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ESTIMATES FOR MULTIPLICATIVE FUNCTIONS, I.

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Abstract. In this paper we give upper estimates and asymptotic results for some classes of multiplicative functions. For the proofs we mainly use tools from the convolution arithmetic of number theoretical functions (see [4]).

1. Introduction

In this paper we consider multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ satisfying

$$\text{(i)} \quad |f(p)| \leq B \;\; \text{for all primes} \;\; p, \; B \geq 1,$$

$$\text{(ii)} \quad \sum_{p, k \geq 2} \frac{|f(p^k)|}{p^k} < \infty.$$

Our aim is to give upper estimates and asymptotic results for

$$M(f, x) := \sum_{n \leq x} f(n) \;\; as \;\; x \to \infty. \tag{1.1}$$

For multiplicative functions f satisfying (i) and (ii) we define the *exponentially multiplicative* function f_0 by (p prime)

$$f_0(p^k) = \frac{(f(p))^k}{k!} \;\; (k \in \mathbb{N}).$$

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Then

$$f = h * f_0,$$

where h is multiplicative and (p prime)

$$h(p^k) = \sum_{\substack{l,m \\ l+m=k}} \frac{(-1)^l}{l!} (f(p))^l f(p^m) \quad (k \in \mathbb{N}). \tag{1.2}$$

Especially $h(p) = 0$. Obviously

$$\begin{aligned}
\sum_{p,k\geq 2} \frac{|h(p)^k|}{p^k} &\leq \sum_{\substack{p,l,m \\ l+m\geq 2}} \frac{1}{l!} \frac{|f(p)|^l |f(p^m)|}{p^{l+m}} = \\
&= \sum_p \Bigg((e^{\frac{|f(p)|}{p}} - 1) \frac{|f(p)|}{p} + e^{\frac{|f(p)|}{p}} - 1 - \frac{|f(p)|}{p} + \\
&\qquad + e^{\frac{|f(p)|}{p}} \sum_{m\geq 2} \frac{|f(p^m)|}{p^m} \Bigg) = \\
&= \sum_p O\left(\frac{|f(p)|^2}{p^2} \right) + O(1) \sum_{m\geq 2} \frac{|f(p^m)|}{p^m} < \\
&< \infty
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty. \tag{1.3}$$

Now we define $\mathcal{M}_1$ by

$$\mathcal{M}_1 := \{ f : \mathbb{N} \to \mathbb{C} : f \ \text{ multiplicative; } \ \text{(i), (ii), (iii) } \ \text{hold} \},$$

where

$$\text{(iii)} \quad \sum_{\substack{p^k \leq x \\ k\geq 2}} |f(p^k)| = O\left(\frac{x}{\log x} \right).$$

Further, let

$$\text{(iv)} \quad \sum_{\substack{p^k \leq x \\ k\geq 2}} \frac{|f(p^k)|}{p^k} \log p^k < \infty,$$

$$\text{(v)} \quad \sum_{\substack{p^k \leq x \\ k\geq 2}} \frac{|f(p^k)|}{p^k} (\log p^k)^2 < \infty$$

and

$$\text{(vi)}\ \exp\left(\sum_{p\le y}\frac{|f(p)|-1}{p}\right) \ll \exp\left(\sum_{p\le x}\frac{|f(p)|-1}{p}\right) \quad if \quad 1\le y\le x.$$

Then we define the spaces

$$\mathcal{M}_3 \subset \mathcal{M}_2 \subset \mathcal{M}_1 \ \text{ and } \ \mathcal{M}_4$$

by

$$\mathcal{M}_2 := \{f\in\mathcal{M}_1,\ \text{(iv) holds}\}$$
$$\mathcal{M}_3 := \{f\in\mathcal{M}_1,\ \text{(v) holds}\}$$
$$\mathcal{M}_4 := \{f:\mathbb{N}\to\mathbb{C} : f \ \text{ multiplicative; (i), (ii), (vi) hold}\}$$

