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APPROXIMATION BY T MEANS OF WALSH–FOURIER SERIES IN LEBESGUE SPACES AND LIPSCHITZ CLASSES

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Abstract. In this paper we present and prove some new results concerning the rate of approximation of T means of functions in $L^p(G)$ and in lip (α, p), for $1 \leq p < \infty$ and $\alpha > 0$. As a corollary, we obtain some new as well as known approximation inequalities.

1. Preliminaries and motivations

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by $Z_2 := \{0, 1\}$ the additive group of integers modulo 2.

Define the group G as the complete direct product of the group Z_2 with the product of the discrete topologies of Z_2 's. The direct product μ of the measures

$$\mu^* (\{j\}) := 1/2 \quad (j \in Z_2)$$

is the Haar measure on G with $\mu(G) = 1$.

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The elements of G are represented by the sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \qquad (x_k \in Z_2).$$

Let $e_n := (x_0 = 0, \ldots, x_{n-1} = 0, x_n = 1, x_{n+1} = 0, x_{n+2} = 0, \ldots)$. It is easy to give a base for the neighborhood of G, namely

$$I_0(x) := G, \quad I_n(x) := \{y \in G, \; y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \; (x \in G, \; n \in \mathbb{N}).$$

The intervals $I_n(x)$ $(n \in \mathbb{N}, \; x \in G)$ are called dyadic intervals.

The norms (or quasi-norms) of the Lebesgue space $L^p(G)$ and the weak Lebesgue space $L^{p,\infty}(G)$, $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{weak-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu (f > \lambda).$$

It is well-known that every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j 2^j, \quad \text{where} \quad n_j \in Z_2 \quad (j \in \mathbb{N})$$

and only a finite number of n_j's differ from zero.

Let

$$|n| := \max\{j \in \mathbb{N}, \; n_j \neq 0\}.$$

Now, we consider the Walsh orthonormal system $\{w_k, \; k \in \mathbb{N}\}$ using the Paley enumeration (see [15]). First define the Rademacher functions as

$$r_k(x) := (-1)^{x_k}, \quad (k \in \mathbb{N}).$$

Next we define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L^2(G)$ (see e.g. [20]).

For $f \in L^1(G)$, we define the Fourier coefficients, partial sums of the Fourier series and Fejér means with respect to the Walsh system in the following way:

$$\begin{aligned}
\widehat{f}(k) &:= \int_G f w_k d\mu, && (k \in \mathbb{N}), \\
S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) w_k, && (n \in \mathbb{N}_+, \; S_0 f := 0), \\
\sigma_n f &:= \frac{1}{n} \sum_{k=1}^{n} S_k f, && (n \in \mathbb{N}_+).
\end{aligned}$$

Recall that (see e.g. [7] and [21]),

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{1.1}$$

$$D_{2^n - j} = D_{2^n} - w_{2^n-1} D_j, \quad (0 \le j < 2^n), \tag{1.2}$$

$$n|K_n| \le 3 \sum_{l=0}^{|n|} 2^l K_{2^l}, \tag{1.3}$$

and

$$\int_G K_n d\mu = 1, \qquad \sup_{n\in\mathbb{N}} \int_G |K_n| \, d\mu = \frac{17}{15}. \tag{1.4}$$

If $n > t$, $t, n \in \mathbb{N}$, then (see [7] and [18])

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & x \in I_t \backslash I_{t+1}, \quad x - e_t \in I_n, \\ \frac{2^n+1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

The n-th Nörlund mean t_n and T mean T_n of the Fourier series of f are defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f \quad \text{ and } \quad T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f,$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

Here $\{q_k, \ k \ge 0\}$ is a sequence of nonnegative numbers, where $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{1.6}$$

Then, T means generated by $\{q_k, \ k \ge 0\}$ is regular if and only if the condition (1.6) is fulfilled (see [18]).

It is evident that

$$T_n f(x) = \int_G f(t) F_n(x+t) \, d\mu(t),$$

where

$$F_n := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k,$$

which are called the kernels of the T means.

