

A NEW PROOF OF A RESULT OF WIRSING FOR MULTIPLICATIVE FUNCTIONS

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*In celebration of the eighty fifth birthday of Professors
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Abstract. In this paper we give a proof of Wirsing’s result [11] for non-negative multiplicative functions.

1. Introduction

In this paper we give a new proof for the following famous result of E. Wirsing ([11], Satz 1.1).

Theorem. *Let g be a multiplicative function which assumes real nonnegative values only. Let*

$$(1.1) \quad \sum_{p \leq x} \frac{g(p) \log p}{p} \sim \alpha \log x, \quad x \rightarrow \infty,$$

hold with a constant $\alpha > 0$. Furthermore, let $g(p) = O(1)$ for all primes p , and let

$$\sum_{p, k \geq 2} p^{-k} g(p^k) < \infty.$$

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Besides this, if $\alpha \leq 1$, then let

$$\sum_{p^k \leq x, k \geq 2} g(p^k) = O(x(\log x)^{-1}).$$

Then

$$\sum_{n \leq x} g(n) \sim \frac{e^{-\gamma\alpha} x}{\Gamma(\alpha) \log x} \prod_{p \leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right)$$

as $x \rightarrow \infty$. Here γ denotes Euler's constant.

In the case of *complex-valued multiplicative functions of modulus ≤ 1* we used a method to give proofs for results of Halász und Wirsing (see [4], [5], [6]). Here we adopt the method to the present situation. Then the idea of the proof is as follows.

Let g as in the theorem. Define an *exponentially multiplicative function* g_0 (see [1] P. Erdős and A. Rényi) by

$$(1.2) \quad g_0(p^k) = \frac{g(p)}{k!} \quad (p \text{ prime}, \quad k \in \mathbb{N}).$$

Then $g = h * g_0$ where

$$\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty.$$

Further, define the multiplicative function τ_α by

$$(1.3) \quad \sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} = \zeta^\alpha(s),$$

where $\zeta(s)$ is Riemann's zeta-function. Then put

$$A(x) := H(1) e^{\alpha c} \exp \left(\sum_{p \leq x} \frac{g(p) - \alpha}{p} \right),$$

where $H(1) = \sum_{n=1}^{\infty} \frac{h(n)}{n}$ and

$$c = \sum_p \left(\frac{1}{p} + \log \left(1 - \frac{1}{p} \right) \right).$$

We define, for $1 \leq u \leq x$,

$$\begin{aligned} M(u) &:= 1 * (g - A(x) \tau_\alpha)(u) = \\ &= \sum_{n \leq u} (g(n) - A(x) \tau_\alpha(n)) \end{aligned}$$

and prove in the first part of the paper by convolution arithmetic

$$|M(x)| \ll \frac{x}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + o \left(\frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right) \text{ as } x \rightarrow \infty.$$

Next, we define an exponentially multiplicative function $\bar{g}_0$ by

$$\bar{g}_0(p^k) = \begin{cases} g_0(p^k) & \text{for } p \leq x, k \geq 1 \\ \alpha/k! & \text{for } p > x, k \geq 1 \end{cases}$$

and put $\bar{g} = h * \bar{g}_0$. Choosing

$$(1.4) \quad K_0(u) = 1 * \Lambda_{\bar{g}_0} * (\bar{g} - A(x)\tau_\alpha)(u)$$

we obtain, for $2 \leq u \leq x$,

$$(1.5) \quad M(u) = \frac{K_0(u)}{\log u} + o \left(\frac{u}{\log u} \exp \left(\sum_{p \leq u} \frac{g(p)}{p} \right) \right).$$

Now

$$(1.6) \quad \begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &\leq \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du =: \\ &=: I_1 + I_2. \end{aligned}$$

Then

$$I_1 \leq \sum_{n \leq x^\varepsilon} \frac{|g(n) - A(x)\tau_\alpha(n)|}{n} \ll \varepsilon \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) + \varepsilon^\alpha \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right)$$

and

$$I_2 \leq \frac{1}{\varepsilon} \left(\frac{1}{\log x} \int_1^x \frac{|K_0(u)|^2}{u^3} du \right)^{\frac{1}{2}} + o \left(\exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

Put $\sigma = 1 + \frac{1}{\log x}$. Then

$$\int_1^x \frac{|K_0(u)|^2}{u^3} du \leq e^2 \int_1^x \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}} du.$$

Since $(s = \sigma + it)$

$$\int_0^\infty K_0(e^u) e^{-us} e^{-iut} dt = \frac{\overline{G}_0'(s)}{\overline{G}_0(s)} (\overline{G}(s) - A(x) \zeta^\alpha(s)),$$

where

$$(1.7) \quad \overline{G}_0(s) = \sum_{n=1}^\infty \frac{\overline{g}_0(n)}{n^s} \quad \text{and} \quad \overline{G}(s) = \sum_{n=1}^\infty \frac{\overline{g}(n)}{n^s},$$

we conclude, by Parseval's equation,

$$(1.8) \quad \int_1^\infty \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}} du = \frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{\overline{G}_0'(s)}{\overline{G}_0(s)} \right|^2 |\overline{G}(s) - A(x) \zeta^\alpha(s)|^2 \frac{dt}{|s|^2}.$$

Using ideas of G. Tenenbaum (see [9], Theorem 4.10 and Theorem 4.13, and [10]) we estimate the last integral in (1.8) by $o\left(\log x \exp\left(2 \sum_{p \leq x} \frac{g(p)}{p}\right)\right)$, which proves our Theorem. ■