It is easy to see that (iv) implies (iii). Further, it is well known that, if (i) and (ii) hold,

$$\sum_{n\le x}|f_0(n)| \ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right) \tag{1.4}$$

and, if (vi) holds,

$$\sum_{n\le x^{\varepsilon}}\frac{|f_0(n)|}{n} \ll \varepsilon\sum_{n\le x}\frac{|f_0(n)|}{n}. \tag{1.5}$$

Remark 1.1. Let

$$\sum_{p\le x}\frac{|f(p)|}{p}\log p \sim \alpha\log x,\quad x\to\infty,$$

hold with a constant $\alpha>1$. Then one can show (see Lemma 2.2. of [7]) that (vi) is valid.

2. Classical results

If $f\in\mathcal{M}_4$ then, by (1.3),

$$\begin{aligned}\Big|\sum_{n\le x} f(n)\Big| \quad &\le \quad \sum_{n\le x}|h(n)|\sum_{m\le\frac{x}{n}}|f_0(m)| \ll \\ &\ll \quad \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right).\end{aligned} \tag{2.1}$$

If $f \in \mathcal{M}_1$ then, by (1.2),

$$\begin{aligned}\sum_{p^k \le x} |h(p^k)| \log p^k &\le \sum_{\substack{p^{v+\mu} \le x\\ v+\mu \ge 2}} \frac{|f(p)|^v}{v!} |f(p^\mu)| \log x \le \\ &\le \sum_{p^\mu \le x} |f(p^\mu)| e^B \log x = O(x)\end{aligned}$$

and (see [7, (3.6)] and A. G. Postnikov [10, p. 201])

$$\begin{aligned}\sum_{n \le x} |h(n)| \log n &\le \sum_{n \le x} |h(n)| \sum_{\substack{p^k \le \frac{x}{n}\\ k \ge 2}} |h(p^k)| \log p^k \ll \\ &\ll x \sum_{n \le x} \frac{|h(n)|}{n} \ll x.\end{aligned} \tag{2.2}$$

By partial summation we obtain

$$\sum_{n \le x} |h(n)| = O\left(\frac{x}{\log x}\right). \tag{2.3}$$

Now, it is easy to show that, if $f \in \mathcal{M}_1$ the estimate (2.1) holds. For this we put, as above, $f = h * f_0$ and obtain

$$\begin{aligned}|M(f)| &\le \sum_{m \le x^{1/2}} |f_0(m)| \sum_{n \le \frac{x}{n}} |h(n)| + \sum_{n \le x^{1/2}} |h(n)| \sum_{m \le \frac{x}{n}} |f_0(m)| =: \\ &=: \Sigma_1 + \Sigma_2.\end{aligned}$$

Then

$$\begin{aligned}\Sigma_1 &\ll \frac{x}{\log x} \sum_{m \le x^{1/2}} \frac{|f_0(m)|}{m} \ll \\ &\ll \frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right)\end{aligned}$$

and

$$\Sigma_2 \ll \frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right) \sum_{n \le x^{\frac{1}{2}}} \frac{|h(n)|}{n},$$

which proves (2.1). Collecting the results we arrive at

Proposition 2.1. *Let $f \in \mathcal{M}_1 \cup \mathcal{M}_4$. Then the estimate (2.1) holds.*

