Annales Univ. Sci. Budapest., Sect. Comp. 55 (2023) 77–96

EXTENDED ZABREJKO THEORY OF THE NEWTON–KANTOROVICH ITERATION AND THE PTÁK–POTRA ERROR ESTIMATES

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Communicated by László Szili

(Received December 17, 2022; accepted May 23, 2023)

Abstract. A new methodology is introduced that allows to extend the application of the Newton–Kantorovich Iteration (NKI) for solving non-linear Banach space valued operator equations containing a non-differentiable term. In particular, the Zabrejko theory is revisited. But the new majorant functions are tighter than before leading to an at least as weak semi-local convergence analysis for NKI. Moreover, the Pták–Potra error estimates are replaced with tighter ones. Furthermore, the region that determines the uniqueness of the solution is more precise. The novelty of this article is that the aforementioned benefits are obtained under the same or even weaker conditions. All these are also verified using numerical examples where the conditions are verified under the new approach but not under the earlier ones. The introduced methodology can apply analogously on other iterations containing the inverses of linear operators.

1. Introduction

Let B_1, B_2 stand for Banach spaces, let $U(x_0, l)$ denote the open ball with center $x_0 \in B_1$ and of radius $l > 0$. The ball $U[x_0, l]$ stands for the complement

Key words and phrases: Newton–Kantorovich Iteration, Semi-local Convergence, Pták–Potra estimates, Banach space, Majorant function.

2010 Mathematics Subject Classification: 65K10, 47H04, 65G99.

https://doi.org/10.71352/ac.55.077

of the ball $U(x_0, l)$. Moreover the notation $\mathscr{L}(B_1, B_2)$ stands for the space of continuous linear operators from B_1 into B_2.

We are concerned with the solvability of the non-linear equation

$$F_1(x) + F_2(x) = 0, \tag{1.1}$$

where the operators $F_1 : \Omega \subset B_1 \to B_2$, $F_2 : \Omega \to B_2$ and $\Omega = U[x_0, R]$ for some $R > 0$. The semi-local convergence of the Newton–Kantorovich Iterations (NKI)

$$x_0 \in \Omega, \quad x_{k+1} = x_k - F_1'(x_k)^{-1}(F_1(x_k) + F_2(x_k)) \quad k = 0, 1, 2, \ldots \tag{1.2}$$

has been established in the elegant article in [19]. The conditions employed are:

(I) F_1 is differentiable in the Fréchet-sense for each $w_1, w_2 \in \Omega$ and

$$\|F_1'(w_2) - F_1'(w_1)\| \le h_1(s)\|w_2 - w_1\|, \quad 0 < s < R, \tag{1.3}$$

where the function $h_1 : [0, s) \to \mathbb{R}$ is non-decreasing and continuous on the interval $M = [0, R]$;

(II) The operator F_2 satisfies the condition

$$\|F_2(w_2) - F_2(w_1)\| \le \epsilon_1(s)\|w_2 - w_1\| \quad (0 < s < R), \tag{1.4}$$

where the function $\epsilon_1 : [0, s) \to \mathbb{R}$ is non-decreasing and continuous on the interval M.

Moreover, the convergence analysis requires the introduction of auxiliary parameters

$$\alpha = \|F_1'(x_0)^{-1}(F_1(x_0) + F_2(x_0))\|,$$

$$\beta = \|F_1'(x_0)^{-1}\| \ \big(\text{provided that } F_1'(x_0)^{-1} \in \mathscr{L}(B_2, B_1)\big),$$

the functions

$$v_1(s) = \int_0^1 h_1(t)dt,$$

$$\varphi_1(s) = \alpha + \beta \int_0^s v_1(t)dt - s,$$

$$\psi_1(s) = \beta \int_0^s \epsilon_1(t)dt$$

$$\text{and} \quad q_1(s) = \varphi_1(s) + \psi_1(s).$$

Then, the semi-local convergence of NKI is based on the study of the solutions of the equation

$$q_1(s) = 0. \tag{1.5}$$

As an example in the interesting case when the functions h_1 and ϵ_1 are constants: $h_1(s) = h_1, \epsilon_1(s) = \epsilon_1$, we have

$$\varphi_1(s) = \frac{\beta h_1 s^2}{2} - s + \alpha,$$
$$\psi_1(s) = \epsilon_1 s.$$

Then, the following conditions are needed [19]

$$2\alpha\beta h_1 \leq (1-\epsilon_1)^2 \tag{1.6}$$

and

(1.7)

$$1-\epsilon_1 - \sqrt{(1-\epsilon_1)^2 - 2\alpha\beta h_1} \quad \leq \quad \beta h_1 R \quad \leq \quad 1-\epsilon_1 - \sqrt{(1-\epsilon_1)^2 - 2\alpha\beta h_1}.$$

If $F_2 = 0$, then $\epsilon_1 = 0$, and the condition (1.6) reduces to

$$2\alpha\beta h_1 \leq 1. \tag{1.8}$$

The inequation (1.8) is famous for its simplicity and clarity, Newton–Kantorovich condition for the semi-local convergence of Newton's method (NM)

$$x_0 \in \Omega, \quad x_{k+1} = x_k - F_1'(x_k)^{-1} F_1(x_k) \quad k = 0, 1, 2, \ldots . \tag{1.9}$$

The following concerns exist with the convergence analysis presented in [19].

