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PARAMETRIZED AND TRIGONOMETRIC DERIVED UNIFORM APPROXIMATION BY VARIOUS SMOOTH SINGULAR INTEGRAL OPERATORS

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Abstract. In this work we continue with the study of smooth Gauss-Weierstrass, Poisson-Cauchy and Trigonometric singular integral operators that started in [3], see there chapters 10–14. This time the foundation of our research is a trigonometric Taylor's formula. We prove the parametrized univariate uniform convergence of our operators to the unit operator with rates via Jackson type parametrized inequalities involving the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not in general positive.

1. Introduction

We are motivated by [2], [3] chapters 10–14, and [4], [1]. We use a trigonometric new Taylor formula from [4], see also [1]. Here we consider some very general operators, the smooth Gauss–Weierstrass, Poisson–Cauchy and trigonometric singular integral operators over the real line and we study further their parametrized and uniform convergence properties quantitatively. We establish related parametrized inequalities involving the first modulus of continuity. We provide detailed proofs.

Other important motivating articles on the topic are [6]–[10].

Key words and phrases: Gauss–Weierstrass, Poisson–Cauchy and Trigonometric smooth singular integrals parametrized approximation, modulus of continuity, trigonometric Taylor formula.

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For the history of the topic we mention about the monograph [5] of 2012, which was the first complete source to deal exclusively with the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there studied quantitatively the basic approximation properties of the general Picard, Gauss–Weierstrass and Poisson–Cauchy singular integral operators over the real line, which are not positive linear operators. In particular they studied the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of higher order modulus of smoothness of the high order derivative of the involved function. Some of these inequalities are proven to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore they gave asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss–Weierstrass and Poisson-Cauchy singular integral operators. These properties were studied with respect to L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz type functions approximation was studied separately and in detail. Furthermore they presented the corresponding general approximation theory of general singular integral operators with lots of applications to, the under focused till then, trigonometric singular integral.

2.

About Part I: on Gauss–Weierstrass smooth singular integrals

By [1], [4], for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$, and $f \in C^4(\mathbb{R})$, $a, x \in \mathbb{R}$, we have the following general trigonometric Taylor formula:

$$
\begin{aligned}
(1)\qquad f(x) - f(a) = {} & f'(a)\left(\frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right) + \\
& + f''(a)\left(\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2-\alpha^2}\right) + \\
& + f'''(a)\left(\frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right) + \\
& + \frac{2\left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right) - \alpha^2 \sin^2\left(\frac{\beta(x-a)}{2}\right)\right) + \\
& + \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left[\left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right) - \right. \\
& \left. - \left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a) + \alpha^2\beta^2 f(a)\right)\right] \cdot \\
& \cdot \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))\right] dt.
\end{aligned}
$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, ..., r. \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & n = 0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

Here we consider all $f, f', f'', f''', f^{(4)} \in C_U(\mathbb{R}) \cap C_B(\mathbb{R})$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Gauss–Weierstrass operators

$$W_{r,\xi}(f, x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-\frac{t^2}{\xi}} dt, \tag{4}$$

see [5], $W_{r,\xi}$ are not in general positive operators, see [5].

We notice by

$$\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1, \tag{5}$$

that

$$W_{r,\xi}(c, x) = c, \text{ where } c \text{ is a constant}, \tag{6}$$

and

$$W_{r,\xi}(f, x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \tag{7}$$

Denote by

$$\omega_1(f, h) := \sup_{\substack{x,y \in \mathbb{R} \\ |x-y| \le \delta}} |f(x) - f(y)|, \tag{8}$$

the first modulus of continuity of f.

By (1) we get that

$$f(x+jt)-f(x)=f'(x)\left(\frac{\beta^3\sin(\alpha jt)-\alpha^3\sin(\beta jt)}{\alpha\beta(\beta^2-\alpha^2)}\right)+ \tag{9}$$

$$+f''(x)\left(\frac{\cos(\alpha jt)-\cos(\beta jt)}{\beta^2-\alpha^2}\right)+$$

$$+f'''(x)\left(\frac{\beta\sin(\alpha jt)-\alpha\sin(\beta jt)}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$

$$+\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\left(\beta^2\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$+\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_x^{x+jt}\left[\left(f^{(4)}(s)+\left(\alpha^2+\beta^2\right)f''(s)+\alpha^2\beta^2f(s)\right)-\right.$$

$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2f(x)\right)\right]\cdot$$

$$\cdot\left[\beta\sin(\alpha(x+jt-s))-\alpha\sin(\beta(x+jt-s))\right]ds,$$

or better,

$$f(x+jt)-f(x)=f'(x)\left(\frac{\beta^3\sin(\alpha jt)-\alpha^3\sin(\beta jt)}{\alpha\beta(\beta^2-\alpha^2)}\right)+ \tag{10}$$

$$+f''(x)\left(\frac{\cos(\alpha jt)-\cos(\beta jt)}{\beta^2-\alpha^2}\right)+$$

$$+f'''(x)\left(\frac{\beta\sin(\alpha jt)-\alpha\sin(\beta jt)}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$

$$+\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\left(\beta^2\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$+\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_0^{jt}\left[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2f(x+z)\right)-\right.$$

$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2f(x)\right)\right]\cdot$$

$$\cdot\left[\beta\sin(\alpha(jt-z))-\alpha\sin(\beta(jt-z))\right]dz.$$

Furthermore it holds

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]= \tag{11}$$

$$= \left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin\left(\alpha jt\right)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin\left(\beta jt\right)\right]+$$

$$+\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos\left(\alpha jt\right)-\sum_{j=0}^{r}\alpha_j\cos\left(\beta jt\right)\right]+$$

$$+\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin\left(\alpha jt\right)-\alpha\sum_{j=0}^{r}\alpha_j\sin\left(\beta jt\right)\right]+$$

$$+\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{\left(\alpha\beta\right)^2\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$+\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\cdot$$

$$\cdot\sum_{j=0}^{r}\alpha_j\int_0^{jt}\left[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\right)-\right.$$