3. Estimates by convolution arithmetic

Using identities in the convolution arithmetic Indlekofer (see [4] and [7]) could prove estimates for $M(f,x)$ in the case $f \in \mathcal{M}_1 \cup \mathcal{M}_4$.
For this define, if the arithmetical function f is given, the arithmetical function $L_0 f$ by $L_0 f(n) = f(n)\log n \ \ (n \in \mathbb{N})$. With the function M ($M(u) = M(f,u)$ for $u \geq 1$), the identity function $\mathbf{1}$ and L, defined by $L(u) = \log u \ \ (u \geq 1)$, we use the convolutions

$$\begin{aligned} (L * f)(u) &= \sum_{n \leq u} (\log \frac{u}{n}) f(n), \\ (M * f)(u) &= \sum_{n \leq u} M(\frac{u}{n}) f(n), \\ (\mathbf{1} * f)(u) &= \sum_{n \leq u} f(n) \end{aligned}$$

and define Λ_f by

$$(L_0 f)(n) = (f * \Lambda_f)(n).$$

Then (see [4, p. 135]) $M = \mathbf{1} * f = \mathbf{1} * (h * f_0)$ and

$$L^2 M = M * (\Lambda_{f_0} * \Lambda_{f_0} + L_0 \Lambda_{f_0}) + (R_1 + R_2) * \Lambda_{f_0} + L(R_1 + R_2), \tag{3.1}$$

where

$$\begin{aligned} R_1 &= L * f, \\ R_2 &= \mathbf{1} * L_0 h * f_0. \end{aligned} \tag{3.2}$$

Now let $\mathcal{M}_1'$ be the set of $f \in \mathcal{M}_1$ such that

$$\sum_{n \leq x^{\varepsilon}} \frac{|f_0(n)|}{n} \leq \delta(\varepsilon) \sum_{n \leq x} \frac{|f_0(n)|}{n}, \tag{3.3}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.
Then we prove

Theorem 3.1. *Let $f \in \mathcal{M}_1' \cup \mathcal{M}_4$. Then*

$$|M(x)| \leq 2B^2 \int_1^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p} \right) \right) \tag{3.4}$$

as $x \to \infty$.

Proof. We follow the lines of proof in [7, §3]. Then

$$|\Lambda_{f_0} * \Lambda_{f_0} + L_0\Lambda_{f_0}| \le B^2(\Lambda * \Lambda + L_0\Lambda),$$

and we obtain by Lemma 2.4 of [7] the first summand of the right side of (3.4). For $R_1(x)$ we conclude as in (3.4) of [7]

$$|R_1(x)| = O\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\right). \tag{3.5}$$

In the case

$$R_2(x) = \sum_{n\le x} h(n)\log n \sum_{m\le \frac{x}{n}} f_0(m)$$

we show, if $f \in \mathcal{M}_4$, by (1.3)

$$\begin{aligned} |R_2(x)| &\ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\sum_{n\le x}\frac{|h(n)|\log n}{n} = \\ &= o\left(x\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\right). \end{aligned} \tag{3.6}$$

If $f \in \mathcal{M}_1'$, then

$$\begin{aligned} R_2(x) &= \sum_{m\le x^\varepsilon} f_0(m)\sum_{n\le\frac{x}{m}} h(n)\log n + \sum_{n\le x^{1-\varepsilon}} h(n)\log n \sum_{m\le\frac{x}{n}} f_0(m) := \\ &:= \Sigma_1 + \Sigma_2. \end{aligned}$$

By (2.2)

$$|\Sigma_1| \ll x\sum_{m\le x^\varepsilon}\frac{|f_0(m)|}{m} \le \delta(\varepsilon)x\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)$$

and

$$|\Sigma_2| \ll \frac{1}{\varepsilon}\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\sum_{n\le x^{1-\varepsilon}}\frac{|h(n)|\log n}{n}$$

which implies by (1.3)

$$|R_2(x)| = o\left(x\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)\right). \tag{3.7}$$

Consider

$$(R_1 * \Lambda_{f_0})(x) = (L * (f * \Lambda_{f_0}))(x) = \sum_{n \le x} (\log \frac{x}{n})(f * \Lambda_{f_0})(n).$$

