

## ON THE EQUATION $F(p^k) = F(p^k - 1) + 1$

Imre Kátaı, Bui Minh Mai Khanh and Bui Minh Phong

(Budapest, Hungary)

*To the memory of Professor Pavel Varbanets*

*and Dr. Margit Kovács*

Communicated by Gábor Farkas

(Received April 17, 2024; accepted June 28, 2024)

**Abstract.** We prove that if an odd positive integer  $k$  and a completely multiplicative function  $F : \mathbb{N} \rightarrow \mathbb{C}$  satisfy the conditions

$$F(p^2) = F(p^2 - 1) + 1 \quad \text{and} \quad F(p^k) = F(p^k - 1) + 1 \quad \text{for every prime } p,$$

then  $F$  is the identity function. We also investigate completely functions  $F : \mathbb{N} \rightarrow \mathbb{R}$  such that  $F(p^2) = F(p^2 - 1) + 1$  is satisfied for every prime  $p$ .

### 1. Introduction

In the following let  $\mathcal{P}$ ,  $\mathbb{N}$ , and  $\mathbb{C}$  denote the set of primes, positive integers and complex numbers, respectively. We denote by  $\mathcal{M}$  ( $\mathcal{M}^*$ ) the set of all complex-valued multiplicative (completely multiplicative) functions, respectively.

Let  $\mathbb{I}(n) = n$  for every  $n \in \mathbb{N}$  and we define the function  $\mathbb{A}, \mathbb{B} \in \mathcal{M}^*$  as follows

$$\begin{cases} \mathbb{A}(2) = 0, \quad \mathbb{A}(3) = -1, \quad \mathbb{A}(p)^2 = 1 & \text{for every } p \in \mathcal{P}, p > 3 \\ \mathbb{B}(2) = -1, \quad \mathbb{B}(3) = 0, \quad \mathbb{B}(p)^2 = 1 & \text{for every } p \in \mathcal{P}, p > 3. \end{cases}$$

---

*Key words and phrases:* Completely multiplicative function, the identity function.

*2010 Mathematics Subject Classification:* primary 11A07, 11A25, secondary 11N25, 11N64.

<https://doi.org/10.71352/ac.57.047>

In the following for a  $k \in \mathbb{N}$  and  $\mathbb{T} \subseteq \mathbb{C}$  we denote by  $\mathcal{F}_k(\mathbb{T})$  the set of all  $F \in \mathcal{M}^*$  such that  $F(\mathbb{N}) \subseteq \mathbb{T}$  and

$$(1.1) \quad F(p^k) = F(p^k - 1) + 1 \quad \text{for every } p \in \mathcal{P}.$$

Let  $\mathcal{F}_k := \mathcal{F}_k(\mathbb{C})$ . It is obvious that  $\mathbb{A} \in \mathcal{F}_2$ ,  $\mathbb{B} \in \mathcal{F}_2$  and  $\mathbb{I} \in \mathcal{F}_k$  for every  $k \in \mathbb{N}$ .

In [1] and [2] we are given all solutions  $(D, G, F)$  of the equation

$$G(n) = F(n^2 - 1) + D \quad \text{for every } n \in \mathbb{N},$$

where  $G, F$  are completely multiplicative functions and  $D \in \mathbb{C}$ .

In this paper we would like to give all solutions  $F$  of the equation (1.1).

The question is simple if  $k = 1$ .

**Theorem 1.** *We have*

$$\mathcal{F}_1 = \{\mathbb{I}\}.$$

It seems to be hard even in the case  $k = 2$ . We can solve (1.1) for  $k = 2$ , if additionally some other property is assumed to be hold.

**Conjecture 1.** *For each prime  $Q > 2$  there is a prime  $\pi \in \mathcal{P}$  such that*

$$Q \mid (\pi - 1)(\pi + 1) \quad \text{and} \quad P\left(\frac{(\pi - 1)(\pi + 1)}{Q}\right) < Q,$$

where  $P(n)$  denotes the largest prime divisor of  $n$ .

**Theorem 2.** *If Conjecture 1 holds, then*

$$\mathcal{F}_2(\mathbb{R}) = \{\mathbb{A}, \mathbb{B}, \mathbb{I}\}.$$

**Conjecture 2.** *If  $F \in \mathcal{M}^*$ ,  $F(2) = 2$  and*

$$(1.2) \quad \begin{cases} F(p^2) &= F(p^2 - 1) + 1 \\ F(p^2) &= F(p^2 - 4) + 4 \end{cases}$$

for every  $p \in \mathcal{P}$ , then  $F = \mathbb{I}$ .