$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\right]\cdot$$

$$\cdot\left[\beta\sin\left(\alpha\left(jt-z\right)\right)-\alpha\sin\left(\beta\left(jt-z\right)\right)\right]dz,$$

or better

$$\sum_{j=0}^{r}\alpha_j\left[f\left(x+jt\right)-f(x)\right]= \tag{12}$$

$$= \left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin\left(\alpha jt\right)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin\left(\beta jt\right)\right]+$$

$$+\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos\left(\alpha jt\right)-\sum_{j=0}^{r}\alpha_j\cos\left(\beta jt\right)\right]+$$

$$+\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin\left(\alpha jt\right)-\alpha\sum_{j=0}^{r}\alpha_j\sin\left(\beta jt\right)\right]+$$

$$+\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$+\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\cdot$$

$$\cdot\sum_{j=0}^{r}\alpha_j j\int_0^t\left[\left(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2f(x+jw)\right)-\right.$$

$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2f(x)\right)\right]\cdot$$

$$\cdot\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw.$$

We call

$$R:=R(t):=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j\cdot \tag{13}$$

$$\cdot\int_0^t\left[\left(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2f(x+jw)\right)-\right.$$

$$\left.-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2f(x)\right)\right]\cdot$$

$$\cdot\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw,\quad\forall\ t\in\mathbb{R}.$$

We set

$$E_1(x):=W_{r,\xi}(f,x)-f(x)-\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\cdot \tag{14}$$

$$\cdot\left[\beta^3\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt\right)-\right.$$

$$\left.-\alpha^3\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\beta jt)\,e^{-\frac{t^2}{\xi}}dt\right)\right]-$$

$$-\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\cdot$$

$$\cdot\left[\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\cos(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt\right)-\right.$$

$$\left.-\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\cos(\beta jt)\,e^{-\frac{t^2}{\xi}}dt\right)\right]-$$

$$-\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left[\beta\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt\right)-\right.$$

$$\left.-\alpha\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(\beta jt)\,e^{-\frac{t^2}{\xi}}dt\right)\right]-$$

$$-\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)e^{-\frac{t^2}{\xi}}dt\right)-\right.$$

$$\left.-\alpha^2\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)e^{-\frac{t^2}{\xi}}dt\right)\right]=$$

$$=\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}R(t)\,e^{-\frac{t^2}{\xi}}dt.$$

Next we simplify $E_1(x)$:

We observe that

$$\int_{-\infty}^{\infty}\sin(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt=\int_{-\infty}^{0}\sin(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt+\int_{0}^{\infty}\sin(\alpha jt)\,e^{-\frac{t^2}{\xi}}dt. \tag{15}$$

Notice $-\infty\leq t\leq 0\Rightarrow\infty\geq -t\geq 0$. So that

$$\int_{-\infty}^{0}\sin(\alpha jt)\,e^{-\frac{\left|t^2\right|}{\xi}}dt=-\int_{-\infty}^{0}\sin(\alpha j(-(-t)))\,e^{-\frac{t^2}{\xi}}d(-t)=$$

$$= -\int_{-\infty}^{0} \left(-\sin\left(\alpha j\left(-t\right)\right)\right) e^{-\frac{t^2}{\xi}} d\left(-t\right) = \int_{-\infty}^{0} \sin\left(\alpha j\left(-t\right)\right) e^{-\frac{t^2}{\xi}} d\left(-t\right) = \tag{16}$$

$$= \int_{\infty}^{0} \sin\left(\alpha j t\right) e^{-\frac{t^2}{\xi}} dt = -\int_{0}^{\infty} \sin\left(\alpha j t\right) e^{-\frac{t^2}{\xi}} dt.$$

So that

$$\int_{-\infty}^{\infty} \sin\left(\alpha j t\right) e^{-\frac{t^2}{\xi}} dt = 0,$$

and all sine integrals in (14) are zeros.

Furthermore we have that

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{t^2}{\xi}} dt = 2\int_{0}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{t^2}{\xi}} dt =$$

$$= 2\sqrt{\xi}\int_{0}^{\infty} \sin^2\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} =$$

($\frac{t}{\sqrt{\xi}} =: x$, and $\frac{\alpha j\sqrt{\xi}}{2} = \beta_1$)

$$= 2\sqrt{\xi}\int_{0}^{\infty} \sin^2\left(\beta_1 x\right) e^{-x^2} dx = 2\sqrt{\xi}\frac{1}{4}\sqrt{\pi}e^{-\beta_1^2}\left(e^{\beta_1^2} - 1\right) = \tag{17}$$

$$= \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\beta_1^2}\right) = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right).$$

That is

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right), \tag{18}$$

and similarly,

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\beta j t}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2}\left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}}\right). \tag{19}$$

Next, we treat

$$\int_{-\infty}^{\infty} \cos(\alpha j t)\, e^{-\frac{t^2}{\xi}}\, dt = 2\int_{0}^{\infty} \cos(\alpha j t)\, e^{-\frac{t^2}{\xi}}\, dt = \tag{20}$$

$$= 2\sqrt{\xi}\int_{0}^{\infty} \cos\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} =$$

$(\frac{t}{\sqrt{\xi}} =: x$, and $\frac{\alpha j\sqrt{\xi}}{2} =: \beta_1)$

$$= 2\sqrt{\xi}\int_{0}^{\infty} \cos(\beta_1 x)\, e^{-x^2} dx = 2\sqrt{\xi}\frac{1}{2}\sqrt{\pi}e^{-\frac{\beta_1^2}{4}} = \sqrt{\pi\xi}e^{-\frac{\alpha^2 j^2 \xi}{16}}.$$