Observing

$$\sum_{n \le y} |f(n)| \sum_{p \le \frac{y}{n}} |f(p)| \log p \ll y \sum_{n \le y} \frac{|f(n)|}{n} \ll y \exp\left(\sum_{p \le y} \frac{|f(p)|}{p}\right)$$

we conclude (cf. [7, (3.4)])

$$|(R_1 * \Lambda_{f_0})(x)| \ll x \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right). \tag{3.8}$$

Since $L_0 f_0 = f_0 * \Lambda_{f_0}$ we have

$$R_2 * \Lambda_{f_0} = (\mathbf{1} * L_0 h * f_0) * \Lambda_{f_0} = \mathbf{1} * L_0 h * L_0 f_0$$

and

$$\begin{aligned} |(R_2 * \Lambda_{f_0})(x)| &\le \log x(\mathbf{1} * |L_0 h| * |f_0|)(x) = \\ &= o\left(x \log x \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right)\right). \end{aligned} \tag{3.9}$$

This ends the proof of Theorem 3.1. ■

If $f \in \mathcal{M}_3$ we can show

Corollary 3.1. *Let $f \in \mathcal{M}_3$. Then*

$$|M(x)| \le 2B^2 \int_1^x \frac{|M(u)|}{u^2} du + O\left(\frac{x}{\log^2 x} \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right)\right).$$

Proof. For $f \in \mathcal{M}_3$ define the multiplicative function $\tilde{h}$ by

$$\tilde{h}(p^k) = \begin{cases} 0 & if\ k = 1 \\ 4|h(p^k| \log p^k & if\ k \ge 2. \end{cases}$$

Then

$$\sum_{p, k \ge 2} \frac{\tilde{h}(p^k) \log p^k}{p^k} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\tilde{h}(n)}{n} < \infty,$$

$$\sum_{n \le x} \tilde{h}(n) \ll \frac{x}{\log x}.$$

By the elementary inequality $a + b \le ab$ for $a, b \ge 2$ we conclude

$$4 \log n \le \prod_{p^k || n} 4 \log p^k$$

which implies

$$|h(n)| \log n \le \frac{1}{4} \tilde{h}(n).$$

Thus

$$\begin{aligned} |R_2(x)| &\le \sum_{n \le x^{1/2}} |f_0(n)| \sum_{m \le \frac{x}{n}} \tilde{h}(m) + \sum_{m \le x^{1/2}} \tilde{h}(m) \sum_{n \le \frac{x}{m}} |f_0(n)| = \\ &= O\left(\frac{x}{\log x} \exp\left(\sum_{n \le x} \frac{|f(p)|}{p} \right) \right), \end{aligned}$$

which leads as above to the assertion of Corollary 3.1. ■

4. Estimates using generating functions

Let $f = h * f_0 \in \mathcal{M}_1' \cup \mathcal{M}_4$. Put

$$K_0 = \mathbf{1} * \Lambda_{f_0} * f.$$

Then (see (3.2))

$$\begin{aligned} LM &= \mathbf{1} * L_0 f + R_1 = \\ &= \mathbf{1} * \Lambda_f * f + R_1 = \\ &= \mathbf{1} * (\Lambda_h + \Lambda_{f_0}) * f + R_1 = \\ &= K_0 + R_1 + R_2. \end{aligned} \tag{4.1}$$

Now

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &= \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du = \\ &= I_1 + I_2. \end{aligned}$$

Then

$$I_1 \le \sum_{n\le x^\varepsilon} \frac{|f(n)|}{n} \ll \sum_{n\le x^\varepsilon} \frac{|f_0(n)|}{n} \le \delta(\varepsilon)\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)$$

if $f \in \mathcal{M}_1'$ and

$$I_1 \ll \varepsilon \exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right)$$

if $f \in \mathcal{M}_4$. Concerning I_2 we observe (cf. (3.6), (3.7))

$$M(u) = \frac{K_0(u)}{\log u} + o\left(\frac{u}{\log u}\exp\left(\sum_{p\le u}\frac{|f(p)|}{p}\right)\right)$$

if $f \in \mathcal{M}_1' \cup \mathcal{M}_4$. Thus

$$\int_1^x \frac{|M(u)|}{u^2}du = \int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2\log u}du + R(x)\exp\left(\sum_{p\le x}\frac{|f(p)|}{p}\right),$$

where

$$R(x) \ll \max(\varepsilon,\delta(\varepsilon)) + o\left(\log\frac{1}{\varepsilon}\right).$$