By applying Abel transformation we get the following two identities:

$$Q_n := \sum_{k=0}^{n-1} q_k \cdot 1 = \sum_{k=0}^{n-2} (q_k - q_{k+1}) k + q_{n-1}(n-1) \tag{1.7}$$

and

$$T_n f = \frac{1}{Q_n} \left(\sum_{k=0}^{n-2} (q_k - q_{k+1}) k \sigma_k f + q_{n-1}(n-1) \sigma_{n-1} f \right). \tag{1.8}$$

Fejér's theorem shows that (see e.g. [7] and [9]) if one replaces ordinary summation by Fejér means σ_n, then, for any $1 \leq p \leq \infty$, there exists an absolute constant C_p, depending only on p such that the inequality

$$\|\sigma_n f\|_p \leq C_p \|f\|_p$$

holds. Moreover, (see [18]) if $1 \leq p \leq \infty$, $2^N \leq n < 2^{N+1}$, $n, N \in \mathbb{N}$ and $f \in L^p(G)$, then we have the following estimate

$$\|\sigma_n f - f\|_p \leq 3 \sum_{s=0}^{N} \frac{2^s}{2^N} \omega_p \left(1/2^s, f\right). \tag{1.9}$$

It follows that if $f \in \text{lip}\,(\alpha, p)$, i.e.

$$\omega_p \left(1/2^n, f\right) = O\left(1/2^{n\alpha}\right), \text{ as } n \to \infty,$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\left(1/2^N\right), & \text{if} \quad \alpha > 1, \\ O\left(N/2^N\right), & \text{if} \quad \alpha = 1, \\ O\left(1/2^{N\alpha}\right), & \text{if} \quad \alpha < 1. \end{cases}$$

Moreover, (see [18]) if $1 \leq p < \infty$, $f \in L^p(G)$ and

$$\|\sigma_{2^n} f - f\|_p = o\left(1/2^n\right), \text{ as } n \to \infty,$$

then f is a constant function.

Boundedness of maximal operators of Vilenkin-Fejer means and weak-$(1,1)$ type inequality

$$\mu\left(\sigma^* f > \lambda\right) \leq \frac{C}{\lambda} \|f\|_1, \qquad \left(f \in L^1(G), \quad \lambda > 0\right)$$

can be found in Schipp [19] for Walsh series and in Pál and Simon [14] (see also [3], [12], [17], [18] and [24]) for bounded Vilenkin series.

Convergence and summability of Nörlund means were studied by Blahota and Nagy [5] (see also [4] and [13]), Fridli, Manchanda and Siddiqi [6], Persson, Tephnadze and Weisz [18] (see also [16]). Móricz and Siddiqi [11] proved that if $f \in L^p(G)$, where $1 \leq p < \infty$ and $\{q_k, k \in \mathbb{N}\}$ is a sequence of non-negative numbers, such that

$$\frac{n^{\theta-1}}{Q_n^{\theta}} \sum_{k=0}^{n-1} q_k^{\theta} = O(1), \quad \text{for some} \ \ 1 < \theta \leq 2 \tag{1.10}$$

holds, then for any $2^N \leq n < 2^{N+1}$, there exists an absolute constant C_p such that the approximation inequality

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{k=0}^{N-1} 2^k q_{n-2^k} \omega_p \left(\frac{1}{2^k}, f \right) + C_p \omega_p \left(\frac{1}{2^N}, f \right)$$

holds when $(q_k, k \in \mathbb{N})$ is non-decreasing, while the approximation inequality

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{k=0}^{N-1} \left(Q_{n-2^k+1} - Q_{n-2^{k+1}+1} \right) \omega_p \left(\frac{1}{2^k}, f \right) + C_p \omega_p \left(\frac{1}{2^N}, f \right)$$

holds when $\{q_k, k \in \mathbb{N}\}$ is non-increasing.

Areshidze and Tephnadze [2] (see also [1]) proved a similar approximation result for Nörlund means with respect to Walsh system generated by a non-decreasing sequence $\{q_k, k \in \mathbb{N}\}$ in Lebesgue spaces $L^p(G)$ when $1 \leq p < \infty$, without any condition considered in Móricz and Siddiqi [11].