That is

$$\int_{-\infty}^{\infty} \cos(\alpha j t)\, e^{-\frac{t^2}{\xi}}\, dt = \sqrt{\pi\xi}e^{-\frac{\alpha^2 j^2 \xi}{16}}, \tag{21}$$

and similarly,

$$\int_{-\infty}^{\infty} \cos(\beta j t)\, e^{-\frac{t^2}{\xi}}\, dt = \sqrt{\pi\xi}e^{-\frac{\beta^2 j^2 \xi}{16}}. \tag{22}$$

Hence, we have the simplified expression,

$$E_1(x) = W_{r,\xi}(f,x) - f(x) - \tag{23}$$

$$-\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}}\right)\right] -$$

$$-\left(\frac{f^{(4)}(x) + \left(\alpha^2+\beta^2\right) f''(x)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\alpha^2 j^2 \xi}{4}}\right) - \alpha^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\beta^2 j^2 \xi}{4}}\right)\right] =$$

$$= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} R(t)\, e^{-\frac{t^2}{\xi}}\, dt.$$

Remark 2.1. Call the function

$$F := f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f. \tag{24}$$

Then, we get

$$R = R(t) = \frac{1}{\alpha\beta\left(\beta^2 - \alpha^2\right)} \sum_{j=0}^{r} \alpha_j j \int_0^t \left[F(x + jw) - F(x)\right] \cdot \tag{25}$$

$$\cdot \left[\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))\right] dw, \quad \forall\, t \in \mathbb{R}.$$

We isolate and study

$$I := \int_0^t \left[F(x + jw) - F(x)\right] \cdot \tag{26}$$

$$\cdot \left[\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))\right] dw, \quad \forall\, t \in \mathbb{R}.$$

For $t < 0$, we have that

$$|I| = \left| \int_t^0 \left[F(x + jw) - F(x)\right] \left[\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))\right] dw \right| \leq$$

$$\leq \int_t^0 |F(x + jw) - F(x)| \left|\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))\right| dw \leq \tag{27}$$

(by $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$)

$$\leq 2|\alpha||\beta| j \int_t^0 |F(x + jw) - F(x)| (w - t)\, dw =$$

$$= -2|\alpha||\beta| j \int_t^0 |F(x - j(-w)) - F(x)| (-t - (-w))\, d(-w) =$$

($t \leq w \leq 0 \Rightarrow -t \geq -w =: \theta \geq 0$)

$$= -2|\alpha||\beta| j \int_{-t}^0 |F(x - j\theta) - F(x)| (-t - \theta)\, d\theta =$$

$$= 2\left|\alpha\right|\left|\beta\right| j \int_{0}^{-t} \left|F\left(x - j\theta\right) - F\left(x\right)\right| \left(-t-\theta\right) d\theta =$$

$$= 2\left|\alpha\beta\right| j \int_{0}^{|t|} \left|F\left(x + sign\left(t\right) j\theta\right) - F\left(x\right)\right| \left(\left|t\right|-\theta\right) d\theta.$$

So, we have proved that

$$|I| \leq 2\left|\alpha\beta\right| j \int_{0}^{|t|} \left|F\left(x + sign\left(t\right) j\theta\right) - F\left(x\right)\right| \left(\left|t\right|-\theta\right) d\theta, \quad \forall\, t \in \mathbb{R}, \tag{28}$$

and, by (25),

$$\left|R\left(t\right)\right| \leq \frac{2}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} \left|\alpha_j\right| j^2 \int_{0}^{|t|} \left|F\left(x + jsign\left(t\right)\theta\right) - F\left(x\right)\right| \left(\left|t\right|-\theta\right) d\theta, \tag{29}$$

$\forall\, t \in \mathbb{R}$.

By (14), we have

$$E_1\left(x\right) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R\left(t\right) e^{-\frac{t^2}{\xi}} dt. \tag{30}$$

About Part II: On the smooth Poisson–Cauchy singular integral operators ([5])

Let $\overline{\alpha} \in \mathbb{N}$, $\overline{\beta} > \frac{1}{2\overline{\alpha}}$ and $f \in C^2\left(\mathbb{R}\right)$. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$M_{r,\xi}\left(f; x\right) = W \int_{-\infty}^{\infty} \frac{\sum_{j=0}^{r} \alpha_j f\left(x + jt\right)}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt, \tag{31}$$

where the constant is defined as

$$W = \frac{\Gamma\left(\overline{\beta}\right) \overline{\alpha} \xi^{2\overline{\alpha}\overline{\beta}-1}}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \frac{1}{2\overline{\alpha}}\right)}. \tag{32}$$

We assume that $M_{r,\xi}\left(f; x\right) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$M_{r,\xi}\left(f; x\right) = W \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} f\left(x + jt\right) \frac{1}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} dt \right). \tag{33}$$

We notice by $W\int\limits_{-\infty}^{\infty}\frac{1}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt=1$ that $M_{r,\xi}\left(c;x\right)=c$, c constant, and

$$M_{r,\xi}\left(f;x\right)-f\left(x\right)=W\left(\sum_{j=0}^{r}\alpha_j\int\limits_{-\infty}^{\infty}\left[f\left(x+jt\right)-f\left(x\right)\right]\frac{1}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}dt\right).\tag{34}$$

We set

$$\begin{aligned}
E_2\left(x\right) &:= M_{r,\xi}\left(f,x\right)-f\left(x\right)-\left(\frac{f'\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\cdot\\
&\cdot\left[\beta^3\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin\left(\alpha jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)-\right.\\
&\left.-\alpha^3\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin\left(\beta jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]-\left(\frac{f''\left(x\right)}{\beta^2-\alpha^2}\right)\cdot\\
&\cdot\left[\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\cos\left(\alpha jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)-\right.\\
&\left.-\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\cos\left(\beta jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]-\left(\frac{f^{(3)}\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\cdot\\
&\cdot\left[\beta\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin\left(\alpha jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)-\right.\\
&\left.-\alpha\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin\left(\beta jt\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]-\\
&-\left(\frac{2\left(f^{(4)}\left(x\right)+\left(\alpha^2+\beta^2\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^2\left(\beta^2-\alpha^2\right)}\right)\cdot\\
&\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)-\right.\\
&\left.-\alpha^2\sum_{j=0}^{r}\alpha_jW\left(\int\limits_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]=
\end{aligned}\tag{35}$$