2. Notations and some preliminaries

Let g be as in Theorem. Define the exponentially multiplicative function g_0 by (1.2). The generating function $G_0(s)$ of g_0 satisfies

$$(2.1) \quad G_0(s) = \exp\left(\sum_p \frac{g(p)}{p^s}\right).$$

If we write $g = h * g_0$ then $G(s) = H(s)G_0(s)$ with $H(s) = \sum_{n=1}^\infty h(n)n^{-s}$ and

$$h(p) = 0 \text{ (} p \text{ prime)} \text{ and } \sum_{p, k \geq 2} \frac{|h(p^k)|}{p^k} < \infty$$

which implies

$$\sum_{n=1}^\infty \frac{|h(n)|}{n} < \infty.$$

For $\alpha > 0$ we define the multiplicative function τ_α by (1.3).

By Theorem II 5.2 and Theorem II 5.4 of Tenenbaum [9] the following holds.

Lemma 2.1. *Let $0 < a < b$. Then, uniformly for $a < \alpha < b, y \geq 2$ the relation*

$$\sum_{n \leq y} \tau_\alpha(n) = \frac{y}{\Gamma(\alpha)} (\log y)^{\alpha-1} \left\{ 1 + O\left(\frac{1}{\log y}\right) \right\}$$

holds.

Let $s = \sigma + it$. The function $\zeta(s) \exp(-\sum_p \frac{1}{p^s})$ is holomorphic at $s = 1$ and for $\sigma > 1$,

$$\log \zeta(s) - \sum_p \frac{1}{p^s} = -c + O(|s-1|),$$

where

$$c = \sum_p \left(\frac{1}{p} + \log \left(1 - \frac{1}{p} \right) \right).$$

Then

$$(2.2) \quad \zeta^\alpha(s) = e^{-\alpha c} \exp \left(\sum_p \frac{\alpha}{p^s} \right) \{1 + O(|s-1|)\}.$$

Put

$$(2.3) \quad A(x) = H(1) e^{\alpha c} \exp \left(\sum_{p \leq x} \frac{g(p) - \alpha}{p} \right)$$

and ($\xi > 0$)

$$W(\xi) := A(e^\xi).$$

We observe that, because of (1.1), $W(\xi)$ is a slowly oscillating function.

Next, we put

$$(2.4) \quad M(u) := \sum_{n \leq u} g(n) - A(x) \sum_{n \leq u} \tau_\alpha(n).$$

Further, if we define the arithmetic function $L_0 g$ by $L_0 g(n) = (\log n)g(n)$, we introduce Λ_g by

$$L_0 g = g * \Lambda_g.$$

Obviously,

$$(2.5) \quad \Lambda_{g_0}(n) = \begin{cases} g(p) \log p & n = p, \text{ } p \text{ prime;} \\ 0 & n \text{ not prime.} \end{cases}$$

and

$$\Lambda_{\tau_\alpha}(n) = \begin{cases} \alpha \log p & n = p^k, \text{ } p \text{ prime;} \\ 0 & n \neq p^k. \end{cases}$$

This can easily be seen from (2.1) and from the relation

$$\frac{(\zeta^\alpha(s))'}{\zeta^\alpha(s)} = \alpha \frac{\zeta'(s)}{\zeta(s)},$$

respectively.

Consider

$$\sum_{p \leq x^\varepsilon} \frac{g(p)}{p} = \sum_{p \leq x} \frac{g(p)}{p} - \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p}.$$

Obviously, by partial summation,

$$\begin{aligned} \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} &= \sum_{x^\varepsilon < p \leq x} \frac{g(p) - \alpha}{p} \cdot \frac{\log p}{\log p} + \sum_{x^\varepsilon < p \leq x} \frac{\alpha}{p} = \\ &= o(1) + \alpha \log \frac{1}{\varepsilon} \end{aligned}$$

which implies the inequality

$$(2.6) \quad \sum_{n \leq x^\varepsilon} \frac{g_0(n)}{n} \ll \varepsilon^\alpha \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right).$$

Now we prove

Lemma 2.2 *Let $\alpha > 1$. Then there exists a constant K so that*

$$\exp \left(\sum_{p \leq y} \frac{g(p) - 1}{p} \right) \leq K \exp \left(\sum_{p \leq x} \frac{g(p) - 1}{p} \right)$$

for $y \leq x$ and $x \geq x_0$.

Proof. Observe

$$\exp \left(\sum_{p \leq y} \frac{g(p) - 1}{p} \right) \asymp \exp \left(\sum_{p \leq y} \frac{g(p) - \alpha}{p} \right) (\log y)^{\alpha-1}.$$

Putting $\xi = \log x$ we estimate $W(\xi)\xi^{\alpha-1}$ where $\alpha > 1$. Since $W(\xi)$ is slowly oscillating we have, for $\xi \geq \xi_0(\varepsilon)$,

$$W(\eta) \leq (1 + \varepsilon)W(\xi) \text{ for } \frac{\xi}{2} \leq \eta \leq \xi$$

and

$$W(\eta) \leq (1 + \varepsilon)^\nu W(\xi) \text{ if } \eta \in [\frac{\xi}{2^\nu}, \frac{\xi}{2^{\nu-1}}].$$

