Annales Univ. Sci. Budapest., Sect. Comp. 54 (2023) 341–362

ON SOLVABILITY OF BARBASHIN'S TYPE EQUATION AND INITIAL VALUE PROBLEM VIA THE STRUCTURE OF GENERALIZED CONVOLUTIONS

Trinh Tuan (Hanoi, Vietnam)

Communicated by Bui Minh Phong

(Received January 15, 2023; accepted June 1, 2023)

Abstract. The purpose of this article is to study the solvability of Barbashin-type integro-differential equation and initial value problem via the structure of well-known generalized convolutions. We give necessary and sufficient conditions for the unique explicit solution of this class of equations and estimate its boundedness in certain weighted L_1 spaces. The illustrative examples of the obtained results are also considered.

1. Introduction

We concerned with integro-differential equation of the form

$$\frac{\partial f(x,s)}{\partial x} = c(x,s)f(x,s) + \int_a^b K(x,s,\rho)f(x,s)d\rho + g(x,s). \tag{1.1}$$

Here $c : J\times[a,b] \to \mathbb{R}$, $K : J\times[a,b]\times[a,b] \to \mathbb{R}$, and mapping $g : J\times[a,b] \to \mathbb{R}$ are given functions, where J is a bounded or unbounded interval, the function f is unknown. The equation (1.1) was first studied by E.A. Barbashin [2] and

Key words and phrases: convolution, Hartley transform, Laplace transform, Fourier cosine transform, Barbashin integro-differential equations, initial value problem.
2010 Mathematics Subject Classification: 44A35, 44A20, 45E10, 45J05.

https://doi.org/10.71352/ac.54.341

his pupils. For this reason, this is nowadays called integro-differential equation of Barbashin type or simply the Barbashin equation.

The (1.1) has been applied to many fields such as mathematical physics, radiation propagation, mathematical biology and transport problems, e.g., more details refer [1]. One of the characteristics of Barbashin equation is that studying solvability of the equation is heavily dependent on the kernel $K(x,s,\rho)$ of the equation. In many cases, we can reduce equation (1.1) to the form of an ordinary differential equation and use the Cauchy integral operator or evolution operator to study it when the kernel does not depend on x. An example is the stationary integro-differential equation of Barbashin type $\partial f(x,s)/\partial x = c(s)f(x,s) + \int\limits_a^b K(s,\rho)f(x,\rho)d\rho + g(x,s)$, i.e. the kernel K does not depend on x. In some other cases, we need to use the partial integral operator to study this equation (see [1]). However, in the general case of $K(x,s,\rho)$ as an arbitrary kernel, the problem of finding a solution for Barbashin equation remains open. In some other cases, we need to use the partial integral operator to study this equation (see [1]). On the other hand, if we view A as the operator defined by $A := \partial/\partial x - c(x,s)\mathcal{I}$, where $\mathcal{I}$ is the identity operator, then Eq. (1.1) is written in the following form $\mathcal{A}f(x,s) = \int\limits_a^b K(x,s,\rho)f(x,s)d\rho + g(t,s)$.

The goal of this work is to show the solvable of some classes of Integro-differential equations Barbashin's type (1.1) via the structure of convolution and polyconvolution related to the Fourier cosine, Hartley, and Laplace transforms as outlined above. Namely, if we replace $Af(x,s)$ by the operator $Af(x,s) = D(f*k)(x)$ where D is the second-order differential operator having the form $D = \left(I - \frac{d^2}{dx^2}\right)$. Then original equation is reduced to become the equation of Barbashin type

$$\left(1 - \frac{d^2}{dx^2}\right)(f*k)(x) = \int\limits_a^b K(x,u)f(u)du + \psi_i(x), \tag{1.2}$$

where h, ψ_i, $i = 1,2$ are given functions, the kernel $K(x,u)$ are correspond to certain case and f is unknown function. To study the solvability of (1.2), we give two approaches by reducing the original equation to the following linear integro-differential equation releated to Fouriercosine-Laplace generalized convolutions and Hartley-Fouriercosine generalized covolutions. Based on the structure of noted above convolutions, together with the techniques introduced in [7, 13, 15, 17, 20]. Besides, we study the Cauchy-type initial value problem for integro-differential equations related to Hartley convolution. Techniques used here to check the solution of the problem satisfies the initial value theorem come from [9, 19], and [24]. Namely, we follow closely the strategy of the Wiener-Lévy theorem, Schwartz's function classes, and the Tauberian theorem.