$$= M_{r,\xi}(f,x) - f(x) - \left(\frac{2f''(x)}{\beta^2-\alpha^2}\right) W \cdot \tag{36}$$

$$\cdot \left[\sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} (\cos(\alpha j t) - \cos(\beta j t)) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right] -$$

$$-\left(\frac{4\left(f^{(4)}(x) + \left(\alpha^2+\beta^2\right) f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right) W \cdot$$

$$\cdot \left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right) -\right.$$

$$\left. -\alpha^2 \sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} \sin^2\left(\frac{\beta j t}{2}\right) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right] =$$

$$= W \int_{-\infty}^{\infty} R(t) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}. \tag{37}$$

That is

$$E_2(x) = W \int_{-\infty}^{\infty} R(t) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}. \tag{38}$$

About Part III: On the smooth Trigonometric singular integral operators ([5])

Let $\xi > 0$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, $\overline{\beta} \in \mathbb{N}$; we set

$$T_{r,\xi}(f;x) := \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt)\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt, \tag{39}$$

where

$$\lambda := \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}} dt = 2\xi^{1-2\overline{\beta}} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^{2\overline{\beta}} dt = \tag{40}$$

(by [11], p. 210, item 1033)

$$= 2\xi^{1-2\overline{\beta}}\pi\left(-1\right)^{\overline{\beta}}\overline{\beta}\sum_{k=1}^{\overline{\beta}}\left(-1\right)^{k}\frac{k^{2\overline{\beta}-1}}{\left(\overline{\beta}-k\right)!\left(\overline{\beta}+k\right)!}.$$

Denote

$$\lambda_1 := 2\pi\left(-1\right)^{\overline{\beta}}\overline{\beta}\sum_{k=1}^{\overline{\beta}}\left(-1\right)^{k}\frac{k^{2\overline{\beta}-1}}{\left(\overline{\beta}-k\right)!\left(\overline{\beta}+k\right)!}, \tag{41}$$

that is

$$\lambda = \lambda_1\xi^{1-2\overline{\beta}}. \tag{42}$$

We suppose that $T_{r,\xi}\left(f;x\right)\in\mathbb{R}$ for all $x\in\mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt = 1, \tag{43}$$

and $T_{r,\xi}\left(c;x\right)=c$, c constant, and

$$T_{r,\xi}\left(f;x\right)-f\left(x\right)=\frac{1}{\lambda}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}\left[f\left(x+jt\right)-f\left(x\right)\right]\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right). \tag{44}$$

We set

$$E_3\left(x\right):=T_{r,\xi}\left(f,x\right)-f\left(x\right)-\left(\frac{f'\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\cdot$$

$$\cdot\left[\beta^3\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin\left(\alpha jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right.$$

$$\left.-\alpha^3\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin\left(\beta jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]-$$

$$-\left(\frac{f''\left(x\right)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\cos\left(\alpha jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right.$$

$$-\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\cos\left(\beta jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\Bigg]-$$

$$-\left(\frac{f^{(3)}\left(x\right)}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin\left(\alpha jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right. \tag{45}$$

$$\left.-\alpha\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin\left(\beta jt\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]-$$

$$-\left(\frac{2\left(f^{(4)}\left(x\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^{2}\left(\beta^{2}-\alpha^{2}\right)}\right)\cdot$$

$$\cdot\left[\beta^{2}\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin^{2}\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right.$$

$$\left.-\alpha^{2}\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin^{2}\left(\frac{\beta jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]=$$

$$=T_{r,\xi}\left(f,x\right)-f\left(x\right)-\left(\frac{2f''\left(x\right)}{\beta^{2}-\alpha^{2}}\right)\frac{1}{\lambda}\cdot$$

$$\cdot\left[\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\left(\cos\left(\alpha jt\right)-\cos\left(\beta jt\right)\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]-$$

$$-\left(\frac{4\left(f^{(4)}\left(x\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^{2}\left(\beta^{2}-\alpha^{2}\right)}\right)\frac{1}{\lambda}\cdot \tag{46}$$

$$\cdot\left[\beta^{2}\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\sin^{2}\left(\frac{\alpha jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right.$$

$$-\alpha^2 \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} \sin^2 \left(\frac{\beta j t}{2} \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \Bigg] =$$

$$= \frac{1}{\lambda} \int_{-\infty}^{\infty} R(t) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt. \tag{47}$$

That is

$$E_3(x) = \frac{1}{\lambda} \int_{-\infty}^{\infty} R(t) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt. \tag{48}$$

We need

Remark 2.2. By (29) we get $(t \in \mathbb{R})$

$$|R(t)| \leq \frac{2}{|\beta^2 - \alpha^2|} \sum_{j=0}^{r} |\alpha_j| j^2 \int_0^{|t|} |F(x + j sign(t)\theta) - F(x)| (|t| - \theta) d\theta \leq$$

$(\xi > 0)$

$$\leq \frac{2}{|\beta^2 - \alpha^2|} \sum_{j=0}^{r} |\alpha_j| j^2 \int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right) (|t| - \theta) d\theta \leq \tag{49}$$

$$\leq \frac{2\omega_1(F,\xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \int_0^{|t|} \left(1 + \frac{j\theta}{\xi} \right) (|t| - \theta) d\theta \right] =$$

$$= \frac{2\omega_1(F,\xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left[\int_0^{|t|} (|t| - \theta) d\theta + \frac{j}{\xi} \int_0^{|t|} (|t| - \theta)^{2-1} (\theta - 0)^{2-1} d\theta \right] \right] =$$

$$= \frac{2\omega_1(F,\xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{\xi} \frac{|t|^3}{6} \right] \right] =$$

$$= \frac{\omega_1(F,\xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left[t^2 + j \frac{|t|^3}{3\xi} \right] \right].$$

Consequently, for $t \in \mathbb{R}$, we obtain

$$|R(t)| \leq \frac{\omega_1(F,\xi)}{|\beta^2-\alpha^2|}\left[\sum_{j=0}^{r}|\alpha_j|\, j^2\left[t^2+\frac{j}{3\xi}|t|^3\right]\right]. \tag{50}$$

3. Main results

Next we present uniform approximation by the $W_{r,\xi}$ operators.