Thus

$$\begin{aligned} \eta^{\alpha-1} W(\eta) &\leq (\frac{\xi}{2^{\nu-1}})^{\alpha-1} (1 + \varepsilon)^\nu W(\xi) = \\ &= \xi^{\alpha-1} W(\xi) \frac{(1 + \varepsilon)^{\nu-1}}{(2^{\alpha-1})^{\nu-1}} (1 + \varepsilon) \leq \\ &\leq 2\xi^{\alpha-1} W(\xi) \end{aligned}$$

and

$$\eta^{\alpha-1} W(\eta) \leq 2\xi^{\alpha-1} W(\xi), \quad \xi_0 \leq \eta \leq \xi.$$

Therefore

$$\exp\left(\sum_{p \leq y} \frac{g(p) - 1}{p}\right) \leq K \exp\left(\sum_{p \leq x} \frac{g(p) - 1}{p}\right) \text{ for } x_0(\varepsilon) \leq y \leq x$$

and, eventually with some larger constant, this holds for $y \leq x_0 \leq x$. ■

Remark 2.1. If $\alpha > 1$ then an easy consequence of Lemma 2.2 gives

$$\sum_{n \leq x} |h(n)| \sum_{m \leq \frac{x}{n}} g_0(m) \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right).$$

The same estimate is valid in the case $0 < \alpha \leq 1$ since (see (3.6))

$$\sum_{n \leq x} |h(n)| \log n \ll x$$

which implies

$$\sum_{n \leq x} |h(n)| \ll \frac{x}{\log x}.$$

Therefore

$$\sum_{n \leq x} g(n) \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right).$$

For applying Lemma 4.13 from [9] we choose B such that $\alpha < B$ and $|g(p)| \leq B$ for all p , and define

$$(2.7) \quad \varphi_0 := \pi \frac{B}{\alpha} \text{ and } \beta = 1 - \frac{\sin \varphi_0}{\varphi_0}.$$

Observe $\beta > 0$.

Further, put

$$I_1(\varepsilon) = \left\{ t : |t| \leq \frac{1}{\varepsilon \log x} \right\}$$

and

$$I_2(\varepsilon) = \left\{ t : \frac{1}{\varepsilon \log x} < |t| \leq \varepsilon^{-2\alpha} \right\}.$$

Then the following Lemma holds.

Lemma 2.3 *Let $\overline{G}(s)$ be defined by (1.7), where $s = \sigma + it$ ($\sigma > 1$). Then, for every $\varepsilon > 0$,*

$$(2.8) \quad \overline{G}(s) - A(x)\zeta^\alpha(s) \ll \varepsilon \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right) \quad \text{for } t \in I_1(\varepsilon)$$

and

$$(2.9) \quad \overline{G}(s) \ll \varepsilon^{\alpha\beta} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right) \quad \text{for } t \in I_2(\varepsilon).$$

Proof. Since, by (2.2),

$$\frac{\overline{G}(s)}{A(x)\zeta^\alpha(s)} = \{1 + o(1)\} \exp\left(\sum_p \frac{\overline{g}(p)}{p^s} - \sum_{p \leq x} \frac{g(p) - \alpha}{p} + \alpha c - \alpha c - \sum_p \frac{\alpha}{p^s}\right),$$

we consider

$$\begin{aligned} & \exp\left(\sum_p \frac{\overline{g}(p)}{p^s} - \sum_{p \leq x} \frac{g(p) - \alpha}{p} - \sum_p \frac{\alpha}{p^s}\right) = \\ & = \exp\left(\sum_{p \leq x} \frac{g(p) - \alpha}{p^s} - \sum_{p \leq x} \frac{g(p) - \alpha}{p}\right). \end{aligned}$$

By (1.1) and partial summation we conclude, as $x \rightarrow \infty$

$$(2.10) \quad \sum_{x^{\varepsilon^2} < p \leq x} \frac{g(p) - \alpha}{p} = o(1), \quad \sum_{x^{\varepsilon^2} < p \leq x} \frac{g(p) - \alpha}{p^s} = o(1).$$

Obviously

$$\sum_{p \leq x^{\varepsilon^2}} \frac{|g(p) - \alpha|}{p} |1 - p^{-(s-1)}| \ll |s-1| \varepsilon^2 \log x \ll \varepsilon \quad \text{for } t \in I_1(\varepsilon),$$

and thus (2.8) holds.

Next we observe

$$\begin{aligned}
 |\overline{G}(s)| &\ll |\overline{G}_0(s)| \ll \\
 &\ll \exp \left(\operatorname{Re} \sum_{p \leq x} \frac{g(p)}{p^{1+it}} \right) = \\
 &= \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \cos(t \log p) \right).
 \end{aligned}$$

Obviously

$$\begin{aligned}
 \sum_{p \leq x} \frac{g(p)}{p} \cos(t \log p) &\leq \sum_{p \leq x^\varepsilon} \frac{g(p)}{p} + \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} \cos(t \log p) = \\
 &= \sum_{p \leq x} \frac{g(p)}{p} - \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} (1 - \cos(t \log p)),
 \end{aligned}$$

where

$$\sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} (1 - \cos(t \log p)).$$

With $0 < \lambda < 1$ we have

$$\begin{aligned}
 \sum &= (1 - \lambda) \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} - \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} (\cos(t \log p - \lambda)) \geq \\
 &\geq (1 - \lambda) \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} - B \sum_{x^\varepsilon < p \leq x} \frac{(\cos(t \log p) - \lambda)^+}{p} = \\
 &= \Sigma_1 - \Sigma_2.
 \end{aligned}$$

