T_{a,b}(x) + \sum_{a<p\leq b} m_{a,b}(x/p)\frac{h(p)}{p}\ln p + O(m_{a,b}(x)(\ln b)^{1-\delta}) \tag{6.5}$$

for $a < b \leq x$.

First, we split the integral $T_{a,b}(x)$ on the right hand side of (6.5) as

$$\int_1^{b/a^\kappa} \frac{m_{a,-}(t)}{t}\,dt + \int_{b/a^\kappa}^{b^{\kappa+5/2}/a^\kappa} \frac{m_{a,b}(t)}{t}\,dt + \int_{b^{\kappa+5/2}/a^\kappa}^{x} \frac{m_{a,b}(t)}{t}\,dt.$$

Here, the first integral can be computed by Lemma 5.1. Using (6.1) in the second integral we get

$$V_{a,b}(\ln b)\int_1^{\kappa+5/2} j_\kappa(v)\,dv + O(V_{a,b}(\ln b)^{1-\delta})$$

and using (6.2) in the third integral we get

$$V_{a,b}(\ln b)\int_{\kappa+5/2}^{\log_b(xa^\kappa)} j_\kappa(v)(1+\Delta_{a,b}(v))\,dv$$

by a change of variable $v = \log_b(xa^\kappa)$. So we get that the integral $T_{a,b}(x)$ is equal to

$$V_{a,b}(\ln b)\left(\frac{B_\kappa}{\kappa+1} + \int_1^{\log_b(xa^\kappa)} j_\kappa(v)\,dv + \int_{\kappa+5/2}^{\log_b(xa^\kappa)} j_\kappa(v)\Delta_{a,b}(v)\,dv\right) + \\ + O(V_{a,b}(\ln b)^{1-\delta}). \tag{6.6}$$

We look at the sum on the right hand side of (6.5). When $\log_b(xa^\kappa) > \kappa$, then we can write this sum as

$$V_{a,b}\sum_{a<p\leq b} j_\kappa(\log_b(xa^\kappa/p))\frac{h(p)}{p}(\ln p)(1+\Delta_{a,b}(\log_b(xa^\kappa/p))).$$

Splitting the sum and using Lemma 5.2, we can write this as

$$\kappa V_{a,b}(\ln b)\int\limits_{\log_b(xa^\kappa)-1}^{\log_b(xa^\kappa)-\log_b(a)} j_\kappa(v)\,dv+O(V_{a,b}(\ln b)^{1-\delta})+ \\ +V_{a,b}\sum_{a<p\le b} j_\kappa(\log_b(xa^\kappa/p))\frac{h(p)}{p}(\ln p)\Delta_{a,b}(\log_b(xa^\kappa/p)). \tag{6.7}$$

We note that $\log_b(a)\to 0$ as b increases, so for sufficiently large b we have that

$$\int\limits_{\log_b(xa^\kappa)-\log_b(a)}^{\log_b(xa^\kappa)} j_\kappa(v)\,dv \ll (\ln b)^{-\delta} \tag{6.8}$$

thus we can extend the integral in (6.7) up until $\log_b(xa^\kappa)$. Then by combining the terms from (6.6) and this extended integral we get

$$\frac{B_\kappa}{\kappa+1}+\int\limits_{1}^{\log_b(xa^\kappa)} j_\kappa(v)\,dv+\kappa\int\limits_{\log_b(xa^\kappa)-1}^{\log_b(xa^\kappa)} j_\kappa(v)\,dv= \\ =\log_b(xa^\kappa)j_\kappa(\log_b(xa^\kappa)) \tag{6.9}$$

based on (1.1).

Dividing both sides of (6.5) with $V_{a,b}j_\kappa(\log_b(xa^\kappa))\ln xa^\kappa$, relying on (6.6) and (6.7) while using (6.9) we get that $\Delta_{a,b}(\log_b(xa^\kappa))$ is equal to

$$\frac{1}{\log_b(xa^\kappa)j_\kappa(\log_b(xa^\kappa))\ln b}\sum_{a<p\le b} j_\kappa(\log_b(xa^\kappa/p))\frac{h(p)}{p}(\ln p)\Delta_{a,b}(\log_b(xa^\kappa/p))+ \\ +\frac{1}{\log_b(xa^\kappa)j_\kappa(\log_b(xa^\kappa))}\int\limits_{\kappa+5/2}^{\log_b(xa^\kappa)} j_\kappa(v)\Delta_{a,b}(v)\,dv+ \\ +O_\kappa\left(\frac{1}{\log_b(xa^\kappa)j_\kappa(\log_b(xa^\kappa))(\ln b)^\delta}\right).$$