(1) The results are given in non-affine invariant form. The advantages of affine versus non-affine invariant form are well explained in [6].

(2) The ball of convergence is small in general.

(3) The estimates on the error distances $\|x_{k+1} - x_k\|$, $\|x_* - x_k\|$ are pessimistic, where $x_* \in \Omega$ denotes a solution of the equation (1.1).

(4) The uniqueness of the solution ball is small in general.

In this article, we address all the concerns (1)-(4). The main idea is the introduction of the center Lipschitz condition (to be precised in (2.1)). This way a more precise location $\Omega_0 \subseteq \Omega$, where the iterates belong than in [19] is obtained. Then, in Ω_0, the new functions h and ϵ (to be precised in (2.3) and (2.30)) are at least as tight as h_1 and ϵ_1, respectively. Hence, they can replace h_1 and ϵ_1 in the convergence analysis for NKI. It is important to notice that no additional conditions are needed, since the computation of the major functions h_1 and ϵ_1 involves that of h_0, h and ϵ as special cases. This is how we improve on all concerns (1)-(4). The new idea can also apply to other single and multi-step methods with inverses of linear operators analogously [1–5, 7, 9, 10, 12–17].

2. Majorant functions and sequences

Some real functions and sequences are developed since they play a role in the study of the semi-local convergence of NM.

Suppose that there exists an initial point $x_0 \in \Omega$ such that $F_1'(x_0)^{-1} \in \in \mathscr{L}(B_2, B_1)$.

Definition 2.1. The operator F_1' satisfies the center-Lipschitz condition on the set Ω if for each $w \in \Omega$

$$\|F_1'(x_0)^{-1}(F_1'(w) - F_1'(x_0))\| \leq h_0(s)\|w - x_0\|, \tag{2.1}$$

where the function h_0 is non-decreasing on the interval M.

Suppose that the function

$$\int_0^s h_0(t)dt - 1 \tag{2.2}$$

has a unique zero in the interval $(0, R]$. Denote such a zero by $\bar{s}$. Set $M_0 = [0, \bar{s}]$ and $\Omega_0 = U(x_0, \bar{s})$.

Definition 2.2. The operator F_1' satisfies the restricted-Lipschitz condition on the set Ω_0 if for each $w_1, w_2 \in \Omega_0$

$$\|F_1'(x_0)^{-1}(F_1'(w_2) - F_1'(w_1))\| \leq h(s)\|w_2 - w_1\|, \tag{2.3}$$

where the function h is non-decreasing on the interval M_0.

Definition 2.3. The operator F_1' satisfies the Lipschitz condition on the set Ω if for each $w_1, w_2 \in \Omega$

$$\|F_1'(x_0)^{-1}(F_1'(w_2) - F_1'(w_1))\| \leq h_1(s)\|w_2 - w_1\|, \tag{2.4}$$

where the function h_1 is non-decreasing on the interval M.

Remark 2.1. The preceding definitions imply for each $t \in M_0$

$$h_0(t) \leq h_1(t) \tag{2.5}$$

and

$$h(t) \leq h_1(t), \tag{2.6}$$

since

$$M_0 \subseteq M \quad \text{and} \quad \Omega_0 \subseteq \Omega. \tag{2.7}$$

Define the constants

$$\alpha = \|F_1^{'}(x_0)^{-1}(F_1(x_0) + F_2(x_0))\|, \quad \beta = \|F_1^{'}(x_0)^{-1}\|,$$