**Conjecture 3.** *Let  $c$  be a fixed positive number. Then for every  $Q \in \mathcal{P}$ ,  $Q > c$  there exists some  $\pi \in \mathcal{P}$ , for which*

$$A_\pi := (\pi + 1)(\pi - 1)(\pi + 2)(\pi - 2) \equiv 0 \pmod{Q}$$

and

$$\pi + 2 < Q^2, \quad P\left(\frac{A_\pi}{Q}\right) < Q.$$

**Theorem 3.** Assume that Conjecture 3 is true with some  $c > 1$ . Let  $F \in \mathcal{M}^*$ , for which  $F(p) = p$  holds for every prime  $p < c$ . Assume that (1.2) is satisfied. Then  $F = \mathbb{I}$ .

For  $\mathcal{F}_\ell \cap \mathcal{F}_k$  we would like to prove the following

**Conjecture 4.** If  $\ell, k \in \mathbb{N}$  and  $(\ell, k) = 1$ , then

$$\mathcal{F}_\ell \cap \mathcal{F}_k = \{\mathbb{I}\}.$$

In this note we prove Conjecture 4 for the cases  $\ell = 2$  and  $k$  is odd number.

**Theorem 4.** We have

$$\mathcal{F}_2 \cap \mathcal{F}_k = \{\mathbb{I}\} \quad \text{for every } k \in \mathbb{N}, (k, 2) = 1.$$

## 2. Proof of Theorem 1

Since  $\mathbb{I} \in \mathcal{F}_1$ , therefore we shall prove that if  $F \in \mathcal{F}_1$ , then  $F = \mathbb{I}$ .

Let  $F \in \mathcal{F}_1$ . Then  $F \in \mathcal{M}^*$  and

$$(2.1) \quad F(p) = F(p-1) + 1 \quad \text{for every } p \in \mathcal{P},$$

which with  $F(1) = 1$  implies that

$$F(2) = F(1) + 1 = 2, \quad F(3) = F(2) + 1 = 3, \quad F(5) = F(4) + 1 = F(2)^2 + 1 = 5.$$

Assume that  $F(n) = n$  for every  $n < P$ , where  $P > 5$ . We have  $F(P) = P$  if  $P \notin \mathcal{P}$ . Thus we can assume that  $P \in \mathcal{P}$ ,  $P \geq 7$  and  $F(P-1) = P-1$ . We infer from (2.1) that

$$F(P) = F(P-1) + 1 = (P-1) + 1 = P.$$

The proof of Theorem 1 is thus completed. ■

## 3. Proof of Theorem 2

First we prove the following

**Lemma 1.** If  $F \in \mathcal{F}_2$ , then

$$(3.1) \quad (F(2), F(3)) \in \{(0, -1), (-1, 0), (2, 3)\}$$

and

$$(3.2) \quad |F(n)| \leq n \quad \text{for every } n \in \mathbb{N}.$$

**Proof.** Let  $x := F(2)$ . Since  $F \in \mathcal{F}_2$ , we have  $F(p)^2 = F(p-1)F(p+1) + 1$  for every  $p \in \mathcal{P}$ , therefore

$$x^2 = F(2)^2 = F(1)F(3) + 1 = F(3) + 1 \quad \text{and} \quad F(3)^2 = F(2)F(4) + 1 = x^3 + 1.$$

These imply that

$$(x^2 - 1)^2 = x^3 + 1 \quad \text{and so} \quad x^2(x+1)(x-2) = 0.$$

Consequently,  $x \in \{0, -1, 2\}$ . This with  $F(3) = x^2 - 1$  proves (3.1).

Now we prove (3.2). It is clear from (3.1) that (3.2) holds for  $n \in \{1, 2, 3, 4\}$ . Assume that (3.2) holds for every  $n < N$ , where  $N \geq 5$ . Thus,

$$|F(N)| \leq N \quad \text{if} \quad N \notin \mathcal{P}.$$

If  $N \in \mathcal{P}$ , then we infer from  $F \in \mathcal{F}_2$  that

$$\begin{aligned} |F(N)^2| &= |F(N-1)F(N+1) + 1| \leq \\ &\leq |F(N-1)||F(N+1)| + 1 \leq (N-1)(N+1) + 1 = N^2. \end{aligned}$$

This proves (3.2), therefore the proof of Lemma 1 is finished.  $\blacksquare$

Now we prove Theorem 2. It is obvious that  $\{\mathbb{A}, \mathbb{B}, \mathbb{I}\} \subseteq \mathcal{F}_2(\mathbb{R})$ . Now we prove that  $\mathcal{F}_2(\mathbb{R}) \subseteq \{\mathbb{A}, \mathbb{B}, \mathbb{I}\}$ .