By Lemma 5.2, the $a<p\le b^{1/2}$ part of the sum can be written as

$$\Delta^*_{a,b}(\log_b(xa^\kappa)-\log_b(a))\left(\kappa(\ln b)\int\limits_0^{1/2} j_\kappa(\log_b(xa^\kappa)-v)\,dv+O((\ln b)^{1-\delta})\right),$$

where we used (6.8), and the $b^{1/2} < p \le b$ part of the sum as

$$\Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)\Big(\kappa(\ln b)\int\limits_{1/2}^{1} j_\kappa(\log_b(xa^\kappa) - v)\,dv + O((\ln b)^{1-\delta})\Big)$$

and the integral is at most

$$\Delta^*_{a,b}(\log_b(xa^\kappa) - 1)\int\limits_{1}^{\log_b(xa^\kappa)-1} j_\kappa(v)\,dv + \Delta^*_{a,b}(\log_b(xa^\kappa))\int\limits_{\log_b(xa^\kappa)-1}^{\log_b(xa^\kappa)} j_\kappa(v)\,dv.$$

So $|\Delta_{a,b}(\log_b(xa^\kappa))|$ is smaller than or equal to

$$\Delta^*_{a,b}(\log_b(xa^\kappa))\alpha(\log_b(xa^\kappa)) + \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)\beta(\log_b(xa^\kappa)) + O\Big(\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta}\Big), \tag{6.10}$$

where

$$\alpha(v) = \frac{\kappa}{vj_\kappa(v)}\int\limits_{0}^{1/2} j_\kappa(v-t)\,dt + \frac{1}{vj_\kappa(v)}\int\limits_{v-1}^{v} j_\kappa(t)\,dt \le \frac{\kappa+2}{2v} \tag{6.11}$$

see the proof of the main theorem in [14], and

$$\beta(v) = \frac{\kappa}{vj_\kappa(v)}\int\limits_{1/2}^{1} j_\kappa(v-t)\,dt + \frac{1}{vj_\kappa(v)}\int\limits_{1}^{v-1} j_\kappa(t)\,dt.$$

Noting that

$$\beta(v) = 1 - \alpha(v) - \frac{B_\kappa}{(\kappa+1)vj_\kappa(v)} \le 1 - \alpha(v)$$

we have that $|\Delta_{a,b}(\log_b(xa^\kappa))|$ is smaller than or equal to

$$\begin{aligned}&\Delta^*_{a,b}(\log_b(xa^\kappa))\alpha(\log_b(xa^\kappa)) + \\ &+ \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)(1 - \alpha(\log_b(xa^\kappa))) + \\ &+ O\Big(\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta}\Big).\end{aligned} \tag{6.12}$$

We also have that $|\Delta_{a,b}(\log_b(xa^\kappa))|$ is smaller than or equal to

$$\frac{1}{2}(\Delta^*_{a,b}(\log_b(xa^\kappa)) + \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2))+ \\ + O\left(\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta}\right) \tag{6.13}$$

when $\log_b(xa^\kappa) \geq \kappa + 2$, because if we subtract the non-asymptotic part of (6.12) from the non-asymptotic part of (6.13) we get

$$\left(\frac{1}{2} - \alpha(\log_b(xa^\kappa))\right)(\Delta^*_{a,b}(\log_b(xa^\kappa)) - \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)) \geq 0$$

based on $\alpha(\log_b(xa^\kappa)) \leq 1/2$ due to (6.11), and the monotonicity of Δ^*.

When $\kappa \leq \lambda \leq \log_b(xa^\kappa) - 1/2$, we have

$$|\Delta_{a,b}(\lambda)| \leq \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2) \leq \frac{1}{2}(\Delta^*_{a,b}(\log_b(xa^\kappa)) + \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2))$$

and when $\log_b(xa^\kappa) - 1/2 \leq \lambda \leq \log_b(xa^\kappa)$, we have

$$|\Delta_{a,b}(\lambda)| \leq \frac{1}{2}(\Delta^*_{a,b}(\lambda) + \Delta^*_{a,b}(\lambda - 1/2)) + C''\frac{1 + \Delta^*_{a,b}(\lambda)}{\lambda(\ln b)^\delta}$$

which is less than or equal to

$$\frac{1}{2}(\Delta^*_{a,b}(\log_b(xa^\kappa)) + \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)) + \frac{2C''}{3}\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta}$$

by the monotonicity of Δ^*, for some positive C'' constant. Based on these, by taking the supremum we get

$$\Delta^*_{a,b}(\log_b(xa^\kappa)) \leq \frac{1}{2}(\Delta^*_{a,b}(\log_b(xa^\kappa)) + \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2)) + \\ + \frac{2C''}{3}\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta} \tag{6.14}$$

which we can rearrange as

$$\Delta^*_{a,b}(\log_b(xa^\kappa)) \leq \Delta^*_{a,b}(\log_b(xa^\kappa) - 1/2) + \frac{4C''}{3}\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{\log_b(xa^\kappa)(\ln b)^\delta}$$

which we can iterate, to descend with the argument of $\Delta^*_{a,b}$ until ξ to get

$$\Delta^*_{a,b}(\log_b(xa^\kappa)) \leq \Delta^*_{a,b}(\xi) + \frac{4C''}{3}\frac{1 + \Delta^*_{a,b}(\log_b(xa^\kappa))}{(\ln b)^\delta}\ln\log_b(xa^\kappa)$$

where $\kappa < \xi \leq \kappa + 5/2$. Then by (6.3) we get (6.4). ■

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