Theorem 3.1. *It holds* ($\xi > 0$, $x \in \mathbb{R}$)

$$|E_1(x)| \leq \|E_1\|_\infty = \| W_{r,\xi}(f,x) - f(x) - \tag{51}$$

$$-\frac{f''(x)}{(\beta^2-\alpha^2)}\left[\sum_{j=0}^{r}\alpha_j\left(e^{-\frac{\alpha^2 j^2\xi}{16}} - e^{-\frac{\beta^2 j^2\xi}{16}}\right)\right] -$$

$$-\left(\frac{f^{(4)}(x)+(\alpha^2+\beta^2)f''(x)}{(\alpha\beta)^2(\beta^2-\alpha^2)}\right)\cdot$$

$$\cdot\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\alpha^2 j^2\xi}{4}}\right) - \alpha^2\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{\beta^2 j^2\xi}{4}}\right)\right]\Bigg\|_{\infty,x} \leq$$

$$\leq \frac{\sum_{j=0}^{r}|\alpha_j|\, j^2\left(\frac{\sqrt{\pi}}{2}+\frac{j}{3}\right)}{|\beta^2-\alpha^2|\sqrt{\pi}}\omega_1\left(f^{(4)}+(\alpha^2+\beta^2)f''+\alpha^2\beta^2 f, \sqrt{\xi}\right)\cdot\xi =: B_1 \to 0,$$

as $\xi \to 0$.

If $f''(x) = f^{(4)}(x) = 0$, *then*

$$|W_{r,\xi}(f,x) - f(x)| \leq B_1, \tag{52}$$

and $W_{r,\xi}(f,x) \to f(x)$, *as* $\xi \to 0$.

Proof. By (30) we get

$$|E_1(x)| \leq \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}|R(t)|\, e^{-\frac{t^2}{\xi}}dt \leq \tag{53}$$

(apply (50) with ξ to be $\sqrt{\xi}$)

$$\leq \frac{\omega_1\left(F,\sqrt{\xi}\right)}{\left|\beta^2-\alpha^2\right|\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_{-\infty}^{\infty}t^2e^{-\frac{t^2}{\xi}}dt+\frac{j}{3\sqrt{\xi}}\int_{-\infty}^{\infty}|t|^3e^{-\frac{t^2}{\xi}}dt\right]\right]=$$

$$=\frac{2\omega_1\left(F,\sqrt{\xi}\right)}{\left|\beta^2-\alpha^2\right|\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_{0}^{\infty}t^2e^{-\frac{t^2}{\xi}}dt+\frac{j}{3\sqrt{\xi}}\int_{0}^{\infty}t^3e^{-\frac{t^2}{\xi}}dt\right]\right]=$$

$$=\frac{2\omega_1\left(F,\sqrt{\xi}\right)}{\left|\beta^2-\alpha^2\right|\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\left(\sqrt{\xi}\right)^3\int_{0}^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^2e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\left(\frac{t}{\sqrt{\xi}}\right)+\right.\right.$$

$$\left.\left.+\frac{j}{3\sqrt{\xi}}\left(\sqrt{\xi}\right)^4\int_{0}^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^3e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\left(\frac{t}{\sqrt{\xi}}\right)\right]\right]=$$

$$=\frac{2\omega_1\left(F,\sqrt{\xi}\right)\xi}{\left|\beta^2-\alpha^2\right|\sqrt{\pi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_{0}^{\infty}x^2e^{-x^2}dx+\frac{j}{3}\int_{0}^{\infty}x^3e^{-x^2}dx\right]\right]=$$

$$=\frac{2\omega_1\left(F,\sqrt{\xi}\right)\xi}{\left|\beta^2-\alpha^2\right|\sqrt{\pi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{\sqrt{\pi}}{4}+\frac{j}{6}\right]\right].$$

Thus, we obtain

$$|E_1(x)|\leq\frac{\omega_1\left(F,\sqrt{\xi}\right)\xi}{\left|\beta^2-\alpha^2\right|\sqrt{\pi}}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{\sqrt{\pi}}{2}+\frac{j}{3}\right]\right].\tag{54}$$

The claim is proved. ∎

Next we present uniform approximation by the $M_{r,\xi}$ operators.

Theorem 3.2. *It holds* ($\xi>0$, $x\in\mathbb{R}$, $\overline{\beta}>\frac{2}{\overline{\alpha}}$, $\overline{\alpha}\in\mathbb{N}$)

$$|E_2(x)|\leq\|E_2\|_\infty=\|M_{r,\xi}(f,x)-f(x)-\tag{55}$$

$$-\left(\frac{2f''(x)}{\beta^2-\alpha^2}\right)W\left[\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\left(\cos(\alpha jt)-\cos(\beta jt)\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]-$$

$$-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)W\cdot$$

$$\cdot \left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} \sin^2 \left(\frac{\alpha j t}{2} \right) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right) - \right.$$

$$\left. \left. -\alpha^2 \sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} \sin^2 \left(\frac{\beta j t}{2} \right) \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right) \right] \right\|_{\infty, x} \leq$$

$$\leq \left\{ \frac{\left[\sum_{j=1}^{r} |\alpha_j| j^2 \left(\Gamma\left(\frac{3}{2\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \frac{3}{2\overline{\alpha}}\right) + \frac{j}{3} \Gamma\left(\frac{2}{\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \frac{2}{\overline{\alpha}}\right) \right) \right]}{|\beta^2 - \alpha^2| \Gamma\left(\frac{1}{2\overline{\alpha}}\right) \Gamma\left(\overline{\beta} - \frac{1}{2\overline{\alpha}}\right)} \right\} \cdot$$

$$\cdot \omega_1 \left(f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f, \xi \right) \cdot \xi^2 =: B_2 \to 0,$$

as $\xi \to 0$.