and the functions

$$v_0(s) = \int_0^s h_0(t)dt, \tag{2.8}$$

$$v(s) = \int_0^s h(t)dt, \tag{2.9}$$

$$\varphi_0(s) = \alpha + \int_0^s h_0(t)dt - s, \tag{2.10}$$

$$\varphi(s) = \alpha + \int_0^s h(t)dt - s. \tag{2.11}$$

It follows by these definitions and (2.5)–(2.11) that for each $s \in M_0$

$$v_0(s) \leq v(s) \tag{2.12}$$

and

$$\varphi_0(s) \leq \varphi(s). \tag{2.13}$$

Moreover, from the estimate

$$\begin{aligned} \|F_1^{'}(x_0)^{-1}(F_1^{'}(w_2) - F_1^{'}(w_1))\| &\leq \|F_1^{'}(x_0)^{-1}\|\|F_1^{'}(w_2) - F_1^{'}(w_1)\| \leq \\ &\leq \beta\|F_1^{'}(w_2) - F_1^{'}(w_1)\| \end{aligned} \tag{2.14}$$

we get that for each $s \in M_0$

$$v_0(s) \leq \beta v_1(s), \tag{2.15}$$

$$v(s) \leq \beta v_1(s), \tag{2.16}$$

$$h_0(s) \leq \beta h_1(s), \tag{2.17}$$

$$h(s) \leq \beta h_1(s), \tag{2.18}$$

$$\varphi_0(s) \leq \varphi_1(s) \tag{2.19}$$

and

$$\varphi(s) \leq \varphi_1(s). \tag{2.20}$$

The following estimate was given in [8] using (2.4) and the Banach Lemma on linear invertible operators for each $x \in \Omega$.

$$\|F_1^{'}(x)^{-1}\| \leq \frac{\beta}{1 - \beta h_1 \|x - x_0\|}, \tag{2.21}$$

since

$$\|F_1^{'}(x_0)^{-1}\|\|F_1^{'}(x) - F_1^{'}(x_0)\| \leq \beta h_1 \|x - x_0\| \leq \beta h_1(s) < 1 \tag{2.22}$$

was used.

But (2.1) is actually needed which is weaker to obtain instead for each $x \in \Omega_0$

$$\|F_1^{'}(x_0)^{-1}(F_1^{'}(x) - F_1^{'}(x_0))\| \leq h_0(\|x - x_0\|) \leq h_0(\bar{s}) < 1, \tag{2.23}$$

thus

$$\|F_1^{'}(x)^{-1} F_1^{'}(x_0)\| \leq \frac{1}{1 - h_0(\|x - x_0\|)}. \tag{2.24}$$

In view of (2.17) the estimate (2.24) is tighter than (2.21).

From now on we assume that for each $s \in M_0$

$$h_0(s) \leq h(s). \tag{2.25}$$

Notice that if this is not true, then we can use the function $\bar{h}$ which is defined to be the largest of h_0, h in the interval M_0.

Then, the estimate (2.24) can be rewritten as

$$\|F_1^{'}(x)^{-1} F_1^{'}(x_0)\| \leq \frac{1}{1 - h_0(\|x - x_0\|)} \leq \frac{1}{1 - h(\|x - x_0\|)} \tag{2.26}$$

if (2.25) holds and

$$\|F_1^{'}(x)^{-1} F_1^{'}(x_0)\| \leq \frac{1}{1 - h_0(\|x - x_0\|)} \leq \frac{1}{1 - \bar{h}(\|x - x_0\|)}. \tag{2.27}$$

In either case, estimates are less tight than (2.24) but tighter than (2.21), since for each $s \in M_0$

$$\bar{h}(s) \leq h_1(s). \tag{2.28}$$

Definition 2.4. The operator F_2 is restricted-Lipschitz on the set Ω_0 if for each $w_1, w_2 \in \Omega_0$

$$\|F_1'(x_0)^{-1}(F_2(w_2) - F_2(w_1))\| \leq \epsilon(s)\|w_2 - w_1\|, \tag{2.29}$$

where the function ϵ is non-decreasing on the interval M_0.

Definition 2.5. The operator F_2 is Lipschitz on the set Ω if for each $w_1, w_2 \in \Omega$

$$\|F_2(w_2) - F_2(w_1)\| \leq \epsilon_1(s)\|w_2 - w_1\|, \tag{2.30}$$

where the function ϵ_1 is non-decreasing on the interval M.

Remark 2.2. It follows by (2.29), (2.30) that for each $s \in M_0$

$$\epsilon(s) \leq \beta\epsilon_1(s). \tag{2.31}$$

Define the function ψ_2 on the interval M_0 by

$$\psi_2(s) = \int_0^s \epsilon(t)dt. \tag{2.32}$$

Then, we have

$$\psi_2(s) \leq \beta\psi_1(s). \tag{2.33}$$

Consequently, in view of the discussion in the last two remarks h or $\bar{h}$ can replace h_1 in all the results in [19]. However, there are other benefits involving the uniqueness ball and the error estimates.