Now

$$\begin{aligned}\int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2\log u}du &\le \frac{1}{\varepsilon}\frac{1}{\log x}\int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2}du \le \\ &\le \frac{1}{\varepsilon\log x}\left(\int_2^x \frac{|K_0(u)|^2}{u^3}du\right)^{1/2}\left(\int_{x^\varepsilon}^x\frac{du}{u}\right)^{1/2} \le \\ &\le \frac{1}{\varepsilon}\left(\frac{1}{\log x}\int_2^x\frac{|K_0(u)|^2}{u^3}du\right)^{1/2}.\end{aligned}$$

Put $\sigma = 1 + \frac{1}{\log x}$. Then

$$\int_2^x \frac{|K_0(u)|^2}{u^3}du \le e^2\int_2^x \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du.$$

Since ($s = \sigma + it$)

$$\int_0^\infty K_0(e^u)e^{-us}e^{iut}du = \frac{F_0'(s)}{F_0(s)}F'(s),$$

where $F(s) = H(s)F_0(s)$ and

$$F(s) = \sum_{n=1}^\infty \frac{f(n)}{n^s}, \quad H(s) = \sum_{n=1}^\infty \frac{h(n)}{n^s}, \quad F_0(s) = \sum_{n=1}^\infty \frac{f_0(n)}{n^s} \tag{4.2}$$

we conclude, by Parseval's eguality,

$$\int_1^\infty \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{F_0'(s)}{F_0(s)}\right|^2 |F(s)|^2 \frac{dt}{|s|^2}. \tag{4.3}$$

5. Main result

We assume that $f, g \in \mathcal{M}_1' \cup \mathcal{M}_4$ where $g \geq 0$ and $|f| \leq g$. Put $g = h_1 * g_0$ where g_0 is exponentially multiplicative with $g_0(p) = g(p)$ (p prime) and let $G(s) = H_1(s)G_0(s)$. Consider

$$\sum_p \frac{g(p) - Ref(p)p^{-it}}{p}, \quad t \in \mathbb{R}. \tag{5.1}$$

In this paper we deal with the case that (5.1) diverges for every $t \in \mathbb{R}$. Then $|F(s)| = o(G(\sigma))$ uniformly for all $t \leq K$ as $\sigma \to 1$. By (4.3) we have

$$\begin{aligned}\int_{-\infty}^\infty \left|\frac{F_0'(s)}{F_0(s)}\right|^2 \frac{|F(s)|^2}{|s|^2}dt \quad &\leq \quad o(G^2(\sigma)) \int_{|t|\leq K} \left|\frac{F_0'(s)}{F_0(s)}\right| \frac{dt}{|s|^2} + \\ &+G^2(\sigma) \int_{|t|\geq K} \left|\frac{F_0'(s)}{F_0(s)}\right| \frac{dt}{|s|^2}.\end{aligned} \tag{5.2}$$

The integrals in (5.2) can be estimated by

$$\sum_{\substack{k\in\mathbb{Z} \\ |k|\leq K}} \frac{1}{k^2+1} \int_{|t-k|\leq \frac{1}{2}} \left|\frac{F_0'(s)}{F_0(s)}\right|^2 dt$$

and

$$\sum_{\substack{k\in\mathbb{Z}\\|k|\geq K}} \frac{1}{k^2+1} \int\limits_{|t-k|\leq\frac{1}{2}} \left|\frac{F_0'(s)}{F_0(s)}\right|^2 dt,$$