If $f''(x) = f^{(4)}(x) = 0$, *then*

$$|M_{r,\xi}(f, x) - f(x)| \leq B_2, \tag{56}$$

and $M_{r,\xi}(f, x) \to f(x)$, *as* $\xi \to 0$.

Proof. By (38) we get

$$|E_2(x)| \leq W \int_{-\infty}^{\infty} |R(t)| \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \leq \quad \text{(by (50))} \tag{57}$$

$$\leq \frac{W \omega_1(F, \xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left[\int_{-\infty}^{\infty} t^2 \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} + \frac{j}{3\xi} \int_{-\infty}^{\infty} |t|^3 \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right] \right] =$$

$$= \frac{2W \omega_1(F, \xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left[\int_0^{\infty} t^2 \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} + \frac{j}{3\xi} \int_0^{\infty} t^3 \frac{dt}{\left(t^{2\overline{\alpha}} + \xi^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right] \right] =$$

$$= \frac{2W \omega_1(F, \xi)}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^{r} |\alpha_j| j^2 \xi^{3 - 2\overline{\alpha}\overline{\beta}} \left[\int_0^{\infty} \left(\frac{t}{\xi}\right)^2 \frac{d\left(\frac{t}{\xi}\right)}{\left(\left(\frac{t}{\xi}\right)^{2\overline{\alpha}} + 1\right)^{\overline{\beta}}} + \right. \right.$$

$$\left. \left. + \frac{j}{3} \int_0^{\infty} \left(\frac{t}{\xi}\right)^3 \frac{d\left(\frac{t}{\xi}\right)}{\left(\left(\frac{t}{\xi}\right)^{2\overline{\alpha}} + 1\right)^{\overline{\beta}}} \right] \right] =$$

$$= \left(\frac{2\Gamma\left(\overline{\beta}\right)\overline{\alpha}\xi^2}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)} \right) \frac{\omega_1\left(F,\xi\right)}{\left|\beta^2-\alpha^2\right|} \cdot$$

$$\cdot \left[\sum_{j=0}^{r} \left|\alpha_j\right| j^2 \left[\int_0^{\infty} x^2 \frac{dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}} + \frac{j}{3} \int_0^{\infty} x^3 \frac{dx}{\left(1+x^{2\overline{\alpha}}\right)^{\overline{\beta}}} \right] \right] = \tag{58}$$

(by [13], p. 397, formula 595)

$$= \left(\frac{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)\left|\beta^2-\alpha^2\right|} \right) \xi^2\omega_1\left(F,\xi\right) \cdot$$

$$\cdot \left[\sum_{j=0}^{r} \left|\alpha_j\right| j^2 \left[\frac{\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)} + \frac{j}{3} \left(\frac{\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right)}{2\overline{\alpha}\Gamma\left(\overline{\beta}\right)} \right) \right] \right] =$$

$$= \left(\frac{\xi^2\omega_1\left(F,\xi\right)}{\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)\left|\beta^2-\alpha^2\right|} \right) \cdot$$

$$\cdot \left[\sum_{j=0}^{r} \left|\alpha_j\right| j^2 \left[\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right) + \frac{j}{3}\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right) \right] \right].$$

Thus, we derive

$$\left|E_2\left(x\right)\right| \leq \frac{\omega_1\left(F,\xi\right)\xi^2}{\left|\beta^2-\alpha^2\right|\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{1}{2\overline{\alpha}}\right)} \cdot \tag{60}$$

$$\cdot \left[\sum_{j=0}^{r} \left|\alpha_j\right| j^2 \left[\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{3}{2\overline{\alpha}}\right) + \frac{j}{3}\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta}-\frac{2}{\overline{\alpha}}\right) \right] \right].$$

The claim is proved. ∎

It follows the uniform approximation by the $T_{r,\xi}$ operators.

Theorem 3.3. *It holds* ($\xi > 0$, $x \in \mathbb{R}$, $\overline{\beta} \in \mathbb{N} - \{1,2\}$)

$$\left|E_3\left(x\right)\right| \leq \left\|E_3\right\|_{\infty} = \| T_{r,\xi}\left(f,x\right) - f\left(x\right) - \tag{61}$$

$$- \left(\frac{2f''\left(x\right)}{\beta^2-\alpha^2} \right) \frac{1}{\lambda} \left[\sum_{j=0}^{r} \alpha_j \left(\int_0^{\infty} \left(\cos\left(\alpha j t\right) - \cos\left(\beta j t\right)\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\overline{\beta}} dt \right) \right] -$$

$$
-\left(\frac{4\left(f^{(4)}(x)+\left(\alpha^{2}+\beta^{2}\right) f^{\prime \prime}(x)\right)}{(\alpha \beta)^{2}\left(\beta^{2}-\alpha^{2}\right)}\right) \frac{1}{\lambda} \cdot
$$

$$
\cdot\left[\beta^{2} \sum_{j=0}^{r} \alpha_{j}\left(\int_{0}^{\infty} \sin ^{2}\left(\frac{\alpha j t}{2}\right)\left(\frac{\sin \left(\frac{t}{\xi}\right)}{t}\right)^{2 \bar{\beta}} d t\right)-\right.
$$

$$
\left.\left.-\alpha^{2} \sum_{j=0}^{r} \alpha_{j}\left(\int_{0}^{\infty} \sin ^{2}\left(\frac{\beta j t}{2}\right)\left(\frac{\sin \left(\frac{t}{\xi}\right)}{t}\right)^{2 \bar{\beta}} d t\right)\right]\right\|_{\infty, x} \leq
$$

