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REMARKS ON DYADIC ANALYSIS

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Abstract. About 100 years ago, N.J. Walsh's fundamental paper [25] was published, in which he introduced the digital version of the trigonometric system. In remembrance of this and the 50th anniversary of Walsh's death, the authors of the paper [2] presented the role of Walsh functions in dyadic analysis and technical applications. 35 years ago, a collaboration between researchers from the Department of Numerical Analysis, Eötvös Loránd University and Professor W.R. Wade (University of Tennessee, USA) resulted in the publication of the first monograph on dyadic analysis. This provided an overview of the significant results in the field before 1990. Since then, several promising results have been achieved that may determine the future direction of research. This paper provides a brief overview of these results.

In the commemorations prepared for the anniversary of the department's establishment, we present in detail our contributions to the achievements in the field. Here, the author only highlights the following. Regarding the Vilenkin generalization of the Walsh system, the interpretation of the concept of the conjugate function and the proof of the corresponding fundamental inequalities were significant [21]. The author of the [11] paper introduced the dyadic analogues of Hermite functions as eigenfunctions of the dyadic derivative and pointed out their application possibilities. In harmonic analysis, the examination of multiplier operators and the corresponding filtering procedures in signal processing is a central theme of research. The strong approximation, two-sided Sidon-type inequalities, and Hardy-type spaces related to this have proven to be of fundamental importance in both the trigonometric and dyadic cases [4]. It would be worthwhile to extend these results to Malmquist–Takenaka systems. New, significant results have also been achieved in the extension of multivariable dyadic analysis, traditional stochastic structures, and function spaces [26, 27]. The [6] paper provides insights into the studies related to the direct product of finite, non-commutative groups.

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1. Introduction

Nowadays, one of the determining factors in the development of mathematics is computer science. Computers have entered every area of life, and digitization has become a fashionable buzzword.

In the field of harmonic analysis, digitization began with a paper written by Walsh in 1923 [25]. In this work, the author introduced a system of functions that depends only on the binary digits of the independent variable, takes only the values 1 and -1, and mimics the sign changes of the trigonometric sine and cosine systems. This system of functions, named after Walsh, can represent the first 2^N terms without error on N-length bytes. Since the 1960s, this property has led engineers working in communication technology to replace the traditionally used trigonometric system with the Walsh system [7, 8].

Studies related to Walsh series, which are nowadays often included in the topic of dyadic analysis, have influenced the development of several branches of mathematics, such as probability theory and martingale theory [14, 16]. They have played a significant role in solving problems related to bases and in the development of fundamental concepts and procedures [14].

The results in this field up to 1990 are summarized in the monograph *Walsh series, an introduction to dyadic harmonic analysis* (Adam Hilger, Bristol-New York, 1990) [14]. Since then, numerous results have been achieved in this area. In this compilation, we draw attention to some problems where the digital approach may prove useful and may serve as a starting point for further investigations.

In computers, information is represented by bit sequences (bytes), and classes of bit sequences are used to model tasks. Various operations (algebraic, logical, etc.) are interpreted among the elements of these classes. From the point of view of numerical and algebraic calculations, it is useful to consider continuous algebraic structures which form a complete space and are closed under arithmetic operations.