By partial summation

$$\Sigma_1 \geq (1 - \lambda) \alpha \log \frac{1}{\varepsilon} + O(1),$$

since (cf. (2.10))

$$\begin{aligned}
 \sum_{x^\varepsilon < p \leq x} \frac{g(p)}{p} &= \sum_{x^\varepsilon < p \leq x} \frac{g(p) - \alpha}{p} + \sum_{x^\varepsilon < p \leq x} \frac{\alpha}{p} \geq \\
 &\geq \alpha \log \frac{1}{\varepsilon} + O(1).
 \end{aligned}$$

Next we use ideas of G. Tenenbaum (see [9], Lemma 4.13 and [10]). By [9], Lemma 4.13, we have

$$\Sigma_2 = Bm \log \frac{1}{\varepsilon} + O(1),$$

where (see (2.7))

$$\begin{aligned}
 m &= \frac{1}{2\pi} \int_{-\varphi_0}^{\varphi_0} (\cos \varphi - \lambda) d\varphi = \\
 &= \frac{\sin \varphi_0 - \varphi_0 \lambda}{\pi} = \\
 &= \frac{\varphi_0}{\pi} \left(\frac{\sin \varphi_0}{\varphi_0} - \lambda \right) = \\
 &= \frac{\alpha}{B} (1 - \lambda - \beta),
 \end{aligned}$$

i.e.

$$-\Sigma_2 = (-\alpha(1 - \lambda) + \beta) \log \frac{1}{\varepsilon}.$$

Then, together with (2.10),

$$\sum \geq (1 - \lambda)\alpha \log \frac{1}{\varepsilon} - \alpha(1 - \lambda) \log \frac{1}{\varepsilon} + O(1) + \beta\alpha \log \frac{1}{\varepsilon},$$

and

$$-\sum \leq -\alpha\beta \log \frac{1}{\varepsilon} + O(1) \ll \alpha\beta \log \varepsilon,$$

which proves (2.9). ■

Remark 2.2. A trivial estimate for $G(s)$ is given by

$$|\overline{G}(s)| \ll \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \quad \text{for all } \sigma > 1.$$

Remark 2.3. We use the following notations:

$$M^*(x) := \sum_{n \leq x} g(n), \quad m^*(x) := \sum_{n \leq x} \frac{g(n)}{n}$$

and

$$\beta^*(t) := e^{-t} M^*(e^t), \quad L^*(t) = t^{-\alpha} m^*(e^t).$$

Then the assertion of Theorem can be written as (cf. [3], p.147)

$$\begin{aligned}
 M^*(x) &\sim \alpha x (\log x)^{\alpha-1} L^*(\log x), \\
 (2.11) \quad m^*(x) &\sim \frac{e^{\gamma\alpha}}{\Gamma(\alpha+1)} \prod_{p \leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right).
 \end{aligned}$$

The function L^* is slowly oscillating. Let

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

be the generating Dirichlet series of g . Then integration by parts shows that, for $s = \sigma + it, \sigma > 1$, we have

$$s^{-1}G(s) = \int_0^{\beta} \beta^*(u) e^{-u(\sigma-1)} e^{-iut} du,$$

which implies that if (2.11) holds, then

$$G(s) = \frac{\alpha L^*\left(\frac{1}{\sigma-1}\right)}{(s-1)^\alpha} + o\left(\frac{L^*\left(\frac{1}{\sigma-1}\right)}{(\sigma-1)^\alpha}\right)$$

as $\sigma \rightarrow 1^+$, which holds uniformly on each bounded interval $-K \leq t \leq K$.

Lemma 2.4. *Let $M(u)$ be defined by (2.4), and let $\{a_n\}$ be a sequence of positive numbers a_n such that*

$$\sum_{n \leq x} a_n = 2x \log x + O(x).$$

Then

$$\sum_{n \leq x} |M\left(\frac{x}{n}\right)| a_n = 2 \int_1^x |M\left(\frac{x}{u}\right)| \log u du + O\left(x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right).$$

Proof. Put $c_1 = 0$ and, for $n \geq 2$,

$$c_n = a_n - 2 \int_{n-1}^n \log t dt.$$

Then

$$\begin{aligned} C(x) &:= \sum_{n \leq x} c_n = \sum_{n \leq x} a_n - 2 \int_1^{[x]} \log t dt = \\ &= O(x). \end{aligned}$$

We write

$$\begin{aligned}
\sum_{n \leq x} |M(\frac{x}{n})| a_n &= \sum_{n \leq x} |M(\frac{x}{n})| a_n - 2 \sum_{2 \leq n \leq x} |M(\frac{x}{n})| \int_{n-1}^n \log t dt + \\
&+ 2 \sum_{2 \leq n \leq x} |M(\frac{x}{n})| \int_{n-1}^n \log t dt - 2 \int_1^x |M(\frac{x}{t})| \log t dt = \\
&= \Sigma_1 + \Sigma_2 + 2 \int_1^x |M(\frac{x}{t})| \log t dt.
\end{aligned}$$

By partial summation

$$\begin{aligned}
|\Sigma_1| &= \left| \sum_{2 \leq n \leq x-1} C(n) \{ |M(\frac{x}{n})| - |M(\frac{x}{n+1})| \} + C(x) |M(\frac{x}{[x]})| + |M(x)| \right| \ll \\
&\ll \sum_{n \leq x-1} n \left| |M(\frac{x}{n})| - |M(\frac{x}{n+1})| \right| + O(x) + |M(x)| \ll \\
&\ll \sum_{n \leq x-1} n |M(\frac{x}{n}) - M(\frac{x}{n+1})| + O(x) + |M(x)|.
\end{aligned}$$