$$
\leq \frac{2 \omega_{1}\left(f^{(4)}+\left(\alpha^{2}+\beta^{2}\right) f^{\prime \prime}+\alpha^{2} \beta^{2} f, \xi\right) \xi^{2}}{\lambda_{1}\left|\beta^{2}-\alpha^{2}\right|} \cdot
$$

$$
\cdot\left[\sum_{j=1}^{r}\left|\alpha_{j}\right| j^{2}\left[\left(\frac{\pi(-1)^{\bar{\beta}-1}(2 \bar{\beta})!}{8(2 \bar{\beta}-3)!}\left(\sum_{k=1}^{\bar{\beta}}(-1)^{k} \frac{k^{2 \bar{\beta}-3}}{(\bar{\beta}-k)!(\bar{\beta}+k)!}\right)\right)+\right.\right.
$$

$$
\left.\left.+\frac{j}{3}\left(\frac{-(2 \bar{\beta})!}{8(2 \bar{\beta}-4)!}\left(\sum_{k=1}^{\bar{\beta}} \frac{(-1)^{\bar{\beta}-k} k^{2 \bar{\beta}-k} \ln (2 k)}{(\bar{\beta}-k)!(\bar{\beta}+k)!}\right)\right)\right]\right]=: B_{3} \rightarrow 0,
$$

as $\xi \rightarrow 0$.

If $f^{\prime \prime}(x)=f^{(4)}(x)=0,$ *then*

$$
|T_{r, \xi}(f, x)-f(x)| \leq B_{3}, \tag{62}
$$

and $T_{r, \xi}(f, x) \rightarrow f(x),$ *as* $\xi \rightarrow 0$.

Proof. By (48) we get

$$
\left|E_{3}(x)\right| \leq \frac{1}{\lambda} \int_{-\infty}^{\infty}|R(t)|\left(\frac{\sin \left(\frac{t}{\xi}\right)}{t}\right)^{2 \bar{\beta}} d t \leq \qquad \text{(by (50))} \tag{63}
$$

$$
\leq \frac{\omega_{1}(F, \xi)}{\lambda\left|\beta^{2}-\alpha^{2}\right|}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right| j^{2}\left[\int_{-\infty}^{\infty} t^{2}\left(\frac{\sin \left(\frac{t}{\xi}\right)}{t}\right)^{2 \bar{\beta}} d t+\right.\right.
$$

$$
\left.\left.+\frac{j}{3 \xi} \int_{-\infty}^{\infty}|t|^{3}\left(\frac{\sin \left(\frac{t}{\xi}\right)}{t}\right)^{2 \bar{\beta}} d t\right]\right]=
$$

$$= \frac{2\omega_1(F,\xi)}{\lambda|\beta^2-\alpha^2|}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\int_0^{\infty}t^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt+\right.\right.$$

$$\left.\left.+\frac{j}{3\xi}\int_0^{\infty}t^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right]\right]=$$

$$= \frac{2\omega_1(F,\xi)}{\lambda|\beta^2-\alpha^2|}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\xi^{3-2\overline{\beta}}\left[\int_0^{\infty}\left(\frac{t}{\xi}\right)^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}}\right)^{2\overline{\beta}}d\left(\frac{t}{\xi}\right)+\right.\right.$$

$$\left.\left.+\frac{j}{3}\int_0^{\infty}\left(\frac{t}{\xi}\right)^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}}\right)^{2\overline{\beta}}d\left(\frac{t}{\xi}\right)\right]\right]=$$

$$= \frac{2\omega_1(F,\xi)\,\xi^2}{\lambda_1|\beta^2-\alpha^2|}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\int_0^{\infty}x^2\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx+\frac{j}{3}\int_0^{\infty}x^3\left(\frac{\sin x}{x}\right)^{2\overline{\beta}}dx\right]\right]=$$

(by [11], p. 210, formula 1033)

$$= \frac{2\omega_1(F,\xi)\,\xi^2}{\lambda_1|\beta^2-\alpha^2|}\cdot$$

$$(64)\qquad \cdot\left[\sum_{j=1}^{r}|\alpha_j|\,j^2\left[\left(\frac{\pi(-1)^{\overline{\beta}-1}(2\overline{\beta})!}{8(2\overline{\beta}-3)!}\left(\sum_{k=1}^{\overline{\beta}}(-1)^k\frac{k^{2\overline{\beta}-3}}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)\right)+\right.\right.$$

$$\left.\left.+\frac{j}{3}\left(\frac{-(2\overline{\beta})!}{8(2\overline{\beta}-4)!}\left(\sum_{k=1}^{\overline{\beta}}\frac{(-1)^{\overline{\beta}-k}k^{2\overline{\beta}-k}\ln(2k)}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)\right)\right]\right].$$

The claim is proved. ■

We make

Remark 3.1. Let $\overline{E_i(x)}$ be $E_i(x)$, $i=1,2,3$, when $\alpha=2$ and $\beta=1$.

We have

$$(65)\qquad \overline{E_1(x)} = W_{r,\xi}(f,x)-f(x)+\frac{f''(x)}{3}\left[\sum_{j=0}^{r}\alpha_j\left(e^{-\frac{j^2\xi}{4}}-e^{-\frac{j^2\xi}{16}}\right)\right]+$$

$$+\left(\frac{f^{(4)}(x)+5f''(x)}{12}\right)\left[\sum_{j=0}^{r}\alpha_j\left(1-e^{-j^2\xi}\right)-4\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{j^2\xi}{4}}\right)\right].$$

We also have

(66)
$$\overline{E_2(x)} := M_{r,\xi}(f,x)-f(x)+\frac{2f''(x)}{3}W\cdot$$
$$\cdot\left[\sum_{j=0}^{r}\alpha_j\left(\int_0^{\infty}(\cos(2jt)-\cos(jt))\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right]+$$
$$+\left(\frac{4\left(f^{(4)}(x)+5f''(x)\right)}{12}\right)W\cdot$$
$$\cdot\left[\sum_{j=0}^{r}\alpha_j\left(\int_0^{\infty}\sin^2(jt)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)-\right.$$
$$\left.-4\sum_{j=0}^{r}\alpha_j\left(\int_0^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{dt}{\left(t^{2\overline{\alpha}}+\xi^{2\overline{\alpha}}\right)^{\overline{\beta}}}\right)\right].$$