The organization of this paper is presented as follows. Besides the introduction, the article has four sections. Section 2 is devoted to the presentation of notions about the Fourier-Laplace convolution, convolution of Hartley transform, and the Hartley-Fourier cosine polyconvolution. The main results of the paper are presented in sections 3 and 4 along with illustrative numerical examples to ensure validity and effectiveness.

2. Preliminaries

To begin with, we briefly recall the convolutions structures that will be used later. These concepts together with related results can be easily found in [3, 4, 5, 7, 10, 11, 13, 14, 20, 21]. Denote by (F) is the Fourier integral transform ([5]) of a function f defined by $(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-ixy} f(x)dx$ with $y \in \mathbb{R}$, where $e^{-ixy} = \frac{1+i}{2}\,\mathrm{Cas}(xy) + \frac{1-i}{2}\,\mathrm{Cas}(-xy)$. The concept of "Harley transform" was proposed as an alternative to the Fourier transform by R. V. L. Hartley in 1942 (see [3]). Compared to the Fourier transforms, then the Hartley transform has the advantage of transforming real functions to real functions as opposed to requiring complex numbers (refer [10]). The Hartley transform is an alternate means of analyzing a given function in terms of its sinusoids. This transform is its own inverse and an efficient computational tool where real-value functions. The Hartley transform of a function is a spectral transform and can be obtained from the Fourier transform by replacing the exponential kernel $exp(-ixy)$ by $\mathrm{Cas}(-xy)$. The Hartley transform of a function f can be expressed as either

$$\left(H_{\{1\} \atop \{2\}} f\right)(y) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)\,\mathrm{cas}(\pm xy)dx, \quad y \in \mathbb{R}. \tag{2.1}$$

Here the function cas is understood as $\mathrm{cas}(\pm xy) = \cos(xy) \pm \sin(xy)$. The Fourier cosine transforms of the function f, denoted by (F_c), are defined by the integral formulas ([14]) as follows

$$\left(F_c f\right)(y) := \sqrt{\frac{2}{\pi}} \int\limits_{0}^{+\infty} f(x)\cos(xy)dx, \quad y > 0, \tag{2.2}$$

The Fourier cosine transform agrees with the Fourier transform if $f(x)$ is an even function. In general, the even part of the Fourier transform of $f(x)$ equals the even part of the Fourier cosine transform of $f(x)$ in the specified region.

Definition 2.1. The convolution of Fourier cosine integral transform [20] defined by

$$\left(f \underset{1}{*} g\right)(x) := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left[g(x+y) + g(|x-y|)\right] f(y)dy, \ x > 0. \tag{2.3}$$

And suppose that f, g are functions belonging to the $L_p(\mathbb{R}_+)$ space, then $\left(f \underset{1}{*} g\right) \in L_p(\mathbb{R}_+)$ with $p = 1, 2$ and the following factorization property holds (readers refer prove to the detail in [20]).

$$F_c\left(f \underset{1}{*} g\right)(x) = (F_c f)(x)\,(F_c g)(x). \tag{2.4}$$

The Laplace transform is denoted by $(\mathcal{L})$ and defined by the integral formula [11, 21]

$$(\mathcal{L}f)(y) := \int_0^{\infty} e^{-xy} f(x)dx, \ \Re y > 0. \tag{2.5}$$

Notation $H(\mathbb{R}_+)$ is the set of all function f such that $(\mathcal{L}f)$ belongs to $L_2(\mathbb{R}_+)$, namely $H(\mathbb{R}_+) = \{f : (\mathcal{L}f) \in L_2(\mathbb{R}_+)\}$ which implies that $L_2(\mathbb{R}_+) \subset H(\mathbb{R}_+)$.