Using

$$|M(y)| \leq M^*(y) := \sum_{n \leq y} |g(n) - A(x)\tau_\alpha(n)|$$

we conclude

$$\begin{aligned}
|\Sigma_1| &\ll \sum_{n \leq x} n \left(M^*(\frac{x}{n}) - M^*(\frac{x}{n+1}) \right) + O(x) + M^*(x) = \\
&= \sum_{n \leq x} M^*(\frac{x}{n}) + O(x) + M^*(x) = \\
&= \sum_{m \leq x} |g(m) - A(x)\tau_\alpha(m)| \sum_{n \leq \frac{x}{m}} 1 + O(x) + M^*(x) = \\
&= O \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).
\end{aligned}$$

For the estimate of $\sum_2$ we observe

$$\begin{aligned}
& M\left(\frac{x}{n}\right) \left| \int_{n-1}^n \log t dt - \int_{n-1}^n \left| M\left(\frac{x}{t}\right) \right| \log t dt \right| \leq \\
& \leq \int_{n-1}^n \left| M\left(\frac{x}{n}\right) \right| - \left| M\left(\frac{x}{t}\right) \right| \log t dt \leq \\
& \leq \int_{n-1}^n \left(M^*\left(\frac{x}{t}\right) - M^*\left(\frac{x}{n}\right) \right) \log t dt \leq \\
& \leq (n-1) \left(M^*\left(\frac{x}{n-1}\right) - M^*\left(\frac{x}{n}\right) \right).
\end{aligned}$$

Then

$$\begin{aligned}
\Sigma_2 & \leq \sum_{n \leq x-1} n \left(M^*\left(\frac{x}{n}\right) - M^*\left(\frac{x}{n+1}\right) \right) + O(\log x) \ll \\
& \ll x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right),
\end{aligned}$$

and the assertion of Lemma 2.4 holds. ■

3. Proof of Theorem; First Step

With the function M defined in (2.4), the identity function $\mathbf{1}$ and L given by $L(u) = \log u$ ($u \geq 1$) we use, for arithmetical functions f , the convolutions

$$\begin{aligned}
L * f(x) &= \sum_{n \leq x} \left(\log \frac{x}{n} \right) f(n), \\
M * f(x) &= \sum_{n \leq x} \left(M\left(\frac{x}{n}\right) \right) f(n), \\
\mathbf{1} * f(x) &= \sum_{n \leq x} f(n).
\end{aligned}$$

Then the proof of Theorem 2 in Indlekofer [5] gives

$$(3.1) \quad L^2 M = M * (\Lambda_{g_0} * \Lambda_{g_0} + L_0 \Lambda_{g_0}) + (R_1 + R_2 + R_3) * \Lambda_{g_0} + L(R_1 + R_2 + R_3),$$

where

$$\begin{aligned} R_1 &= L * (g - A(x)\tau_\alpha), \\ R_2 &= \mathbf{1} * (L_0 h * g_0), \\ R_3 &= -\mathbf{1} * A(x)\tau_\alpha * (\Lambda_{\tau_\alpha} - \Lambda_{g_0}). \end{aligned}$$

In the **first step** we shall prove

$$\begin{aligned} (3.2) \quad |M(x)| &\ll \\ &\ll \frac{1}{\log x} \int_1^x |M(\frac{x}{u})| du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) = \\ &= \frac{x}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) \end{aligned}$$

as $x \rightarrow \infty$.

Since $g(p) = O(1)$ we obtain by a crude estimate (see (2.5))

$$\Lambda_{g_0} * \Lambda_{g_0} + L_0 \Lambda_{g_0} \ll \Lambda * \Lambda + L_0 \Lambda.$$

Putting $a_n = (\Lambda * \Lambda)(n) + \Lambda(n)(\log n)$ we use Selberg's formula

$$(3.3) \quad \sum_{n \leq x} a_n = 2x \log x + O(x)$$

and get, by (3.1) and Lemma 2.4, the first estimate in (3.2).

For $R_1(x)$ we obtain

$$\begin{aligned} (3.4) \quad |R_1(x)| &\ll \int_2^x \frac{\sum_{n \leq u} |g(n) + A(x)\tau_\alpha(n)|}{u} du \ll \\ &\ll \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right) \int_2^x \frac{du}{\log u} \ll \\ &\ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right). \end{aligned}$$

In the case

$$R_2(x) = \sum_{n \leq x} h(n) \log n \sum_{m \leq \frac{x}{n}} g_0(m)$$

we use, if $\alpha > 1$, Lemma 2.2 and conclude

$$(3.5) \quad |R_2(x)| \ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \sum_{n \leq x} \frac{|h(n)| \log n}{n} =$$

$$= o \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right)$$

since $\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty$.

In the case $0 < \alpha \leq 1$ we have

$$\sum_{\substack{p^k \leq x \\ k \geq 2}} |h(p^k)| \log p^k = O(x)$$

and (see Postnikov [8], p. 201)

$$(3.6) \quad \sum_{n \leq x} |h(n)| \log n \ll \sum_{n \leq x} |h(n)| \sum_{\substack{p^k \leq x \\ k \geq 2}} |h(p^k)| \log p^k \ll$$

$$\ll x \sum_{n \leq x} \frac{|h(n)|}{n} \ll x.$$

Then

$$\begin{aligned} R_2(x) &= \sum_{n \leq x} h(n) \log n \sum_{m \leq \frac{x}{n}} g_0(m) = \\ &= \sum_{m \leq x^\varepsilon} g_0(m) \sum_{n \leq \frac{x}{m}} h(n) \log n + \\ &\quad + \sum_{n \leq x^{1-\varepsilon}} h(n) \log n \sum_{m \leq \frac{x}{n}} g_0(m) =: \\ &=: \Sigma_1 + \Sigma_2. \end{aligned}$$