Let $\mathbb{B}$ denote the set of (left-finite) bit sequences $\dot{x} = (x_n \in \mathbb{Z}_2, n \in \mathbb{Z})$ $(\mathbb{Z}_2 := \{0,1\}, \lim_{n\to-\infty} x_n = 0)$. For every $\dot{x} \in \mathbb{B}$, $\dot{x} \neq \dot{o} := (o_n = 0, n \in \mathbb{Z})$, there exists a number $\pi(x) \in \mathbb{Z}$ such that $x_n = 1$ if $n = \pi(x)$ and $x_n = 0$ if $n < \pi(x)$. Using the mapping $\dot{x} \to x := \sum_{n=\pi(x)}^{\infty} x_n 2^{-(n+1)}$ (using the binary expansion of the number $x \in \mathbb{R}_+$), we associate a non-negative real number with the elements of $\mathbb{B}$. Let $\mathbb{B}_0$ be the set of right-finite bit sequences $\dot{x} = (x_n, n \in \mathbb{Z}) \in \mathbb{B}$ $(\lim_{n\to\infty} x_n = 0)$. The image of these, $\mathbb{Q} := \{x : \dot{x} \in \mathbb{B}_0\} \subset \mathbb{R}_+$, is called dyadic rational numbers. The mapping $\dot{x} \to x$ is not bijective: each number $x \in \mathbb{Q}$ has two preimages. Let $\mathbb{B}_1$ denote the (countable) set of bit sequences for which $\lim_{n\to\infty} x_n = 1$. Then the mapping $\dot{x} \to x$ is a bijection between the sets $\mathbb{B} \setminus \mathbb{B}_1$ and $\mathbb{R}_+$. Based on this, in many cases, the set of bit sequences can be identified with non-negative real numbers, and the $L^p(\mathbb{B})$ spaces with the $L^p(\mathbb{R}_+)$ spaces.

For any $a = k2^{-n} \in \mathbb{Q}, k \in \mathbb{N}, n \in \mathbb{Z}$, introduce the sets of the form $I_n(a) := \{x \in \mathbb{R}_+ : a \le x < a+2^{-n}\}$ (and their preimages), the dyadic intervals of $\mathbb{R}_+$ (or $\mathbb{B}$). These not only serve as the basis for representing numbers and interval arithmetic but also enable the introduction of an important stochastic structure, called dyadic martingales [14].

On the set $\mathbb{B}$, starting from the set $\mathbb{B}_0$ identified with dyadic rational numbers, we can introduce three algebraic structures (three fields), each of which plays a decisive role in describing the real world. The first structure is the dyadic (or 2-series) field [22], in which the operations of addition and multiplication are defined by the formulas

$$\dot{x} \oplus \dot{y} = (x_n + y_n \pmod 2, n \in \mathbb{Z}),$$
$$\dot{x} \odot \dot{y} = (z_n, n \in \mathbb{Z}),\ z_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k} \pmod 2\ (n \in \mathbb{Z}).$$

The set of bytes $\mathbb{B}$ with these operations forms a field, where the zero element is the sequence $\dot{o}$, the unit element is the sequence $\dot{e} = (\delta_{k0}, k \in \mathbb{Z})$, each element is its own additive inverse, and every element different from $\dot{o}$ has a multiplicative inverse. The function $\|\dot{x}\| := 2^{-\pi(x)}$ ($\dot{x} \in \mathbb{B}$) is a (non-Euclidean) norm:

$$\|\dot{x} \oplus \dot{y}\| \le \max\{\|\dot{x}\|, \|\dot{y}\|\}, \quad \|\dot{x} \odot \dot{y}\| = \|\dot{x}\|\|\dot{y}\| \quad (\dot{x}, \dot{y} \in \mathbb{B}).$$

From the above, it follows that the field operations are continuous, $\mathbb{B}_0$ is dense in $\mathbb{B}$, and the field $\mathbb{B}$ is complete. The operation $\oplus$ is identical to the EOR logical operation. On this basis, the dyadic field will also be referred to as the logical field. The statement formulated now can be interpreted as embedding the algebraic structure $(\mathbb{B}_0, \oplus, \odot)$ (ring) into a complete field.

The preimages of the intervals $I_n(0)$ denoted by $\dot{I}_n(0)$ ($n \in \mathbb{Z}$), are subgroups of $(\mathbb{B}, \oplus)$. Specifically, the subgroup $\mathbb{D} = \dot{I}_0(0)$ is called the Cantor group.

On the set of dyadic rational numbers $\mathbb{Q} \equiv \mathbb{B}_0$, the usual operations of $+$ and $\cdot$ (multiplication and addition) are defined, as well as the Euclidean distance $\rho_E(x, y) := |x - y|$ ($x, y \in \mathbb{Q}$). It is known that $\mathbb{Q}$ can be embedded in a complete field (with respect to the ρ_E metric) in which $\mathbb{Q}$ is a dense subset everywhere. Through this process, starting from $\mathbb{B}_0$, we can reach the field of real numbers $(\mathbb{R}, +, \cdot)$ and the Euclidean geometry compatible with it. One consequence of this is that segments of arbitrary (large and small) lengths exist. This property of real numbers is usually highlighted under the name of the Euclidean axiom.