Goginava [8] proved that if t_n are Nörlund means generated by non-increasing sequence $\{q_k, k \in \mathbb{N}\}$ satisfying the condition

$$\sup_{N \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_{2^N}^p} \sum_{j=1}^{N} Q_{2^j}^p 2^{j(p-1)} < \infty, \tag{1.11}$$

for some $p \in [1/2, 1)$, then there exists an absolute constant C, such that the weak–$(1,1)$ type inequality

$$\mu\left(t^* f > \lambda\right) \leq \frac{C}{\lambda} \|f\|_1, \qquad \left(f \in L^1(G), \quad \lambda > 0\right) \tag{1.12}$$

holds.

It was also proved (see [18]) that inequality (1.12) also holds for any Nörlund mean generated by non-decreasing sequence $(q_k, k \in \mathbb{N})$.

It follows from these results that if $f \in L^p(G)$, where $1 \leq p < \infty$ and either $\{q_k, k \in \mathbb{N}\}$ is a sequence of non-negative and non-increasing numbers, such that condition (1.11) is fulfilled or the sequence $\{q_k, k \in \mathbb{N}\}$ is non-decreasing, then

$$\lim_{n\to\infty} \|t_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty.$$

Tutberidze [23] (see also [18] and [22]) proved that if T_n are T means generated by either non-increasing sequence $\{q_k, k \in \mathbb{N}\}$ or non-decreasing sequence $\{q_k, k \in \mathbb{N}\}$ satisfying the condition

$$\frac{q_0}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty, \tag{1.13}$$

then there exists an absolute constant C, such that

$$\|T^* f\|_{weak-L_1} \leq C \|f\|_1, \qquad \left(f \in L^1(G)\right) \tag{1.14}$$

holds. It follows from these results that if $f \in L^p(G)$, where $1 \leq p < \infty$ and either the sequence $\{q_k, k \in \mathbb{N}\}$ is non-increasing, or $\{q_k, k \in \mathbb{N}\}$ is a sequence of non-decreasing numbers, such that condition (1.11) is fulfilled, then

$$\lim_{n\to\infty} \|T_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty.$$

In Móricz and Rhoades [10] it was proved that if $f \in L^p(G)$, where $1 \leq$ $\leq p < \infty$ and T_n are regular T means generated by a non-increasing sequence $\{q_k,\ k \in \mathbb{N}\}$, then, for any $2^N \leq n < 2^{N+1}$ we have the following estimate

$$\|T_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{s=0}^{N-1} 2^s q_{2^s} \omega_p\left(1/2^s, f\right) + C_p \omega_p\left(1/2^N, f\right). \tag{1.15}$$

In the case when the sequence $\{q_k,\ k \in \mathbb{N}\}$ is non-decreasing and satisfies the condition

$$\frac{q_{k-1}}{Q_k} = O\left(\frac{1}{k}\right), \quad \text{as} \quad k \to \infty, \tag{1.16}$$

then

$$\|T_n f - f\|_p \leq C_p \sum_{j=0}^{N-1} 2^{j-N} \omega_p\left(1/2^j, f\right) + C_p \omega_p\left(1/2^N, f\right). \tag{1.17}$$

In this paper we use a new and simpler approach to prove somewhat improved versions of the inequalities in (1.15) and (1.17) for T means with respect to the Walsh system (see Theorems 2.1 and 2.2). We also prove a new inequality for the subsequences $\{T_{2^n}\}$ means when the sequence $\{q_k,\ k \in \mathbb{N}\}$ is non-decreasing and where the restrictive (1.16) is omitted (see Theorem 2.3).

The main results and some of their consequences are presented in Section 2 while the proofs are given in Section 3.

2. The main results

Our first main results are the following improved version of some results in [10]:

Theorem 2.1. *Let $f \in L^p(G)$, where $1 \leq p < \infty$ and T_n are regular T means generated by a non-increasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $2^N < n \leq 2^{N+1}$ we have the following inequality:*

$$\|T_n f - f\|_p \leq \frac{12}{Q_n} \sum_{s=0}^{N-1} 2^s q_{2^s} \omega_p \left(1/2^s, f\right) + 12 \omega_p \left(1/2^N, f\right). \tag{2.1}$$

Theorem 2.2. *Let $f \in L^p(G)$, where $1 \leq p < \infty$ and T_n are T means generated by non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n, N \in \mathbb{N}$, $2^N < n \leq$ $\leq 2^{N+1}$ we have the following inequality*

$$\|T_n f - f\|_p \leq \frac{18 q_{n-1}}{Q_n} \sum_{j=0}^{N-1} 2^j \omega_p \left(1/2^j, f\right) + \frac{8 q_{n-1}}{Q_n} 2^N \omega_p \left(1/2^N, f\right). \tag{2.2}$$