Definition 2.2. (See [13]) The Fourier-Laplace convolution is defined by

$$\left(f \underset{2}{*} k\right)(x) := \tag{2.6}$$
$$:= \frac{1}{\pi} \int_0^{+\infty} \int_0^{+\infty} \left[\frac{v}{v^2 + (x-u)^2} + \frac{v}{v^2 + (x+u)^2}\right] f(u)k(v)dudv, \quad x > 0.$$

Suppose that $f \in L_2(\mathbb{R}_+)$ and k is function belonging to the $H(\mathbb{R}_+)$, then the Parseval's identity holds true

$$\left(f \underset{2}{*} k\right)(x) = F_c\left((F_c f)(y)(\mathcal{L}k)(y)\right)(x). \tag{2.7}$$

Readers can refer to the detailed proof in [13].

Definition 2.3. The Hartley-Fourier cosine polyonvolution [22] of three functions f, k_1, k_2 is denoted by $\underset{3}{*}(f, k_1, k_2)$ and defined

$$\left(\underset{3}{*}(f, k_1, k_2)\right)(x) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_0^{+\infty} f(u)k_1(v)\Big[k_2(-x+u+v) + \tag{2.8}$$
$$+ k_2(x-u-v) + k_2(-x+u-v) + k_2(x-u+v) + k_2(-x-u+v) -$$
$$- k_2(x+u-v) + k_2(-x-u-v) - k_2(x+u+v)\Big]dudv.$$

Let f, k_2 are given functions in $L_2(\mathbb{R})$ and $k_1 \in L_2(\mathbb{R}_+)$, then the following Parseval's identity holds

$$\left(\underset{3}{*}(f, k_1, k_2)\right)(x) = H_1\Big((H_1 f)(y)(F_c k_1)(|y|)(H_2 k_2)(y)\Big). \tag{2.9}$$

To prove this, the readers must refer to detail in [22].

According to results in [7] p.364, the authors used integral transform

$$(T_1 f)(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \cos\left(xy + \frac{\pi}{4}\right) f(y)dy \quad \text{and}$$

$$(T_2 f)(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \sin\left(xy + \frac{\pi}{4}\right) f(y)dy$$

to study the convolutions $(f \underset{T_1}{*} g)$ and $(f \underset{T_2}{*} g)$. It's obvious that: $\cos\left(xy + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\,\mathrm{cas}(-xy)$ and $\sin\left(xy + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\,\mathrm{cas}(xy)$. Therefore, $(T_1 f) \equiv (H_2 f)$ and $(T_2 f) \equiv (H_1 f)$, where $H_{\left\{\frac{1}{2}\right\}}$ is the Hartley transform defined by formula (2.1). Consequently, the Theorem 3.5 page 337 in [7] and the Theorem 3.14 page 378 in [7] can be rewritten in the following form.

Definition 2.4. The convolution of Hartley transform for two function f, g is defined by

$$\left(f \underset{4}{*} g\right)(x) := \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\big[g(x+y) + g(x-y) + g(-x+y) - g(-x-y)\big]dy, \tag{2.10}$$

where $x \in \mathbb{R}$. Assume that f, g are functions belonging to the $L_1(\mathbb{R})$, then $\left(f \underset{4}{*} g\right) \in L_1(\mathbb{R})$, see detail in [7] and

$$H_{\left\{\frac{1}{2}\right\}}\left(f \underset{4}{*} g\right)(y) = \left(H_{\left\{\frac{1}{2}\right\}} f\right)(y)\ \left(H_{\left\{\frac{1}{2}\right\}} g\right)(y), \ y \in \mathbb{R}, \tag{2.11}$$

3. Integro-differential equations of Barbashin's type

• We reduce the original equation to the following linear integro-differential equation of Barbashin type related to the Fourier cosine (F_c), Laplace $(\mathcal{L})$ generalized convolutions. Indeed, let f, k be given functions, in Eq.(1.2) we choose $(a,b) = (0,+\infty)$ and using operator $\left(1 - \frac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x)$ in the left-hand side. Moreover, the kernel $K(x,u,v)$ is chosen independent on v as $K(x,u) = -\frac{1}{\sqrt{2\pi}}\big[g(x+u) + g(|x-u|)\big]$. Applying the formulas (2.3) then the equation (1.2) can be rewritten in the form

$$\left(1 - \frac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x) + \left(f \underset{1}{*} g\right)(x) = \Psi_1(x). \tag{3.1}$$

Here, k, g, Ψ_1 are given functions, f is unknown function and $(.\underset{2}{*}.)$, $(.\underset{1}{*}.)$ are determined in the formula (2.6) and (2.3) respectively.