By (2.6)

$$\Sigma_1 \ll x \sum_{n \leq x^\varepsilon} \frac{g_0(n)}{n} \ll \varepsilon^\alpha x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right)$$

and as in (3.5),

$$\begin{aligned}\Sigma_2 &\ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right) \sum_{n \leq x^{1-\varepsilon}} \frac{|h(n)| \log n}{n} = \\ &= o(x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)).\end{aligned}$$

Let us consider $R_3(x)$. We have

$$\begin{aligned}(3.7) \quad R_3(x) &= A(x) \sum_{n \leq x} \tau_\alpha(n) \sum_{m \leq \frac{x}{n}} (\Lambda_{\tau_\alpha}(m) - \Lambda_{g_0}(m)) = \\ &= A(x) \left\{ \sum_{n \leq x^\varepsilon} \dots + \sum_{x^\varepsilon < n \leq x} \dots \right\} =: \\ &=: A(x)(\Sigma'_1 + \Sigma'_2).\end{aligned}$$

Then

$$\begin{aligned}|\Sigma'_1| &= \sum_{n \leq x^\varepsilon} \tau_\alpha(n) \sum_{m \leq \frac{x}{n}} |\Lambda_{\tau_\alpha}(m) - \Lambda_{g_0}(m)| \ll \\ &\ll x \sum_{n \leq x^\varepsilon} \frac{\tau_\alpha(n)}{n} \ll \varepsilon^\alpha x (\log x)^\alpha,\end{aligned}$$

i.e.

$$(3.8) \quad A(x)\Sigma'_1 \ll \varepsilon^\alpha x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right).$$

For the estimate of Σ'_2 we use Lemma 2.1. Then

$$\Sigma'_2 = \frac{x}{\Gamma(\alpha)} \sum_{m \leq x^{1-\varepsilon}} \frac{\Lambda_{\tau_\alpha}(m) - \Lambda_{g_0}(m)}{m} (\log \frac{x}{m})^{\alpha-1} \{1 + O(\frac{1}{\varepsilon \log x})\}.$$

Since

$$\sum_{m \leq y} \frac{\Lambda_{\tau_\alpha}(m) - \Lambda_{g_0}(m)}{m} = \sum_{p \leq y} \frac{\alpha \log p - g(p) \log p}{p} + \sum_{\substack{p^k \leq y \\ k \geq 2}} \frac{\alpha \log p}{p^k} = o(\log y) + O(1)$$

we conclude

$$(3.9) \quad \Sigma'_2 = o(x(\log x)^\alpha)$$

by partial summation. By (3.7), (3.8) and (3.9)

$$R_3(x) = o \left(x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

Consider

$$R_1 * \Lambda_{g_0}(x) = \sum_{n \leq x} \log \frac{x}{n} (g - A(x)\tau_\alpha) * \Lambda_{g_0}(n).$$

Observing

$$\begin{aligned} & \sum_{n \leq y} |g(n) - A(x)\tau_\alpha(n)| \sum_{p \leq \frac{y}{n}} g(p) \log p \ll \\ & \ll y \sum_{n \leq y} \frac{|g(n) - A(x)\tau_\alpha(n)|}{n} \ll y \exp \left(\sum_{p \leq y} \frac{g(p)}{p} \right) \end{aligned}$$

we conclude (cf. (3.4))

$$(3.10) \quad R_1 * \Lambda_{g_0}(x) \ll x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right).$$

Since $L_0 g_0 = g_0 * \Lambda_{g_0}$ we have

$$(3.11) \quad R_2 * \Lambda_{g_0}(x) \leq \log x R_2(x) = o \left(x \log x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

Let us return to

$$R_3 * \Lambda_{g_0}(x) = \sum_{n \leq x} (A(x)\tau_\alpha * (\Lambda_{\tau_\alpha} - \Lambda_{g_0}) * \Lambda_{g_0})(n).$$

First, we observe

$$\sum_{n \leq y} (\Lambda_{\tau_\alpha} * \Lambda_{g_0} - \Lambda_{g_0} * \Lambda_{g_0})(n) = O(y \log y).$$

Next, we consider

$$\begin{aligned} & \sum_{n \leq y} \left(\frac{\Lambda_{\tau_\alpha} * \Lambda_{g_0} - \Lambda_{g_0} * \Lambda_{g_0}}{n} \right)(n) = \\ & = \sum_{pp' \leq y} \frac{g(p) \log p}{p} \left(\frac{g(p') - \alpha}{p'} \log p' \right) + \sum_{\substack{pp'^k \leq y \\ k \geq 2}} \frac{g(p) \log p \log p'}{p p'^k} = \\ & = \sum_{pp' \leq y} \frac{g(p) \log p}{p} \left(\frac{g(p') - \alpha}{p'} \log p' \right) + O(\log y) =: \sum + O(y). \end{aligned}$$

We split $\sum$ into

$$\sum = \sum_{p' \leq y} \dots + \sum_{p \leq y^{1-\varepsilon}} \dots =: \Sigma_1 + \Sigma_2.$$

Then obviously

$$\Sigma_1 \ll \varepsilon (\log y)^2$$

and

$$\Sigma_2 = \sum_{p \leq y^{1-\varepsilon}} \frac{g(p) \log p}{p} o\left(\log \frac{y}{p}\right) = o((\log y)^2).$$

Then, arguing as in (3.8) and (3.9) we prove

$$(3.12) \quad R_3 * \Lambda_{g_0}(x) = o\left(x \log x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right).$$

Collecting the results (3.3) with Lemma 2.1, (3.4), (3.5), (3.7), (3.10), (3.11) and (3.12) we obtain (3.2).