In addition, if the sequence $\{q_k,\ k \in \mathbb{N}\}$ satisfies the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad as \quad n \to \infty. \tag{2.3}$$

then

$$\|T_n f - f\|_p \leq C_p \sum_{j=0}^{N} 2^{j-N} \omega_p \left(1/2^j, f\right). \tag{2.4}$$

Finally, we state the third main result for the non-decreasing sequences again but only for subsequences T_{2^n} of T means but without any restrictions.

Theorem 2.3. *Let $f \in L^p(G)$, where $1 \leq p < \infty$ and T_k are T means generated by a non-decreasing sequence $\{q_k,\ k \in \mathbb{N}\}$. Then, for any $n \in \mathbb{N}$, the following inequality holds:*

$$\begin{aligned} &\|T_{2^n} f - f\|_p \leq \\ &\leq \sum_{s=0}^{n-1} \frac{2^s}{2^n} \omega_p \left(1/2^s, f\right) + \frac{3}{q_0} \sum_{s=0}^{n-1} \frac{(n-s) q_{2^n - 2^s}}{2^{n-s}} \omega_p \left(1/2^s, f\right) + 2 \omega_p \left(1/2^n, f\right). \end{aligned} \tag{2.5}$$

As a consequence we obtain the following similar result proved in Móricz and Rhoades [10]:

Corollary 2.1. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers, while in case when the sequence is non-decreasing it is assumed that also the condition (2.3) is satisfied. If $f \in \operatorname{lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then*

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1, \\ O(n^{-1}), & \text{if} \quad \alpha > 1, \end{cases} \tag{2.6}$$

We also obtain the following results proved in the same paper:

Corollary 2.2. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that*

$$q_k \asymp k^{-\beta} \quad \textit{for some} \quad 0 < \beta \leq 1$$

is satisfied. If $f \in \operatorname{lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad \alpha + \beta < 1, \\ O(n^{-(1-\beta)}\log n + n^{-\alpha}), & \text{if} \quad \alpha + \beta = 1, \\ O(n^{-(1-\beta)}), & \text{if} \quad \alpha + \beta > 1,\ \beta > 1, \\ O((\log n)^{-1}), & \text{if} \quad \beta = 1. \end{cases}$$

Corollary 2.3. *Let $\{q_k,\ k \geq 0\}$ be a sequence of non-negative and non-increasing numbers such that*

$$q_k \asymp (\log k)^{-\beta} \quad \textit{for some} \quad \beta > 0$$

is satisfied. If $f \in \operatorname{lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p < \infty$, then

$$\|T_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1,\ \beta > 0, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1,\ 0 < \beta < 1, \\ O(n^{-1}\log n \log\log n), & \text{if} \quad \alpha = \beta = 1, \\ O(n^{-1}(\log n)^{\beta}), & \text{if} \quad \alpha > 1,\ \beta > 0. \end{cases} \tag{2.7}$$

We also obtain the following convergence result:

Corollary 2.4. *Let $f \in L^p(G)$, where $1 \leq p < \infty$ and $\{q_k,\ k \geq 0\}$ is a sequence of non-negative and non-increasing numbers, while in the case where the sequence is non-decreasing, it is also assumed that the condition (2.3) is satisfied. Then,*

$$\lim_{n\to\infty} \|T_n f - f\|_p \to 0, \quad \textit{as} \quad n \to \infty.$$

3. Proofs

Proof of Theorem 2.1. Let $2^N < n \le 2^{N+1}$ and $\{q_k,\ k \in \mathbb{N}\}$ is a sequence of non-increasing numbers. By combining identities (1.7) and (1.8) we find that

$$\begin{aligned}
(3.1) \qquad & \|T_n f - f\|_p \le \\
\le \; & \frac{1}{Q_n}\left(\sum_{j=0}^{n-2}(q_j - q_{j+1}) j \|\sigma_j f - f\|_p + q_{n-1}(n-1)\|\sigma_{n-1} f - f\|_p\right) := \\
:= \; & I_1 + I_2.
\end{aligned}$$