3. Semi-local convergence

The following result is the extension of the Theorem 1 in [19]. Define the function q on the interval M_0 by

$$q(s) = \varphi(s) + \psi_2(s). \tag{3.1}$$

Theorem 3.1. *Suppose:*

(i) *The conditions* (2.1) *and* (2.3) *hold on* Ω *and* Ω_0 *respectively.*

(ii) *The function* q *has a unique zero* ρ *in the interval* M_0 *and*

$$q(\bar{s}) \leq 0, \tag{3.2}$$

where $\bar{s}$ *is given by* (2.2).

Set $M_1 = [0, \rho)$ *and* $\Omega_1 = U(x_0, \rho)$.
Then, the following assertions hold:

(1) *The equation* (1.1) *has a solution* $x_* \in \Omega_1$. *This solution is unique in* Ω.

(2) *The iterates* x_k *are well defined in* Ω_1 *and satisfy for each* $k = 0, 1, \dots$

$$\|x_{k+1} - x_k\| \le \rho_{k+1} - \rho_k \tag{3.3}$$

and

$$\|x_* - x_k\| \le \rho - \rho_k. \tag{3.4}$$

Here, the sequence $\{\rho_k\}$ *is monotonically increasing with*

$$\rho_{k+1} = \rho_k - \frac{q(\rho_k)}{\varphi'(\rho_k)} \tag{3.5}$$

and

$$\lim_{k \to +\infty} \rho_k = \rho. \tag{3.6}$$

The proofs of specializations of this theorem when $F_2 = 0$ were provided in [18]. But the convergence criteria considered were rather challenging to verify. That is why this theorem was shown in [19, Theorem 1] but with h_1 replacing h in the above extended version of our Theorem 3.1.

Concerning the uniqueness ball, we present:

Proposition 3.2. *Suppose:*

(1) *The conditions* (2.1) *and* (2.3) *hold on* Ω *and* Ω_0, *respectively.*

(2) *The function* q *has a unique zero in the interval* $[0, \bar{s}]$ *and* $q(\bar{s}) \le 0$.

Then, the following assertions hold:

(i) *The equation* $F_1(x) = 0$ *has a unique solution in* Ω_0 *with* $x_* \in \Omega_1$.

(ii) *The modified Newton's method defined for each* $k = 0, 1, 2, \dots$ *by*

$$y_{k+1} = y_k - F_1^{'}(y_0)^{-1} F_1(y_k), \quad y_0 = x_0$$

converges to x_* *with*

$$\|y_{k+1} - y_k\| \le \rho_{k+1}^0 - \rho_k^0$$

and

$$\|x_* - y_k\| \le \rho - \rho_k^0,$$

where

$$\rho^0_{k+1} = d^0(\rho^0_k), \quad \rho^0_0 = 0$$

and

$$d^0(s) = s + q(s).$$

Proof. Simply replace h_1 by h in the proof of Proposition 2 in [19]. ■

Remark 3.1. Define the functions on the interval M by

$$\varphi_1(s) = \alpha + \beta \int_0^s h_1(t)dt - s,$$

$$\psi_1(s) = \beta \int_0^1 \epsilon_1(t)dt,$$

$$q_1(s) = \varphi_1(s) + \psi_1(s),$$
$$d_1(s) = s + q_1(s)$$

and the sequence

$$\rho^1_{k+1} = d^1(\rho^1_k), \quad \rho^1_0 = 0.$$

The hypotheses in Proposition 2 are: Suppose that the function q_1 has a unique zero ρ^1 in the interval M and

$$q_1(R) \leq 0.$$

Then, the solution is unique in $\Omega_2 = U(x_0, \rho^1)$.

But, we have that for each $s \in M$

$$q(s) \leq q_1(s).$$

In particular, we get

$$q(\rho^1) \leq q_1(\rho^1) = 0$$

by the definition of ρ^1. Consequently, we conclude that

$$\rho \leq \rho^1.$$

Therefore, the Proposition 3.2 gives a smaller ball where the solution x_* is unique, since

$$\Omega_1 \subseteq \Omega_2.$$

Moreover, a simple induction argument shows that

$$0 \leq \rho^0_{k+1} - \rho^0_k \leq \rho^1_{k+1} - \rho^1_k.$$

Hence, the new error estimates are tighter.

We conclude advantages hold for the modified Newton method (see also the numerical examples) and under weaker conditions, where

$$\rho^1_{k+1} = \rho^1_k - \frac{q_1(\rho^1_k)}{\varphi^1_1(\rho^1_k)}. \tag{3.7}$$

In order to study the convergence of the sequence $\{\rho_k\}$ given by the formula (3.5), let

$$y(s) = -\frac{q(s)}{\varphi'(s)}.$$

Then, (3.5) can be rewritten as

$$\rho_{k+1} = \rho_k + y(s). \tag{3.8}$$

Some properties of this sequence are provided in the next result.