On the set $\mathbb{B}_0$, introduce the mapping $\pi_-(x) \in \mathbb{Z}$, where $x_n = 1$ if $n = \pi_-(x)$ and $x_n = 0$ if $n > \pi_-(x)$ $(x \in \mathbb{B}_0)$. On the semigroup $(\mathbb{Q}, +) \equiv (\mathbb{B}_0, +)$, the mapping $\|x\|_* := 2^{-\pi_-(x)}$ $(x \in \mathbb{B}_0)$ is a non-Euclidean norm:

$$\|x + y\|_* \leq \max\{\|x\|_*, \|y\|_*\} \quad (x, y \in \mathbb{B}_0).$$

The algebraic structure $(\mathbb{B}_0, +, \cdot) \equiv (\mathbb{Q}, +, \cdot)$ (semiring?) can be embedded in a complete field $(\mathbb{B}_*, +, \cdot)$ with respect to the norm $\| \cdot \|_*$, in which $\mathbb{B}_0$ is dense. The field thus obtained is called the 2-adic field in the literature. Hereafter, we will refer to this as the arithmetic field [22].

According to some views, in physics, sizes smaller than the Planck constant and, in astronomy, sizes larger than the radius of the universe have no physical reality. The Archimedean axiom of real numbers ensures that segments of any length, both very long and very short, exist. This is often used to argue that real numbers should be replaced with non-Archimedean fields [24].

Among other reasons, this is the background for mentioning the arithmetic field in addition to the logical field [23]. The book *Transforms in Normed Fields* [15] was written with the intention of transferring the Fourier techniques, which form the basis of wavelet constructions, to the mentioned fields.

In this compilation, we only deal with the logical field. In $\mathbb{R}$, by using the logarithmic function, we can establish a connection between the additive and multiplicative structures. It follows that there is essentially no difference between the corresponding Fourier and Mellin transformations. In the case of logical and arithmetic fields, the relationship between the additive and multiplicative structures is not so simple, consequently, the examination of the counterparts of the Mellin transformation requires new, special considerations [15]. It would be worthwhile to study these types of Fourier transformations and their applications in depth.

2. The Rademacher and Walsh systems

In a paper written 100 years ago [25], *Walsh* introduced a system that has since been named after him, which, in today's fashionable terminology, can be considered the *digital version of the trigonometric system* (see Figure 1). The Walsh functions take only the values 1 and -1 and the number of sign changes matches that of the trigonometric system. Furthermore they form a complete orthonormal system in the space $L^2[0, 1]$. Hereafter, using the mapping $\dot{x} \to x$, we identify the dyadic group $\mathbb{D}$ with the interval $[0, 1)$.

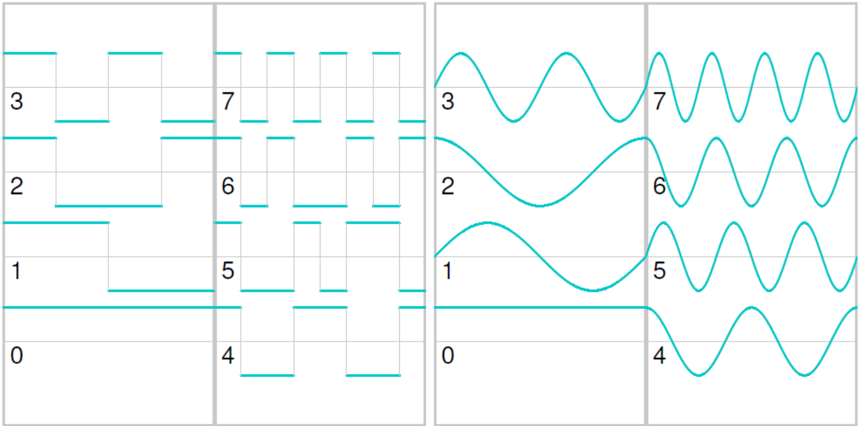

Figure 1. The Walsh and the trigonometric systems.