By using the inequality (1.9) for I_1 we can conclude that

$$\begin{aligned}
(3.2) \qquad I_1 \; & \le \frac{1}{Q_n}\sum_{j=0}^{2^{N+1}-1}(q_j - q_{j+1}) j \|\sigma_j f - f\|_p \le \\
& \le \frac{3}{Q_n}\sum_{k=0}^{N}\sum_{j=2^k}^{2^{k+1}-1}\left(q_j - q_{j+1}\right) j \sum_{s=0}^{k}\frac{2^s}{2^k}\omega_p\left(1/2^s, f\right) \le \\
& \le \frac{3}{Q_n}\sum_{k=0}^{N}2^{k+1}\sum_{j=2^k}^{2^{k+1}-1}\left(q_j - q_{j+1}\right)\sum_{s=0}^{k}\frac{2^s}{2^k}\omega_p\left(1/2^s, f\right) \le \\
& \le \frac{6}{Q_n}\sum_{k=0}^{N}\left(q_{2^k} - q_{2^{k+1}}\right)\sum_{s=0}^{k}2^s\omega_p\left(1/2^s, f\right) \le \\
& \le \frac{6}{Q_n}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s, f\right)\sum_{k=s}^{N}\left(q_{2^k} - q_{2^{k+1}}\right) \le \\
& \le \frac{6}{Q_n}\sum_{s=0}^{N}2^s q_{2^s}\omega_p\left(1/2^s, f\right) \le \\
& \le \frac{6}{Q_n}\sum_{s=0}^{N-1}2^s q_{2^s}\omega_p\left(1/2^s, f\right) + 6\omega_p\left(1/2^N, f\right).
\end{aligned}$$

For I_2 we have that

$$\begin{aligned}
(3.3) \qquad I_2 \; & \le \frac{3q_{n-1}2^{N+1}}{Q_n}\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s, f\right) \le \\
& \le \frac{6}{Q_n}\sum_{s=0}^{N-1}2^s q_{2^s}\omega_p\left(1/2^s, f\right) + 6\omega_p\left(1/2^N, f\right).
\end{aligned}$$

By combining (3.1), (3.2) and (3.3) we obtain that (2.1) holds so the proof is complete. ∎

Proof of Theorem 2.2. Let $2^N < n \leq 2^{N+1}$. Since $\{q_k,\ k \in \mathbb{N}\}$ is a sequence of non-decreasing numbers, by combining (1.7) and (1.8) we find that

$$\begin{aligned}
&\|T_n f - f\|_p \leq \qquad (3.4)\\
&\leq \frac{1}{Q_n}\left(\sum_{k=0}^{n-2}(q_{k+1}-q_k)k\|\sigma_k f - f\|_p + q_{n-1}(n-1)\|\sigma_{n-1}f - f\|_p\right) := \\
&:= II_1 + II_2.
\end{aligned}$$

For II_1 we have that

$$\begin{aligned}
II_1 &= \frac{1}{Q_n}\sum_{j=0}^{2^N-1}(q_{j+1}-q_j)j\|\sigma_j f - f\|_p + \\
&+ \frac{1}{Q_n}\sum_{j=2^N}^{n-2}(q_{j+1}-q_j)j\|\sigma_j f - f\|_p := II_1^1 + II_1^2.
\end{aligned} \tag{3.5}$$

Analogously to (3.2) we get that

$$\begin{aligned}
II_1^1 &\leq \frac{6}{Q_n}\sum_{k=0}^{N-1}\left(q_{2^{k+1}} - q_{2^k}\right)\sum_{s=0}^{k} 2^s \omega_p\left(1/2^s, f\right) \leq \\
&\leq \frac{6}{Q_n}\sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right)\sum_{k=s}^{N-1}\left(q_{2^{k+1}} - q_{2^k}\right) = \\
&= \frac{6}{Q_n}\sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right)\left(q_{2^N} - q_{2^s}\right) \leq \\
&\leq \frac{6q_{2^N}}{Q_n}\sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right) \leq \\
&\leq \frac{6q_{n-1}}{Q_n}\sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right).
\end{aligned} \tag{3.6}$$