Proposition 3.3. *Suppose that the conditions of Theorem 3.1 hold. Then, the sequence $\{\rho_k\}$ given by (3.5) (or (3.8)) is monotonically increasing and converges to ρ.*

Proof. Simply exchange (3.5), h by (3.7), h_1 in the proof of Proposition 3 in [19]. ∎

Proof of Theorem 3.1. Notice that in view of (2.21) and (2.26) the function h_1 can be exchanged by h in the proof of Theorem 1 (see also Proposition 4) in [19]. ∎

Remark 3.2. We have as in Remark 3.1 that for each $k = 0, 1, 2, \ldots$

$$0 \leq \rho_{k+1} - \rho_k \leq \rho^1_{k+1} - \rho^1_k.$$

Hence, the aforementioned advantages hold for NM.

The rest of the improvements involve Pták–Potra estimates.

Suppose that the function $y(s)$ is strictly monotonically increasing on the interval M_1. Then, for $\alpha = y(0)$, the function $y(s)$ has an inverse function $z(s) = y(s)^{-1}$ on the interval $M_2 = [0, \alpha]$. As in [19] define the function

$$T(s) = y(s + z(s)).$$

Then, we can write for each $k = 1, 2, \ldots$

$$\rho_{k+1} - \rho_k = T(\rho_k - \rho_{k-1}).$$

Moreover, define the iterates of the function $T(s)$ for $k = 0, 1, 2, \dots$ as

$$T^{(0)}(s) = s, \quad T^{(k+1)}(s) = T(T^{(k)}(s))$$

and the function

$$Q(s) = \sum_{k=0}^{\infty} T^{(k)}(s).$$

Then, we have the corresponding results to the ones in [19].

Proposition 3.4. *Suppose that the function $y(s)$ is strictly monotonically decreasing on the interval M_0. Then, the following error estimates hold for each $k = 0, 1, 2, \dots$*

$$\rho_{k+1} - \rho_k = T^{(k)}(\alpha)$$

and

$$\rho - \rho_k = Q(T^{(k)}(\alpha)).$$

Theorem 3.5. *Suppose that the conditions of the Theorem 3.1 hold. Then, the conclusions of it also hold for each $k = 0, 1, 2, \dots$*

$$\|x_{k+1} - x_k\| \leq T^{(k)}(\alpha)$$

and

$$\|x_* - x_k\| \leq Q(T^{(k)}(\alpha)).$$

Proposition 3.6. *Suppose that the conditions of the Theorem 3.1 hold. Then, the function $y(s)$ is strictly monotonically decreasing on the interval $[0, \rho]$. Hence, this function is a bijection from $[0, \rho]$ to $[0, \alpha]$.*

It follows by the last two Propositions that the Theorem 3.1 can be rewritten as

Theorem 3.7. *Suppose:*

(i) *The function $q(s)$ has a unique zero ρ in the interval $[0, \bar{s}]$; and*

(ii) $q(\bar{s}) \leq 0$.

Then, the following assertions hold:

The equation $F_1(x) = 0$ has a solution $x_* \in \Omega_1$, which is unique in Ω_0 and the iterates $x_k \in \Omega_1$;

For $k = 0, 1, 2, \dots$

$$\|x_{k+1} - x_k\| \leq T^{(k)}(\alpha)$$

and

$$\|x_* - x_k\| \leq Q(T^{(k)}(\alpha)).$$

Remark 3.3. The proofs of the last four results is given from the corresponding ones in [19] if the functions $y(s)$ replaces the function $y_1(s)$ defined by

$$y_1(s) = -\frac{q_1(s)}{\varphi_1'(s)}.$$

Denote by T_1, Q_1 the corresponding operators to T and Q respectively. Then, from the preceding relationships between the functions q, φ, q_1, ψ_1 we conclude that

$$T^{(k)}(\alpha) \leq T_1^{(k)}(\alpha)$$

and

$$Q(T^{(k)}(\alpha)) \leq Q_1(T_1^{(k)}(\alpha)).$$

Consequently, the new error bounds are at least as tight as the ones in [19].