Theorem 3.1. *Suppose that Ψ_1, g are given functions belonging to $L_2(\mathbb{R}_+)$ and $k \in H(\mathbb{R}_+)$ satisfying the following conditions*

$$(1+y^2)|(\mathcal{L}k)(y)| < +\infty, \tag{3.2}$$

and

$$\frac{(F_c\Psi_1)(y)}{(1+y^2)(\mathcal{L}k) + (F_c g)(y)} \in L_2(\mathbb{R}_+), \tag{3.3}$$

where $\mathcal{L}$ and F_c are defined respectively by formula (2.5), (2.2). Then equation (3.1) has a unique solution belonging to $L_2(\mathbb{R}_+)$ and is represented as follow

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{(F_c\Psi_1)(y)}{(1+y^2)(\mathcal{L}k)(y) + (F_c g)(y)} \cos(xy)\, dy,\ \ x > 0.$$

In order to prove Theorem 3.1, we need the following auxiliary lemma.

Lemma 3.1. *Let f is a function belonging to $L_2(\mathbb{R}_+)$ space, such that the condition $(1+y^2)f(y) \in L_2(\mathbb{R}_+)$ holds, then we have the following equality*

$$\left(1 - \frac{d^2}{dx^2}\right)(F_c f)(x) = F_c\big((1+y^2)f(y)\big)(x) \in L_2(\mathbb{R}_+), \quad y > 0, \tag{3.4}$$

and

$$\left(1 - \frac{d^2}{dx^2}\right)\left(H_{\{\frac{1}{2}\}} f\right)(x) = H_{\{\frac{1}{2}\}}\big((1+y^2)f(y)\big)(x) \in L_2(\mathbb{R}), \quad y \in \mathbb{R}. \tag{3.5}$$

Proof. Titchmarch's previous results on the Fourier transform tell us that $f(y), yf(y), \ldots, y^n f(y) \in L_2(\mathbb{R})$ if and only if

$$(Ff)(x), \frac{d}{dx}(Ff)(x), \ldots, \frac{d^n}{dx^n}(Ff)(x)$$

belong to $L_2(\mathbb{R})$ (see Theorem 68, p.92, [14]), then above assertion is still true for F_c and $H_{\{1 \atop 2\}}$. On the other hand, since

$$\frac{d}{dx}\left(F_c f\right)(x) = \sqrt{\frac{2}{\pi}}\frac{d}{dx}\left(\int\limits_0^{+\infty} f(y)\cos(xy)dy\right) = F_s\Big((-y)f(y)\Big)(x) \in L_2(\mathbb{R}_+),$$

then

(3.6)
$$\frac{d^2}{dx^2}\left(F_c f\right)(x) = \frac{d}{dx}\left(\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}(-y)f(y)\sin(xy)dy\right) = F_c(-y^2 f(y)) \in L_2(\mathbb{R}_+).$$

From (3.6), we obtain $\left(1-\frac{d^2}{dx^2}\right)\left(F_c f(y)\right)(x) = F_c\big((1+y^2)f(y)\big)(x) \in L_2(\mathbb{R}_+)$. Moreover

$$\frac{d}{dx}\operatorname{cas}(\pm xy) = \frac{d}{dx}(\cos xy \pm \sin xy) = \pm y\operatorname{cas}(\mp xy),$$

then

$$\frac{d}{dx}\left(H_{\{1 \atop 2\}} f\right)(x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\left(\int\limits_{-\infty}^{+\infty} f(y)\operatorname{cas}(\pm xy)dy\right) =$$
$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \pm y f(y)\operatorname{cas}(\mp xy)dy = H_{\{2 \atop 1\}}(\pm y f(y)) \in L_2(\mathbb{R}),$$

and

(3.7)
$$\begin{aligned}\frac{d^2}{dx^2}\left(H_{\{1 \atop 2\}} f\right)(x) &= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\left(\int\limits_{-\infty}^{+\infty} \pm y f(y)\operatorname{cas}(\mp xy)dy\right) = \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} -y^2 f(y)\operatorname{cas}(\pm xy)dy = \\ &= H_{\{1 \atop 2\}}(-y^2 f(y))(x) \in L_2(\mathbb{R}).\end{aligned}$$