Let  $F \in \mathcal{F}_2(\mathbb{R})$ . According (3.1), we need to consider three cases:

**Case 1.**  $(F(2), F(3)) = (0, -1)$ . In this case,  $p^2 - 1 \equiv 0 \pmod{2}$  for every  $p \in \mathcal{P}$ ,  $p > 2$ , and so  $F(p^2 - 1) = 0$  for every  $p \geq 3$ . Consequently

$$F(p)^2 = F(p^2) = F(p^2 - 1) + 1 = 1 \quad \text{for every} \quad p \in \mathcal{P}, p \geq 3,$$

which proves that  $F = \mathbb{A}$ .

**Case 2.**  $(F(2), F(3)) = (-1, 0)$ . It is easily seen that

$$p^2 - 1 \equiv 0 \pmod{3} \quad \text{and so} \quad F(p^2 - 1) = 0 \quad \text{for every} \quad p \in \mathcal{P}, p \neq 3.$$

This implies that

$$F(p)^2 = F(p^2) = F(p^2 - 1) + 1 = 1 \quad \text{for every} \quad p \in \mathcal{P}, p \geq 3,$$

which proves that  $F = \mathbb{B}$ .

**Case 3.**  $(F(2), F(3)) = (2, 3)$ .

Now we prove  $F = \mathbb{I}$ . It is clear that  $F(n) = n$  holds for  $n \in \{1, 2, 3, 4\}$ . Assume that  $F(n) = n$  holds for every  $n < Q$ , where  $Q \geq 5$ . Assuming, by contradiction, that  $F(Q) \neq Q$ . Then  $Q \in \mathcal{P}$  and

$$\begin{aligned} F(Q)^2 &= F(Q^2) = F(Q^2 - 1) + 1 = F(2)^2 F\left(\frac{Q-1}{2}\right) F\left(\frac{Q+1}{2}\right) + 1 = \\ &= 4 \frac{Q-1}{2} \frac{Q+1}{2} + 1 = Q^2, \end{aligned}$$

consequently  $F(Q) = -Q$ .

On the other hand, it follows from Conjecture 1 that there exists  $\pi \in \mathcal{P}$  such that

$$Q \mid (\pi - 1)(\pi + 1) \quad \text{and} \quad P\left(\frac{(\pi - 1)(\pi + 1)}{Q}\right) < Q.$$

Consequently

$$F\left(\frac{\pi^2 - 1}{Q}\right) = \frac{\pi^2 - 1}{Q}$$

and

$$\begin{aligned} F(\pi)^2 &= F(\pi^2) = F(\pi^2 - 1) + 1 = F(Q)F\left(\frac{\pi^2 - 1}{Q}\right) + 1 = -Q\frac{\pi^2 - 1}{Q} + 1 = \\ &= -\pi^2 + 2 < 0. \end{aligned}$$

This is impossible by our assumption  $F \in \mathcal{F}_2(\mathbb{R})$ .

Theorem 2 is proved. ■

#### 4. Proof of Theorem 3

Assume that  $F(n) = n$  holds for every  $n < N$ . If  $N \leq c$  or  $N$  is not a prime, then  $F(N) = N$  is true as well. Assume that  $N = Q \in \mathcal{P}$  and  $Q > c$ . Since  $P((Q - 1)(Q + 1)) < Q$ , then (1.1) implies that  $F(Q) = Q^2$ .

Let  $\pi$  be according to Conjecture 3. Thus  $\pi < Q^2, Q \mid A_\pi, P(\frac{A_\pi}{Q}) < Q$ . We have two cases:

Case a):  $Q \mid (\pi - 1)(\pi + 1)$ ,

Case b):  $Q \mid (\pi - 2)(\pi + 2)$ .

In case a) we have  $F(\pi^2) = F(\pi^2 - 4) + 4 = (\pi^2 - 4) + 4 = \pi^2$ , since  $P(\pi^2 - 4) < Q$ . Then we infer from the following relation

$$\pi^2 = F(\pi^2) = F(\pi^2 - 1) + 1 = F\left(\frac{\pi^2 - 1}{Q}\right)F(Q) + 1 = \frac{\pi^2 - 1}{Q}F(Q) + 1$$

that  $F(Q) = Q$ .