Walsh defined his system with a complex recursion that is difficult to handle. The Walsh system is a subsystem of the system $(r_k, k \in \mathbb{N})$ introduced by Rademacher in 1922 (see Figure 2). The values of the functions r_k can be expressed using the binary digits of the number $x \in [0,1)$:

$$r_k(x) = (-1)^{x_k} \quad \left(x = \sum_{k=0}^{\infty} x_k/2^{k+1} \in [0,1) \right).$$

Studies related to the Rademacher system later contributed to the theoretical foundations of probability theory [16].

In a fundamental paper written in 1932 [10], *Paley* demonstrated that the Walsh functions, apart from the order, can be generated using the Rademacher functions. Specifically, Paley introduced all possible finite products formed from the Rademacher functions. Using the binary digits $n_k \in \{0,1\}$ of the number

$$\dot{n} \to n = \sum_{k=0}^{\infty} n_k 2^k \in \mathbb{N},$$

we can write them in the form

$$w_n = r_0^{n_0} r_1^{n_1} \dots r_k^{n_k} \dots .$$

Using the terminology that has since become widespread, we say that the $W = (w_n, n \in \mathbb{N})$ (the so-called Walsh–Paley system) is the *product system of the Rademacher system.* It can be shown that the original Walsh system itself can also be generated as a product system of the $(r_k r_{k-1}, k \in \mathbb{N})$ $(r_{-1} := 1)$

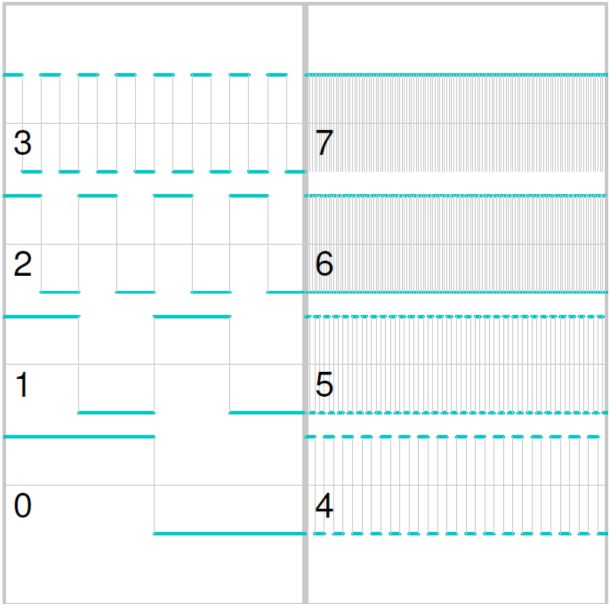

Figure 2. The Rademacher system.

system. From this, it follows that the two systems consist of the same functions, written in different orders.

The digital nature of the connection becomes even more striking when the group $\mathbb{D}$ is written in the form of the direct product

$$\mathbb{D} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \mathbb{Z}_2 \times \cdots ,$$

where $\mathbb{Z}_2 = (\{0,1\}, \oplus, \cdot)$ denotes the finite field with two elements. The subgroup $\mathbb{D}_0 := \mathbb{D} \cap \mathbb{B}_0$ can be identified with the set of natural numbers $\mathbb{N}$ based on the mapping $\dot{n} \to n := \sum_{k=0}^{\infty} n_k 2^k$. Accomrdingly, $\mathbb{D}$ can be considered a (complete) linear normed space (Banach space) over the field $\mathbb{Z}_2$. The function

$$\lfloor n, x \rfloor = \sum_{k \in \mathbb{N}} n_k x_k \pmod 2 \ (n \in \mathbb{N} \equiv \mathbb{D}_0, x \in [0,1) \equiv \mathbb{D})$$

is a $\mathbb{Z}_2$ bilinear mapping on the space $\mathbb{D}_0 \times \mathbb{D}$. The Walsh-Paley system can be expressed using binary digits:

$$w_n(x) = (-1)^{\lfloor n,x \rfloor} \quad (n \in \mathbb{N}, x \in [0,1)).$$

For every $\mathbb{Z}_2$-linear mapping $T : \mathbb{N} \to \mathbb{N}$, there exists a unique $\mathbb{Z}_2$-linear mapping $T^* : \mathbb{D} \to \mathbb{D}$ such that

$$\lfloor Tn, x \rfloor = \lfloor n, T^* x \rfloor \quad (n \in \mathbb{N}, x \in \mathbb{D})$$

is satisfied. If $T : \mathbb{N} \to \mathbb{N}$ is bijective, then $T^* : \mathbb{D} \to \mathbb{D}$ is measure-preserving, and furthermore,

$$w_{Tn}(x) = w_n(T^* x) \quad (n \in \mathbb{N}, x \in \mathbb{D}).$$

Measure-preserving mappings transform orthonormal systems into orthonormal systems. Applying the mapping T^*, we do not obtain a new system, but rather a rearrangement of the original system [18, 14, 12].

We can reach several systems widely used in mathematics with this construction. Specifically, the original Walsh system can be derived from the Walsh–Paley system using the

$$\dot{\mathcal{T}} := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & & & & \end{pmatrix} \tag{2.1}$$

$\mathbb{Z}_2$-linear transformation. With this mapping, we have established a well-manageable passage between the original Walsh system, preferred in technical applications, and the Walsh-Paley system used in mathematical literature.

In the literature, other $\mathbb{Z}_2$-linear mappings are also used. For example, the

$$\begin{aligned} \mathcal{F}_k :&(n_0, n_1, \cdots, n_{k-1}, n_k, n_{k+1}, \cdots) \to \\ &\to (n_{k-1}, n_{k-2}, \cdots, n_0, n_k, n_{k+1}, \cdots) \quad (k \in \mathbb{N}) \end{aligned}$$

bit-reversal transformations define the *Hadamard matrices* and the *Kaczmarz rearrangements*. Note that these transformations also play an important role in FFT algorithms [19, 14].

In mathematics, several mappings have been introduced that can be simply described in the language of bits. An example is the operation of bit interleaving, which is related to the Peano mapping. This was suggested by Frigyes Riesz for the mathematical treatment of multivariable problems [13]. The mapping

$$\dot{x} \circledast \dot{y} := (x_0, y_0, x_1, y_1, \cdots) \quad (x, y \in \mathbb{D})$$

establishes a bijective, measure-preserving correspondence between the sets $\mathbb{D} \times \mathbb{D}$ and $\mathbb{D}$, whose restriction $\circledast : \mathbb{D}_0 \times \mathbb{D}_0 \to \mathbb{D}_0$ is also a bijection. Based on this, there is a relationship between the one- and two-variable Walsh–Paley systems [14]:

$$(w_n \times w_m)(x, y) = w_{n \circledast m}(x \circledast y) \quad (x, y \in \mathbb{D}, m, n \in \mathbb{D}_0).$$

Thus, some questions related to multivariable Walsh analysis can be reduced to uni-variate problems. It would be worthwhile to reconsider Hilbert's construction given for the Peano curve based on the above.

3. Special function and sequence spaces

The Walsh functions are the continuous solutions of the functional equation

$$w(x \oplus y) = w(x)w(y) \quad (x, y \in \mathbb{D}),$$

or, in other terms, they are the characters (trigonometric functions) of the group $(\mathbb{D}, \oplus)$. Following Fine's observation in 1949, investigations related to Walsh functions have been classified into the specific chapter of abstract harmonic analysis known as dyadic analysis [3]. It is worth noting that two years earlier, Vilenkin introduced a class of orthogonal systems that include the Walsh system. Several instructors from our department have significantly contributed to these research efforts [23].

In the monograph [14], most essential questions of harmonic analysis are discussed in detail. Besides the traditional topics (convergence, summation, dyadic derivative, approximation, Walsh–Fourier transform, etc.), we also address related systems (Haar, Franklin, etc.), martingale spaces, and the basis problem. The almost everywhere convergence is approached in a new manner by extending the concept of martingales.