4. Special cases

Let us suppose that the functions h_0, h and ϵ are constants and set $h_0(s) = h_0$, $h(s) = h$ and $\epsilon(s) = \epsilon$. Then, the functions φ, ψ_2 and q_1 reduce to

$$\varphi(s) = \alpha + \frac{hs^2}{2} - s, \quad \psi_2(s) = \epsilon s$$

and

$$q(s) = \alpha + \frac{hs^2}{2} - (1-\epsilon)s.$$

Therefore, the conditions of the two theorems are fulfilled if

$$2\alpha h \leq (1-\epsilon)^2 \tag{4.1}$$

and

$$1-\epsilon-\sqrt{(1-\epsilon)^2-2\alpha h} \leq h\bar{s} \leq 1-\epsilon+\sqrt{(1-\epsilon)^2-2\alpha h}, \tag{4.2}$$

where

$$\rho = \frac{1-\epsilon-\sqrt{(1-\epsilon)^2-2\alpha h}}{h}. \tag{4.3}$$

The right inequality in (4.1) and (4.2) are both strict or both equalities. Then, we connect these conditions to the function $T(s)$ and $Q(s)$.

Proposition 4.1. *Suppose that the functions h_0, h and ϵ are constants and the conditions (4.1) and (4.2) hold. Then, the following assertions hold for*

$$\gamma = \frac{1}{h}\epsilon, \quad \delta = \frac{1}{h}(1 - 2\epsilon - 2\alpha h),$$

$$T(s) = \frac{s^2 + 2\gamma s}{2(\gamma + \sqrt{(s+\gamma)^2 + \delta})}$$

and

$$Q(s) = s + \sqrt{(s+\gamma)^2 + \delta} - \sqrt{\gamma^2 + \delta}.$$

Proposition 4.2. *Under the conditions of the Proposition 4.1, the following assertions hold for*

$$\lambda(s) = \frac{\sqrt{s^2 + \delta} - \sqrt{\delta}}{s},$$

$$T^{(k)}(s) = \frac{2\sqrt{\delta}\lambda^{2^k}(s)}{1 - \lambda^{2^{k+1}}}$$

and

$$Q(T^{(k)}(s)) = \frac{2\sqrt{\delta}\lambda^{2^{k+1}}(s)}{1 - \lambda^{2^k}(s)}.$$

Remark 4.1. (1) The proofs of the last two Propositions are obtained from the corresponding ones in [19] by exchanging T_1, Q_1 by T and Q, respectively.

(2) As in [19], we recognize two special choices of the function ϵ.
Case 1: $\epsilon = \mathbf{0}$. Then, the sequence $\{\rho_k\}$ generated by the formula (3.5) converges faster than any geometric progression of any denominator to ρ.
Case 2: $\epsilon > \mathbf{0}$. Then, the sequence $\{\rho_k\}$ converges to zero not faster than a geometric progression with denominator

$$\mu_1 = \min\left\{\frac{\gamma}{\gamma + \sqrt{\gamma^2 + \delta}}, \frac{\alpha + 2\gamma}{2(\gamma + \sqrt{(\alpha+\gamma)^2 + \delta})}\right\}\mu_0$$

and not slower than a geometric progression with denominator with

$$\mu_2 = \max \mu_0,$$

with $\mu_1, \mu_2 \in [0, \frac{1}{2}]$ if $\delta \geq 0$ and $[\frac{1}{2}, 1]$ if $\delta < 0$, where

$$\mu_0 = \left\{\frac{\gamma}{\gamma + \sqrt{\gamma^2 + \delta}}, \frac{\alpha + 2\gamma}{2(\gamma + \sqrt{(\alpha+\gamma)^2 + \delta})}\right\}.$$

Moreover, the following holds

$$\lim_{n\to+\infty} \frac{\rho-\rho_{k+1}}{\rho-\rho_k} = \frac{\gamma}{\gamma+\sqrt{\gamma^2+\delta}}.$$

That is, the sequence $\{\rho_k\}$ converges to ρ asymptotically as a geometric progression with denominator

$$\frac{\gamma}{\gamma+\sqrt{\gamma^2+\delta}}.$$

(3) As in Pták [11], the estimates $\|x_*-x_k\|$ can be further improved if one works with the ball $U(x_k, \bar{s}-\|x_k-x_0\|) \subset \Omega$ instead of Ω. The corresponding functions to $h(s)$ and $\epsilon(s)$ functions should be

$$h^k(s) = h(\|x_k-x_0\|+s),$$
$$\epsilon^k(s) = \epsilon(\|x_k-x_0\|+s)$$

the numbers

$$\alpha_k = \|x_{k+1}-x_k\|$$

and $\|x_*-x_k\|$ is determined by the zero s_k of the equation

$$\alpha_k + \int_0^s (s-t)h^k(t)dt + \int_0^s \epsilon^k(t)dt - s = 0.$$

5. Numerical examples

The convergence conditions are tested on some examples in the interesting case when the functions h_0, h and ϵ are constants.