The proof in the case b) is similarly. Since  $Q \mid (\pi - 2)(\pi + 2)$  and  $P(\pi^2 - 1) < Q$ , thus  $F(\pi^2) = F(\pi^2 - 1) + 1 = (\pi^2 - 1) + 1 = \pi^2$  and

$$\pi^2 = F(\pi^2) = F(\pi^2 - 4) + 4 = F\left(\frac{\pi^2 - 4}{Q}\right)F(Q) + 4 = \frac{\pi^2 - 4}{Q}F(Q) + 4,$$

which implies that  $F(Q) = Q$ .

By using infinite induction we proved that  $F(Q) = Q$  for every  $Q \in \mathcal{P}$ . The proof of Theorem 3 is completed. ■

## 5. Proof of Theorem 4

**Lemma 2.** *If*

$$(5.1) \quad F \in \mathcal{F}_2 \cap \mathcal{F}_k \quad \text{for some } k \in \mathbb{N}, (k, 2) = 1,$$

*then*

$$F(n) = n \quad \text{for every } n \in \{1, 2, 3, 4\}.$$

**Proof.** We shall prove that if  $F$  satisfies (5.1) then  $(F(2), F(3)) = (2, 3)$ . According (3.1), we need to show that

$$(F(2), F(3)) \notin \{(0, -1), (-1, 0)\}.$$

Assume first that  $(F(2), F(3)) = (0, -1)$ . Since  $(k, 2) = 1$ ,  $F(2) = 0$  and  $F \in \mathcal{F}_k$ , we have

$$-1 = (-1)^k = F(3)^k = F(3^k - 1) + 1 = 1,$$

which is a contradiction.

Now assume that  $(F(2), F(3)) = (-1, 0)$ . Then

$$p^2 - 1 \equiv 0 \pmod{3} \quad \text{and} \quad F(p^2 - 1) = 0 \quad \text{for every } p \in \mathcal{P}, p \neq 3,$$

consequently

$$(5.2) \quad F(p)^2 = F(p-1)F(p+1) + 1 = 1 \quad \text{for every } p \in \mathcal{P}, p \neq 3.$$

On the other hand, we have

$$(5.3) \quad -1 = F(2)^k = F(2^k) = F(2^k - 1) + 1 \quad \text{and so } F(2^k - 1) = -2.$$

Since  $(2^k - 1, 3) = 1$ , the relation (5.2) implies that  $F(2^k - 1) = \pm 1$ , which contradicts to (5.3). The proof of Lemma 2 is completed.  $\blacksquare$

Now we prove Theorem 4.

By using Lemma 2, we assume that  $F(n) = n$  for every  $n < P$ , where  $P > 4$ . We shall prove that  $F(P) = P$ .

Assume by contradiction that  $F(P) \neq P$ . Then  $P \in \mathcal{P}$  and  $P \geq 5$ . It is clear from  $F \in \mathcal{F}_2$  and our assumptions that

$$\begin{aligned} F(P)^2 &= F(P-1)F(P+1) + 1 = F(2)(P-1)F\left(\frac{P+1}{2}\right) + 1 = \\ &= 2(P-1)\left(\frac{P+1}{2}\right) + 1 = P^2. \end{aligned}$$

Thus we infer from our assumption  $F(P) \neq P$  that  $F(P) = -P$ . Then from  $(k, 2) = 1$  and  $F \in \mathcal{F}_k$ , we have

$$-P^k = F(P)^k = F(P^k) = F(P^k - 1) + 1$$

consequently

$$|F(P^k - 1)| = |-P^k - 1| = P^k + 1,$$

which contradicts to (3.2).

Theorem 4 is proved. ■

## References

- [1] **Kátai, I., B.M.M. Khanh and B.M. Phong**, On the equation  $G(n) = F(n^2 - 1) + D$ , *Ann. Univ. Sci. Budapest., Sect. Comp.*, **53** (2022), 159–173.
- [2] **Kátai, I. and B.M. Phong**, Completely multiplicative functions with special properties, *Mathematica Pannonica New Series*, **29/NS3**, Issue 1, (2023), 68–76.

**I. Kátai, B.M.M. Khan and B.M. Phong**

Department of Computer Algebra

Faculty of Informatics

Eötvös Loránd University

H-1117 Budapest

Pázmány Péter sétány 1/C

Hungary

katai@inf.elte.hu

mbuiminh@yahoo.com

bui@inf.elte.hu