In this compilation, we highlight some issues that seem relevant since the publication of the book.

Let $\mathcal{J}_n$ $(n \in \mathbb{N})$ denote the set of dyadic subintervals of length 2^{-n} in the interval $[0, 1)$, and let $\mathcal{A}_n := \sigma(\mathcal{J}_n)$ $(n \in \mathbb{N})$ denote the stochastic basis generated by these. The set $\mathcal{J}$ with the inclusion relation $\subseteq$ is a partially ordered set, or in graph theory terminology, a binary tree.

In mathematics, physics, and the theory of stochastic processes, the time domain is often modeled with a linearly ordered set (e.g., $\mathbb{N}$ or $\mathbb{R}_+$). In describing branching processes, their role is taken over by graphs (e.g., the $\mathcal{J}$ binary tree). Recently (in connection with deep learning), concepts and methods of harmonic analysis (Euclidean spaces, harmonic functions, convolution, etc.) have been successfully applied in connection with graphs. In addition, we demonstrate that the results of dyadic analysis can be incorporated into this perspective in the context of binary trees [1]. In addition to Euclidean structures, other important Banach spaces can be introduced in the sequence space indexed by the elements of the $\mathcal{J}$ tree.

In the space of sequences indexed by $\mathbb{N}$, the ℓ^2 space plays a distinguished role. According to the Riesz-Fischer theorem, every separable Hilbert space is isometrically isomorphic to the sequence space ℓ^2. Assigning the sequence of Fourier coefficients according to some complete orthonormal system to the elements of the Hilbert space, the isomorphism can be explicitly described. Such a characterization does not exist for other types of Banach spaces. However, certain classes of function spaces (such as L^p, H^p, and VMO spaces) can be characterized using certain types of sequence spaces (utilizing Haar, Franklin, wavelet, and Fourier coefficients).

The partial sums of Walsh-Fourier series with index 2^n are identical to the conditional expectation with respect to $\mathcal{A}_n$:

$$S_{2^n} f := \sum_{k=0}^{2^n-1} \langle f, w_k \rangle w_k = E_n f = E(f|\mathcal{A}_n) \quad (f \in L^1[0,1), n \in \mathbb{N}).$$

In his cited work, Paley proved a notable inequality, which was later found to be crucial in the study of martingale theory and the rearrangements of function series [10]. Let

$$Qf := \left(\sum_{n=0}^{\infty} |E_n f - E_{n-1} f|^2 \right)^{1/2} \quad (E_{-1} f := 0, \quad f \in L^1)$$

be the quadratic variation of f. Paley proved that

$$\|Qf\|_p \sim \|f\|_p \quad (1 < p < \infty). \tag{3.1}$$

It is also known that for the maximal function $f^* := \sup_{n \in \mathbb{N}} |E_n f|$,

$$\|f^*\|_p \sim \|f\|_p \quad (1 < p \leq \infty), \ \|f^*\|_1 \sim \|Qf\|_1 \nsim \|f\|_1.$$

Using these, we can define the dyadic Hardy spaces:

$$H^p := \{ f \in L^1 : \|f\|_{H^p} := \|Qf\|_p < \infty \} \quad (1 \leq p < \infty).$$

Then based on (3.1),

$$H^p = L^p \ (1 < p < \infty), \ H^1 \subsetneq L^1.$$

The dual of the H^1 dyadic Hardy space can be given using the average oscillation of the function f:

$$O_n f := \|(E_n |f - E_n f|^2)^{1/2}\|_\infty$$

in the following form:

$$BMO := \{ f \in L^1 : \|f\|_{BMO} := \sup_{n \in \mathbb{N}} O_n f < \infty \}.$$

The space

$$VMO := \{ f \in BMO : \lim_{n \to \infty} O_n f = 0 \}$$

is a (separable, closed) subspace of BMO, whose dual is the H^1 Hardy space.