Moreover, if we apply (1.7) we find that

$$\begin{aligned}
II_1^2 &\leq \frac{3}{Q_n}\sum_{j=2^N}^{n-2}\left(q_{j+1} - q_j\right) j \sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s, f\right) \leq \\
&\leq \frac{3}{Q_n}\sum_{j=0}^{n-2}\left(q_{j+1} - q_j\right) j \sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s, f\right) =
\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&= \frac{3}{Q_n}\left((n-1)q_{n-1}-Q_n\right)\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{3(n-1)q_{n-1}}{Q_n 2^N}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{6q_{n-1}}{Q_n}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{6q_{n-1}}{Q_n}\sum_{s=0}^{N-1}2^s\omega_p\left(1/2^s,f\right)+ \\
&+ \frac{6q_{n-1}}{Q_n}2^N\omega_p\left(1/2^N,f\right).
\end{aligned}
$$

For II_2 we have that

$$
\begin{aligned}
II_2 &\leq \frac{3q_{n-1}2^{N+1}}{Q_n}\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{6q_{n-1}}{Q_n}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right)= \\
&= \frac{6q_{n-1}}{Q_n}\sum_{s=0}^{N-1}2^s\omega_p\left(1/2^s,f\right)+ \\
&+ \frac{6q_{n-1}}{Q_n}2^N\omega_p\left(1/2^N,f\right).
\end{aligned}
\tag{3.8}
$$

By combining (3.4)-(3.8) we obtain the inequality (2.2).

Finally, by using the condition (2.3) we also get the inequality (2.4) so the proof is complete. ■

Proof of Theorem 2.3. By using (1.2) we find that

$$
F_{2^n}=D_{2^n}-\frac{1}{Q_{2^n}}\sum_{k=1}^{2^n}q_{2^n-k}\left(w_{2^n-1}D_k\right) \tag{3.9}
$$

and

$$
T_{2^n}f=D_{2^n}*f-\frac{1}{Q_{2^n}}\sum_{k=1}^{2^n}q_{2^n-k}\left(\left(w_{2^n-1}D_k\right)*f\right). \tag{3.10}
$$

By applying Abel transformation we get that

$$
\begin{aligned}
(3.11) \qquad T_{2^n} f &= D_{2^n} * f - \\
&- \frac{1}{Q_{2^n}} \sum_{j=1}^{2^n-1} \left(q_{2^n-j} - q_{2^n-j-1}\right) j((w_{2^n-1} K_j) * f) - \\
&- \frac{1}{Q_{2^n}} q_0 2^n (w_{2^n-1} K_{2^n} * f)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.12) \quad & T_{2^n} f(x) - f(x) = \\
= & \int_G (f(x+t) - f(x)) F_{2^n}(t) dt = \\
= & \int_G (f(x+t) - f(x)) D_{2^n}(t) dt - \\
- & \frac{1}{Q_{2^n}} \sum_{j=1}^{2^n-1} \left(q_{2^n-j} - q_{2^n-j-1}\right) j \int_G \left(f(x+t) - f(x)\right) w_{2^n-1}(t) K_j(t) dt - \\
- & \frac{1}{Q_{2^n}} q_0 2^n \int_G (f(x+t) - f(x)) w_{2^n-1}(t) K_{2^n}(t) dt := \\
:= & III_1 + III_2 + III_3.
\end{aligned}
$$

By combining generalized Minkowski's inequality and equality (1.1), we find that

$$
(3.13) \qquad \|III_1\|_p \leq \int_{I_n} \|f(\cdot + t) - f(\cdot))\|_p D_{2^n}(t) dt \leq \omega_p \left(1/2^n, f\right).
$$