Remark 3.1. Let g as above. If f is multiplicative and $|f| \leq g$, then

$$\left|\sum_{n \leq x} f(n)\right| \ll \frac{x}{\log x} \int_2^x \frac{\left|\sum_{n \leq u} f(n)\right|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right)$$

and

$$\begin{aligned} & \left|\sum_{n \leq x} f(n) - A_x g(n)\right| \ll \\ & \ll \frac{x}{\log x} \int_2^x \frac{\left|\sum_{n \leq u} f(n) - A_x g(n)\right|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) \end{aligned}$$

if $A_x \neq 0$, $A_x = O(1)$ and

$$\sum_{p \leq x} \frac{\Lambda_f - \Lambda_g}{p} = o(\log x).$$

Remark 3.2. It is easy to see that

$$\begin{aligned} LM &= \mathbf{1} * L_0(g - A(x)\tau_\alpha) + R_1 = \\ &= \mathbf{1} * \Lambda_{g_0} * (g - A(x)\tau_\alpha) + R_1 + R_2 + R_3, \end{aligned}$$

which implies, by (3.4), (3.5) and (3.10),

$$M(u) = \frac{\mathbf{1} * \Lambda_{g_0} * (g - A(x)\tau_\alpha)(u)}{\log u} + o\left(\frac{u}{\log u} \exp\left(\sum_{p \leq u} \frac{g(p)}{p}\right)\right)$$

for $u \leq x$. Since $g(n) = \bar{g}(n)$, $g_0(n) = \bar{g}_0(n)$ for $n \leq x$, we define (cf. (1.4))

$$K_0(u) = \mathbf{1} * \Lambda_{\bar{g}_0} * (\bar{g} - A(x)\tau_\alpha)(u) \quad \text{for } u \leq x.$$

Collecting the estimates (1.5), (1.6) from Section 1 we obtain

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &\leq \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du \ll \\ &\ll (\varepsilon + \varepsilon^\alpha) \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right) + \\ &+ \frac{1}{\varepsilon} \left(\frac{1}{\log x} \int_{-\infty}^{\infty} \left| \frac{\bar{G}'_0(s)}{\bar{G}_0(s)} \right|^2 |\bar{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2} \right) + \\ &+ o\left(\exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Therefore, if we show that

$$\begin{aligned} (3.13) \quad &\int_{-\infty}^{\infty} \left| \frac{\bar{G}'_0(s)}{\bar{G}_0(s)} \right|^2 |\bar{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2} = \\ &= o\left(\log x \exp\left(2 \sum_{p \leq x} \frac{g(p)}{p}\right)\right) \quad \text{as } x \rightarrow \infty \end{aligned}$$

we have finished the proof of our Theorem. ■

4. Proof of Theorem; Second Step

For the estimate of the integral in (3.13) we split the interval $(-\infty, \infty)$ into $I_1(\varepsilon) \cup I_2(\varepsilon)$ and $\{t : \varepsilon^{-2\alpha} < |t|\}$. In the first case we obtain by (2.3)

$$(4.1) \quad \int_{|t| \leq \varepsilon^{-2\alpha}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2} \ll \\ \ll \max(\varepsilon^2, \varepsilon^{2\alpha}, \varepsilon^{2\alpha\beta}) \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \int_{|t| \leq \varepsilon^{-2\alpha}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 \frac{dt}{|s|^2}$$

whereas

$$(4.2) \quad \int_{|t| > \varepsilon^{-2\alpha}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2} \ll \\ \ll \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \int_{|t| > \varepsilon^{-2\alpha}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 \frac{dt}{|s|^2}.$$

The integrals in (4.1) and (4.2) can be estimated by

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \leq [\varepsilon^{-2\alpha}]}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 dt, \\ \sum_{\substack{k \in \mathbb{Z} \\ |k| > [\varepsilon^{-\alpha}]}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{\overline{G}'_0(s)}{\overline{G}_0(s)} \right|^2 dt,$$

respectively. Further, we have

$$\frac{\overline{G}'_0(s)}{\overline{G}_0(s)} = - \sum_p \frac{g(p) \log p}{p^s}, \\ - \sum_p \frac{\log p}{p^s} = \frac{\zeta'(s)}{\zeta(s)} + O(1) \text{ for } \sigma > 1.$$

Since $|f(p)| \leq B$, we use $|a_n| < Bb_n$ in the proof of the Theorem 4.10 in [9] and conclude

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\overline{G}'_0}{\overline{G}_0} \left(1 + \frac{1}{\log x} + ik + it \right) \right|^2 dt \leq 3B^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\zeta'}{\zeta} \left(1 + \frac{1}{\log x} + it \right) \right|^2 dt \ll \log x.$$

Thus, by (4.1), (4.2) and (3.13) the proof of our Theorem is finished. ■

Remark 4.1. Let $g \geq 0$ be a multiplicative function, satisfying $g(p) \leq B$,

$$(4.3) \quad \sum_{p,k \geq 2} g(p^k) p^{-k} < \infty \quad \text{and} \quad \sum_{p^k \leq x, k \geq 2} g(p^k) = O(x(\log x)^{-1}).$$

If, for some $\alpha > 0$,

$$\sum_{n \leq x^\varepsilon} \frac{g(n)}{n} \ll \varepsilon^\alpha \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \quad \text{for every } \varepsilon > 0,$$

then we may apply the methods of the paper in the following context.