Example 5.1. Let $b \in (0, \frac{1}{2}), R = 1-b$ and $x_0 = 1$. Then, the set $\Omega = = U(1, 1-b)$. Define the functions F_1 and F_2 on the ball Ω by

$$F_1(x) = x^3 - b,$$

$$F_2(x) = \frac{1}{100\pi}\sin \pi x.$$

Case $F_2(x) = 0$: Then, the definitions (2.1), (2.3), (2.4) hold if $\beta = \frac{1}{3}$, $h_0 = 3-b, h_1 = 2(2-b)$ and since $\Omega \cap U(x_0, \bar{s}) = U(x_0, \bar{s})$ for $\bar{s} = \frac{1}{3-b}$, we get $h = 2(1+\frac{1}{3-b})$.

Notice that

$$h_0 < h < h_1$$

hold. We also have that $\alpha = \frac{1-b}{3}$. Then, the condition (1.8) is not fulfilled since

$$2h_1\alpha = 2(2(2-b))\frac{1-b}{3} > 1 \quad \text{for each } b \in (0, \frac{1}{2}).$$

But, the new condition (4.1) is fulfilled, since

$$2h\alpha = 2\left(2\left[1 + \frac{1}{3-b}\right]\right)\frac{1-b}{3} < 1 \tag{5.1}$$

for each $b \in (0.461983, \frac{1}{2})$.

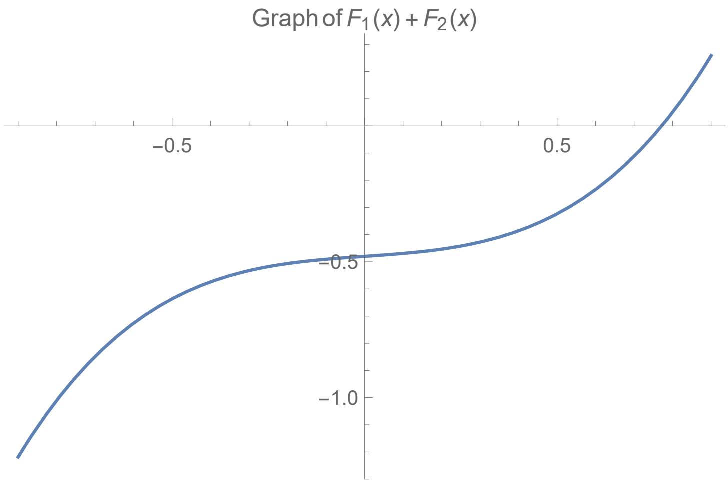

Figure 1. $F_2(x) = 0$

Case $F_2(x) \neq 0$: Then, we have $\epsilon = \frac{1}{100}$. Clearly, again the condition (1.6) does not hold. But the condition (4.1) becomes

$$2\left(2\left[1 + \frac{1}{3-b}\right]\right)\frac{1-b}{3} \leq \left(1 - \frac{1}{100}\right)^2 \tag{5.2}$$

is fulfilled for each $b \in (0.47336, \frac{1}{2})$.

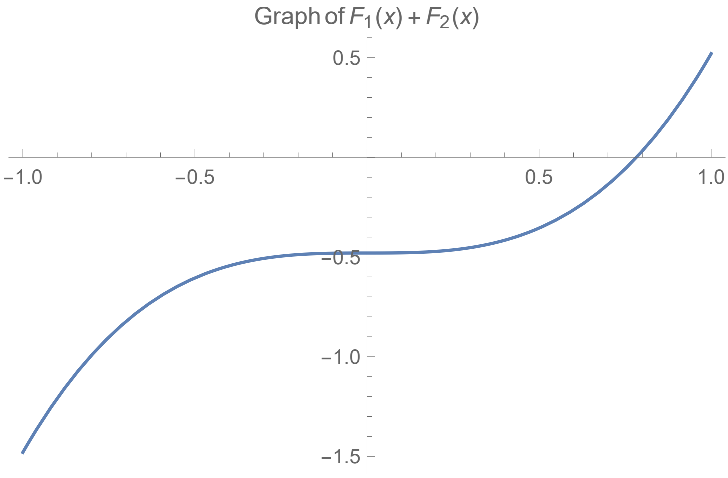

Figure 2. $F_2(x) \neq 0$

Remark 5.1. It is worth noticing that the iterations $\{\rho_k\}$ and $\{\rho_k^1\}$ become for $\rho_0 = \rho_0^1 = 0$, $\rho_1 = \rho_1^1 = \alpha$

$$\rho_{k+1} - \rho_k = \frac{h(\rho_k - \rho_{k-1})^2}{2(1 - h\rho_k)}$$

and

$$\rho_{k+1}^1 - \rho_k^1 = \frac{h_1(\rho_k - \rho_{k-1})^2}{2(1 - h_1\rho_k)}.$$