Since

$$
2^n q_0 \leq Q_{2^n} \ (n \in \mathbb{N}),
$$

by combining (1.5) and generalized Minkowski's inequality we obtain that

$$
\begin{aligned}
(3.14) \qquad \|III_3\|_p &\leq \int_G \|f(\cdot + t) - f(\cdot)\|_p K_{2^n}(t) \, d\mu(t) = \\
&= \int_{I_n} \|f(\cdot + t) - f(\cdot)\|_p K_{2^n}(t) \, d\mu(t) + \\
&+ \sum_{s=0}^{n-1} \int_{I_n(e_s)} \|f(\cdot + t) - f(\cdot)\|_p K_{2^n}(t) \, d\mu(t) \leq
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{I_n} \|f(\cdot+t)-f(\cdot)\|_p \frac{2^n+1}{2} d\mu(t) + \\
&+ \sum_{s=0}^{n-1} 2^s \int_{I_n(e_s)} \|f(\cdot+t)-f(\cdot)\|_p \, d\mu(t) \leq \\
&\leq \omega_p\left(1/2^n, f\right) \int_{I_n} \frac{2^n+1}{2} d\mu(t) + \\
&+ \sum_{s=0}^{n-1} 2^s \int_{I_n(e_s)} \omega_p\left(1/2^s, f\right) d\mu(t) \leq \\
&\leq \omega_p\left(1/2^n, f\right) + \sum_{s=0}^{n-1} \frac{2^s}{2^n} \omega_p\left(1/2^s, f\right).
\end{aligned}
$$

From the estimate (3.14) we can also conclude that

$$
2^n \int_G \|f(\cdot+t)-f(\cdot)\|_p K_{2^n}(t)\, d\mu(t) \leq \sum_{s=0}^{n} 2^s \omega_p\left(1/2^s, f\right). \tag{3.15}
$$

Let $2^k \leq j \leq 2^{k+1}-1$. By combining (1.3) and (3.15) we find that

$$
\begin{aligned}
&\left\| j \int_G |f(\cdot+t)-f(\cdot)| K_j(t)\, d\mu(t) \right\|_p \leq \qquad (3.16) \\
&\leq 3 \sum_{l=0}^{k} 2^l \int_G \|f(\cdot+t)-f(\cdot)\|_p K_{2^l}(t)\, d\mu(t) \leq \\
&\leq 3 \sum_{l=0}^{k} \sum_{s=0}^{l} 2^s \omega_p\left(1/2^s, f\right).
\end{aligned}
$$

According to (1.3) and (3.16) we get that

$$
\begin{aligned}
&\|III_2\|_p \leq \qquad (3.17) \\
&\leq \frac{1}{Q_{2^n}} \sum_{j=0}^{2^n-1} \left(q_{2^n-j}-q_{2^n-j-1}\right) j \int_G \|f(\cdot+t)-f(\cdot)\|_p |K_j(t)| dt \leq \\
&\leq \frac{1}{Q_{2^n}} \sum_{k=0}^{n-1} \sum_{j=2^k}^{2^{k+1}-1} \left(q_{2^n-j}-q_{2^n-j-1}\right) j \int_G \|f(\cdot+t)-f(\cdot)\|_p |K_j(t)| dt \leq
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{3}{Q_{2^n}}\sum_{k=0}^{n-1}\sum_{j=2^k}^{2^{k+1}-1}\left(q_{2^n-j}-q_{2^n-j-1}\right)\sum_{l=0}^{k}\sum_{s=0}^{l}2^s\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{3}{Q_{2^n}}\sum_{k=0}^{n-1}\left(q_{2^n-2^k}-q_{2^n-2^{k+1}}\right)\sum_{l=0}^{k}\sum_{s=0}^{l}2^s\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{3}{Q_{2^n}}\sum_{l=0}^{n-1}\sum_{k=l}^{n-1}\left(q_{2^n-2^k}-q_{2^n-2^{k+1}}\right)\sum_{s=0}^{l}2^s\omega_p\left(1/2^s,f\right)\leq \\
&\leq \frac{3}{Q_{2^n}}\sum_{l=0}^{n-1}q_{2^n-2^l}\sum_{s=0}^{l}2^s\omega_p\left(1/2^s,f\right)\leq \frac{3}{Q_{2^n}}\sum_{s=0}^{n-1}2^s\omega_p\left(1/2^s,f\right)\sum_{l=s}^{n-1}q_{2^n-2^l}\leq \\
&\leq \frac{3}{Q_{2^n}}\sum_{s=0}^{n-1}2^s\omega_p\left(1/2^s,f\right)q_{2^n-2^s}(n-s)\leq \\
&\leq 3\sum_{s=0}^{n-1}\frac{n-s}{2^{n-s}}\frac{q_{2^n-2^s}}{q_0}\omega_p\left(1/2^s,f\right).
\end{aligned}
$$

Finally, by combining (3.12), (3.13), (3.14) and (3.17) we can conclude that (2.5) holds so the proof is complete. ■

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