Furthermore, it is

(67)
$$\overline{E_3(x)} = T_{r,\xi}(f,x)-f(x)+$$
$$+\frac{2f''(x)}{3\lambda}\left[\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}(\cos(2jt)-\cos(jt))\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right]+$$
$$+\left(\frac{4\left(f^{(4)}(x)+5f''(x)\right)}{12\lambda}\right)\cdot$$
$$\cdot\left[\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\sin^2(jt)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)-\right.$$
$$\left.-4\sum_{j=0}^{r}\alpha_j\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\overline{\beta}}dt\right)\right].$$

We finish with the following special results.

Corollary 3.1. (*to Theorem 3.1*) *It holds*

$$\left|\overline{E_1(x)}\right| \leq \left\|\overline{E_1}\right\|_\infty \leq \tag{68}$$

$$\leq \frac{\sum_{j=0}^{r} |\alpha_j| \, j^2 \left(\frac{\sqrt{\pi}}{2} + \frac{j}{3}\right)}{3\sqrt{\pi}} \omega_1 \left(f^{(4)} + 5f'' + 4f, \sqrt{\xi}\right) \cdot \xi \to 0,$$

as $\xi \to 0$.

Corollary 3.2. (*to Theorem 3.2*) *It holds*

$$\left|\overline{E_2(x)}\right| \leq \left\|\overline{E_2}\right\|_\infty \leq \tag{69}$$

$$\leq \left\{ \frac{\left[\sum_{j=1}^{r} |\alpha_j| \, j^2 \left(\Gamma\left(\frac{3}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta} - \frac{3}{2\overline{\alpha}}\right) + \frac{j}{3}\Gamma\left(\frac{2}{\overline{\alpha}}\right)\Gamma\left(\overline{\beta} - \frac{2}{\overline{\alpha}}\right)\right)\right]}{3\Gamma\left(\frac{1}{2\overline{\alpha}}\right)\Gamma\left(\overline{\beta} - \frac{1}{2\overline{\alpha}}\right)} \right\} \cdot$$

$$\cdot \omega_1 \left(f^{(4)} + 5f'' + 4f, \xi\right) \xi^2 \to 0,$$

as $\xi \to 0$.

Corollary 3.3. (*to Theorem 3.3*) *It holds*

$$\left|\overline{E_3(x)}\right| \leq \left\|\overline{E_3}\right\|_\infty \leq \tag{70}$$

$$\leq \frac{2\omega_1 \left(f^{(4)} + 5f'' + 4f, \xi\right) \xi^2}{3\lambda_1} \cdot$$

$$\cdot \left[\sum_{j=1}^{r} |\alpha_j| \, j^2 \left[\left(\frac{\pi (-1)^{\overline{\beta}-1} (2\overline{\beta})!}{8(2\overline{\beta}-3)!} \left(\sum_{k=1}^{\overline{\beta}} (-1)^k \frac{k^{2\overline{\beta}-3}}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)\right) + \right.\right.$$

$$\left.\left. + \frac{j}{3} \left(\frac{-(2\overline{\beta})!}{8(2\overline{\beta}-4)!} \left(\sum_{k=1}^{\overline{\beta}} \frac{(-1)^{\overline{\beta}-k} k^{2\overline{\beta}-k} \ln(2k)}{(\overline{\beta}-k)!(\overline{\beta}+k)!}\right)\right)\right]\right] \to 0,$$

as $\xi \to 0$.

References

[1] **Ali Hasan Ali and Zs. Páles,** Taylor-type expansions in terms of exponential polynomials, *Mathematical Inequalities and Applications*, **25(4)** (2022), 1123–1141.

[2] **Anastassiou, G.A.,** Basic convergence with rates of smooth Picard singular integral operators, *J. Computational Analysis and Appl.*, **8(4)** (2006), 313–334.

[3] **Anastassiou, G.A.,** *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 12, 2011.

[4] **Anastassiou, G.A.,** Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae, *Malaya J. Mathematik*, **11(S)** (2023), 1–26.

[5] **Anastassiou, G.A. and R. Mezei,** *Approximation by Singular Integrals*, Cambridge Scientific Publishers, Cambridge, UK, 2012.

[6] **Ali Aral,** On a generalized Gauss–Weierstrass singular integral, *Fasc. Math. No.*, **35** (2005), 23–33.

[7] **Ali Aral,** Pointwise approximation by the generalization of Picard and Gauss–Weierstrass singular integrals, *J. Concr. Appl. Math.*, **6(4)** (2008), 327–339.

[8] **Ali Aral,** *On generalized Picard integral operators*, Advances in summability and approximation theory, 157-168, Springer, Singapore, 2018.

[9] **Ali Aral, E. Deniz and H. Erbay,** The Picard and Gauss–Weiertrass singular integrals in (p, q)-calculus, *Bull. Malays. Math. Sci. Soc.*, **43(2)** (2020), 1569–1583.

[10] **Ali Aral and S.G. Gal,** q-generalizations of the Picard and Gauss–Weierstrass singular integrals, *Taiwanese J. Math.*, **12(9)** (2008), 2501–2515.

[11] **Edwards, J.,** *A Treatise on the Integral Calculus, Vol II*, Chelsea, New York, 1954.

[12] **Mohapatra, R.N. and R.S. Rodriguez,** On the rate of convergence of singular integrals for Hölder continuous functions, *Math. Nachr.*, **149** (1990), 117–124.

[13] **Zwillinger, D.,** *CRC Standard Mathematical Tables and Formulae*, 30th edn. Chapman & Hall/CRC, Boca Raton (1995).

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