It follows by a inductive argument that

$$\rho_k \leq \rho_k^1,$$

$$0 \leq \rho_{k+1} - \rho_k \leq \rho_{k+1}^1 - \rho_k^1$$

and

$$\rho = \lim_{n\to+\infty} \rho_k \leq \rho^1 = \lim_{n\to+\infty} \rho_k^1.$$

In view of the proof of Theorem 3.1, it follows that the actual majorizing sequence is defined (for $F_2 = 0$) by

$$r_0 = 0, \quad r_1 = \alpha,$$

$$r_{k+1} = r_k + \frac{h(r_k - r_{k-1})^2}{2(1 - h_0 r_k)} \quad \text{for } k = 0, 1, 2, \ldots .$$

The sequence $\{r_k\}$ is tighter than $\{\rho_k\}$, i.e., certainly converges under the conditions of Theorem 3.1. But it also converges under weaker conditions [19]. One such condition is given by

$$2L\alpha \leq 1, \tag{5.3}$$

where $L = \frac{1}{8}(4h_0 + \sqrt{hh_0 + 8h_0^2} + \sqrt{hh_0})$.

k	0	1	2	3	4	5	6	7
ρ_k	0	0.173333	0.254701	0.286762	0.293981	0.294388	0.294389	0.294389
r_k	0	0.173333	0.247848	0.273063	0.276807	0.276893	0.276893	0.276893

Table 1. Estimates for Example (5.1)

The sequence $\{r_k\}$ is tighter than $\{\rho_k\}$, as can be seen from the preceding table.

Let $\mathcal{S}$ denote the space of continuous operators on $[0,1]$ with max-norm.

Example 5.2. Let $B_1 = B_2 = \mathcal{S}$ and $\Omega_* = U[0,d]$ for some $d > 1$. Define the non-linear integral operator on the ball Ω_* as

$$F_1(v)(r) = -z(r) + v(r) - c\int_0^1 \Lambda(r_1, r_2)v^3(r_2)dr_2 \quad \text{for each } v \in \mathcal{S} \text{ and } r_1 \in [0,1]$$

provided that $z \in \mathcal{S}, c \in \mathbb{R}$ and Λ is the Green's function given as

$$\Lambda(r_1, r_2) = \begin{cases} (1-r_1)r_2, & \text{for} \quad r_2 \leq r_1 \\ r_2(1-r_1), & \text{for} \quad r_1 \leq r_2. \end{cases}$$

It follows by the definition of the operator F_1 that the derivative is

$$F_1'(x)(\bar{v})(r) = \bar{v}(r) - 3c\int_0^1 \Lambda(r_1, r_2)x^2(r_2)z(r_2)dr_2$$

for each $\bar{v} \in \mathcal{S}$ with $r_2 \in [0,1]$. Pick $z(r) = x_0(r) = 1$ and $|c| < \frac{8}{3}$. We need the calculations

$$\|I - F_1'(x_0(r))\| \leq \frac{8}{3}|c|, \quad F_1'(x_0)^{-1} \in \mathscr{L}(B_2, B_1),$$

$$\|F_1'(x_0)^{-1}\| \leq \frac{8}{8-3|c|}, \quad \alpha = \frac{|c|}{8-3|c|},$$

$$h_0 = \frac{12|c|}{8-3|c|}, \quad h = h_1 = \frac{6d|c|}{8-3|c|}.$$

Choose $d = 3$.

Case $F_2(x) = 0$: Then, the condition (1.8) of Theorem 1 in [19] gives

$$2\beta h_1 \alpha > 1.102065600$$

but the condition (5.3) gives

$$2h\alpha = .9715205068 = c_0.$$

Hence, the new Theorem 3.1 can apply to solve (1.1) and guarantees a solution $x_* \in \Omega_*$ of the equation $F_1(x(r)) = 0$.

Case $F_2(x) \neq 0$: Set $F_2(x) = \frac{1}{m}(1 - |x(r)|)$ for some $m > \frac{1}{1-\sqrt{c_0}}$. Then, we have that $\epsilon = \frac{1}{m}$ and

$$2h\alpha \leq (1-\epsilon)^2.$$

That is the condition (1.8) is satisfied. Hence, the new Theorem 3.1 asserts the existence of a solution x_* of the equation $F_1(x(r)) + F_2(x(r)) = 0$.

6. Conclusion

In this paper, we investigate the potential of extending the NKI to nonlinear Banach space valued operator equations with a non-differentiable term. The proposed methodology results in a semi-local convergence analysis for NKI that is at least as weak as the previous one because the new majorant functions are tighter than the previous ones. Furthermore, stricter error estimates replace the Pták–Potra estimates. The region that determines the solution's uniqueness is also more precisely defined.

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