The Walsh system is closely related to the orthonormal system introduced by Alfred Haar in 1910, which later played a significant role in solving some problems related to bases and became the starting point for research on wavelets [16, 17, 14]. For indexing Haar functions, it is practical to use dyadic intervals besides natural numbers, associating the Haar function with its support. In the following, we restrict ourselves to functions $f \in L^1[0,1)$ whose integral over the interval $[0,1)$ is 0. This set is denoted by L^1_0. Let $h_0 = 1$, and for $I = [k/2^n, (k+1)/2^n) \in \mathcal{J}$ $(0 \le k < 2^n, n \in \mathbb{N})$, we introduce the notation

$$h_{2^n+k} := h_I := 2^{n/2} r_n \chi_I = |I|^{-1/2} r_n \chi_I \tag{3.2}$$

where χ_I is the characteristic function of the interval I and $|I| := 2^{-n}$ is the length of the interval I. The partial sums of Haar–Fourier and Walsh–Fourier series with index 2^n are equal, consequently, the quadratic variation can be simply expressed with the Haar–Fourier coefficients $f_J := \langle f, h_J \rangle$ $(J \in \mathcal{J})$:

$$(Qf)(x) = \Big(\sum_{x \in J \in \mathcal{J}} |f_J|^2 \ |J|^{-1} \Big)^{1/2} \quad (f \in L^1_0, x \in [0,1)). \tag{3.3}$$

The Haar system is a basis in the space of continuous functions on the group $\mathbb{D}$, denoted by $C = C(\mathbb{D})$, in the spaces L^p $(1 \le p < \infty)$, and in the H^1 and VMO spaces. It also follows from the Paley theorem (3.1) that any rearrangement of the Haar system is also a basis in the L^p spaces for $1 < p < \infty$.

Using the integrals of the Haar functions, Faber, Schauder, Franklin, and Ciesielski constructed function systems with good convergence properties in the space of smooth functions, addressing several problems related to bases. Banach's famous question, whether every separable Banach space has a basis, remained unanswered for a long time [14, 16]. The answer was provided by Enflo in 1973, who constructed a separable Banach space without a basis. In Enflo's construction, the Walsh system played an important role. In [20, 14], we constructed a VMO-type space that has no basis. The space in question is generated by a special (non-linearly ordered) stochastic basis. It would be worth examining how the existence of a basis in such VMO-type spaces relates to other properties (e.g., mixing) of the stochastic basis.

For the sequence $\mathfrak{a} = (a_I, I \in \mathcal{J})$ and parameters $0 < p \le \infty$, we introduce the following weighted mixed norms:

$$\begin{aligned} \|\mathfrak{a}\|_{\mathfrak{h}^p} &= \sup_{n \in \mathbb{N}} \Big(\sum_{|I|=2^{-n}} |I| \Big(\sum_{I \subseteq J \in \mathcal{J}} |a_J|^2 \ |J|^{-1} \Big)^{p/2} \Big)^{1/p}, \\ \|\mathfrak{a}\|_{\mathfrak{bmo}} &:= \sup_{J \in \mathcal{J}} \ \Big(|J|^{-1} \sum_{I \subseteq J} |a_I|^2 \Big)^{1/2}. \end{aligned} \tag{3.4}$$

Let $\mathfrak{f} := (f_I, I \in \mathcal{J})$ denote the Haar-Fourier coefficients of the function $f \in L_0^1$. Then, based on (3.3) and (3.4),

$$\|Qf\|_p = \|\mathfrak{f}\|_{\mathfrak{h}_p} \quad (0 < p < \infty), \quad \|f\|_{BMO} = \|\mathfrak{f}\|_{\mathfrak{b}mo}.$$

It follows that the Haar–Fourier transform $f \to \mathfrak{f} = (f_J,\ J \in \mathcal{J})$ is an isomorphism between the H^p and $\mathfrak{h}_p$ $(0 < p < \infty)$ sequence spaces, as well as between the BMO and $\mathfrak{b}mo$ spaces. Specifically, the elements of the $\mathfrak{h}_p$ $(1 < p < \infty)$ sequence space can characterize the L^p spaces. We note that by using Franklin or wavelet Fourier coefficients instead of Haar-Fourier coefficients, a similar characterization can be given for the classical H^1 and BMO spaces [14, 9].

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