respectively. Further, we have

$$\frac{F_0'(s)}{F(s)} = -\sum_p \frac{f(p)\log p}{p^s}$$

and

$$-\sum_p \frac{\log p}{p^s} = -\frac{\zeta'(s)}{\zeta(s)} + O(1)$$

for $\sigma > 1$. Since $|f(p)| \leq Bb_n$ in the proof of Theorem 4.10 in G. Tenenbaum [11] we conclude

$$\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+ik+it)\right|^2 dt \leq 3B^2 \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\zeta'}{\zeta}(1+\frac{1}{\log x}+ik+it)\right|^2 dt \ll \log x.$$

Thus the right-hand side of (5.2) can be estimated by

$$o(\log x(G(\sigma)^2) + O(\log x(G(\sigma)^2) \sum_{|k|\geq K} \frac{1}{k^2+1}. \tag{5.3}$$

Since

$$G(\sigma) \ll \exp\left(\sum_{p\leq x} \frac{g(p}{p}\right)$$

we finish the proof of

Theorem 5.1. *Let $f, g \in \mathcal{M}_1 \cup \mathcal{M}_4$ such that $|f| \leq g$. If (5.1) diverges for all $t \in \mathbb{R}$ then*

$$\sum_{n\leq x} f(n) = o\left(\frac{x}{\log x} \exp\left(\sum_{p\leq x} \frac{g(p}{p}\right)\right).$$

Remark 5.1. (i) If $g = \mathbf{1}$ and $|f(n)| \leq 1$ for all $n \in \mathbb{N}$ then Theorem 5.1. contains the corresponding results of G. Halász [2], K.-H. Indlekofer [3], [4], [5]. Especially, if μ denotes the Möbius function we obtain the Prime Number Theorem in the form

$$\sum_{n\leq x} \mu(n) = o(x) \quad as \quad x \to \infty.$$

(ii) If there is $\mathbb{P}_1 \subset \mathbb{P}$ such that, for some $\alpha > 0$,

$$\sum_{\substack{p\le x\\ p\in\mathbb{P}_1}} \frac{g(p)\log p}{p} \sim \alpha \log x \quad as \quad x\to\infty \tag{5.4}$$

then

$$\sum_{n\le x^{\varepsilon}} \frac{g_0(n)}{n} \ll \varepsilon^{\alpha} exp\left(\sum_{p\le x}\frac{g(p}{p}\right)$$

and Theorem 5.1 contains the corresponding results of P.D.T.A. Elliott [1], K.-H. Indlekofer, I. Kátai, R. Wagner [6], E. Kaya [9].

Remark 5.2. Let $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$ and $g_0 = g_1 * g_2$, where

$$g_i(p) = \begin{cases} g(p) & if\ p\in\mathbb{P}_i \\ 0 & if\ p\notin\mathbb{P}_i \end{cases} \quad (i=1,2).$$

If (5.4) holds for some $\alpha > 0$, then

$$\sum_{n\le x} g_0(n) \ge \sum_{n\le x^{1/2}} g_2(n) \sum_{m\le \frac{x}{n}} g_1(m).$$

A Result of E. Wirsing [12] gives for α typical inner sum the asymptotic estimate

$$\{1+o(1)\}\frac{e^{-\gamma\alpha}}{\Gamma(\alpha)}\frac{x}{n\log\frac{x}{n}}\prod_{p\le\frac{x}{n}}\left(1+\frac{g_1(p)}{p}\right).$$

Thus the double sum exceeds a constant multiple of

$$\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{g_1(p)}{p}\right)\sum_{n\le x^{1/2}}\frac{g_2(n)}{n}.$$

Then

$$\sum_{n\le x} g(n) \gg \sum_{n\le x} g_0(n) \gg \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(n)}{n}\right).$$

Remark 5.3. In [8] we deal with the case that (5.1) converges for some $t = t_0 \in \mathbb{R}$.

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