Put $f = h_1 * f_0$ where f_0 is exponentially multiplicative and $F(s) = H_1(s)F_0(s)$. Assume that

$$(4.4) \quad \sum_p \frac{g(p) - \operatorname{Ref}(p)p^{it}}{p}$$

converges for some $t = t_0$. For simplicity let $t_0 = 0$.

Put

$$A_x = \frac{H_1(1)}{H(1)} \exp \left(- \sum_{p \leq x} \frac{g(p) - \operatorname{Ref}(p)}{p} \right)$$

A_x has the form $AW(\log x)$, where A is non-zero constant and $W(\xi)$ is a non-vanishing slowly oscillating function of ξ . If (4.4) diverges for all $t \in \mathbb{R}$ then we choose $A_x = 0$.

Put $M = \mathbf{1} * (f - A_x g)$ $f = h_1 * f_0$, where f_0 is exponentially multiplicative. Then, as above,

$$LM = \mathbf{1} * L_0(f - A_x g) + R'_1$$

where $R'_1 = L * (f - A_x g)$ and

$$R'_1(x) \ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right).$$

Further,

$$\mathbf{1} * L_0(f - A_x g) = \mathbf{1} * \Lambda_{f_0} * (f - A_x g) + R'_2 + R'_3$$

$$R'_2 = \mathbf{1} * \Lambda_{h_1} * (f - A_x g)$$

and

$$(4.5) \quad R'_3 = \mathbf{1} * A_x g * (\Lambda_f - \Lambda_g).$$

If $A_x \neq 0$, then

$$\sum_p \frac{g(p) - \text{Ref}(p)}{p} \text{ converges,}$$

which implies

$$\sum_p \frac{|g(p) - f(p)|^2}{p} < \infty$$

and

$$R'_3(x) = O\left(\varepsilon^\alpha x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) + o\left(x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right).$$

In any case

$$R'_2(x) = o\left(x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right).$$

Now, in the same way as above we obtain

$$(4.6) \quad \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du \ll \ll \frac{1}{\varepsilon} \left(\frac{1}{\log x} \int_{-\infty}^{\infty} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 |F(s) - A_x G(s)|^2 \frac{dt}{|s|^2} \right)^{\frac{1}{2}} + o\left(x \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right).$$

First observe

$$(4.7) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{F'_0}{F_0} (1 + \sigma + ik + it) \right|^2 dt \ll \log x.$$

If $A_x = 0$ then

$$(4.8) \quad F(s) = o(G(\sigma)) = o\left(\exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) \text{ for } t \in I_1(\varepsilon) \cup I_2(\varepsilon).$$

If (4.5) holds then

$$(4.9) \quad F(s) - A_x G(s) = o(G(\sigma)) = o\left(\exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right)\right) \text{ for } t \in I_1(\varepsilon)$$

and

$$(4.10) \quad \max(|F(s)|, G(s)) \ll \varepsilon^{\alpha\beta} \left(\exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right) \right) \quad \text{for } t \in I_2(\varepsilon).$$

From this we conclude

Proposition. *Let g be multiplicative, $0 \leq g(p) \leq B$,*

$$\sum_{p, k \geq 2} g(p^k) p^{-k} \quad \text{and} \quad \sum_{p^k \leq x, k \geq 2} g(p^k) = O\left(\frac{x}{\log x}\right).$$

If $f : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative, $|f| \leq g$ satisfies (4.3) then (4.6) together with (4.7), ..., (4.10) holds.

References

- [1] **Erdős, P. and A. Rényi**, On the mean value of nonnegative multiplicative arithmetical functions, *Michigan Math. J.*, **12** (1965), 321–338.
- [2] **Halász, G.**, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta. Mat. Sci. Hung.*, **19** (1986), 365–403.
- [3] **Indlekofer, K.-H., I. Kátai and R. Wagner**, A comparative result for multiplicative functions, *Liet. Matem. Rink.*, **41(2)** (2001), 183–201.
- [4] **Indlekofer, K.-H.**, On the prime number theorem, *Annales Univ. Sci. Budapest., Sect. Comp.*, **27** (2007), 167–185.
- [5] **Indlekofer, K.-H.**, Identities in the convolution arithmetic of number theoretical functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **28** (2008), 303–325.
- [6] **Indlekofer, K.-H.**, On a quantitative form of Wirsing’s mean-value theorem for multiplicative functions, *Publ. Math. Debrecen*, **75(1-2)** (2009), 105–122.
- [7] **Kaya, E. and R. Wagner**, On some results of Indlekofer for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **48** (2018), 17–29.
- [8] **Postnikov, A. G.**, *Introduction to Analytic Number Theory*, Translation of mathematical monographs, Vol. **68.**, American Math. Soc. (1988).
- [9] **Tenenbaum, G.**, *Introduction to Analytic and Probabilistic Number Theory*, Graduate Studies in Mathematics, Vol. **163.**, American Math. Soc. (2015).

- [10] **Tenenbaum, G.**, Moyennes effectives de fonctions multiplicatives complexes, *Ramanujan Journal*, Springer-Verlag **44(3)** (2017), 641–701.
- [11] **Wirsing, E.**, Das asymptotische Verhalten von Summen über multiplikative Funktionen II, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 411–467.

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