From (3.7), we deduce that $\left(1-\frac{d^2}{dx^2}\right)\left(H_{\{1 \atop 2\}} f(y)\right)(x) = H_{\{1 \atop 2\}}\big((1+y^2)f(y)\big) \in$ $\in L_2(\mathbb{R}),\ \ y \in \mathbb{R}.$ ∎

Proof of Theorem 3.1. Since k is a function belonging to $H(\mathbb{R}_+)$ then $\mathcal{L}k \in L_2(\mathbb{R}_+)$, using the condition (3.2), (2.7) and formula (3.4) in lemma 3.1 with together unitary property of Fourier cosin transform on the $L_2(\mathbb{R})$, we obtain

$$\left(1-\frac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x) = (1+y^2)(F_cf)(y)(\mathcal{L}k)(y) \in L_2(\mathbb{R}_+). \tag{3.8}$$

In the $L_2(\mathbb{R}_+)$ space, applying (F_c) transform on both sides of the equation (3.1) and simultaneous use of formulas (3.8), (2.4), we obtain

$$(1+y^2)(F_cf)(y)(\mathcal{L}k)(y) + (F_cf)(y)(F_cg)(y) = (F_c\Psi_1)(y).$$

Under the condition (3.3), we obtain

$$(F_cf)(y) = \frac{(F_c\Psi_1)(y)}{(1+y^2)(\mathcal{L}k)(y)+(F_cg)(y)} \in L_2(\mathbb{R}_+).$$

Therefore $f(x)$ belongs to $L_2(\mathbb{R}_+)$ space, via the inverse formula of Fourier cosine transform, we represent explicitly the solution in the following form $f(x) =$ $= \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \frac{(F_c\Psi_1)(y)}{(1+y^2)(\mathcal{L}k)(y)+(F_cg)(y)} \cos(xy)\, dy$, $x > 0$. ■

If compared with the results in [16, 17], in this theorem, we use generalized convolution associated with the Fourier cosine-Laplace transforms. Therefore, it is difficult to show the functional space because the Laplace transform is not an isometric (unitary) isomorphism mapping on $L_2(\mathbb{R}_+)$. The novelties is that, we constructed the functional space $H(\mathbb{R}_+) = \{f : (\mathcal{L}f) \in L_2(\mathbb{R}_+)\}$ for satisfies the conditions of the problem, which implies that $L_2(\mathbb{R}_+) \subset H(\mathbb{R}_+)$. An example is given below to illustrate the above result.

Example 3.2. Let $k(x) = i \sin x$ is function belonging to $H(\mathbb{R}_+)$. Using formula (3.2.9) in [6], we obtain $(\mathcal{L}k)(y) = \frac{i}{1+y^2} \in L_2(\mathbb{R})$, which means $(1+y^2)|(\mathcal{L}k)(y)| = 1 < +\infty$. We choose $\Psi_1 = g = \sqrt{\frac{\pi}{2}} e^{-x} \in L_2(\mathbb{R}_+)$. Then $\frac{(F_c\Psi_1)(y)}{(1+y^2)(\mathcal{L}k)(y)+(F_cg)(y)} = \frac{1}{1+i(1+y^2)}$ belongs to $L_2(\mathbb{R}_+)$. The solution in this case based on the formula 1, page 615 in [6] applied with $a = \sqrt{1-i}$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \frac{1}{1+i(1+y^2)} \cos(xy) dy = \frac{\sqrt{(1-i)^3}}{2} \sqrt{\frac{\pi}{2}}\, e^{-\sqrt{1-i}\ x} \in L_2(\mathbb{R}_+).$$

• The remaining approach is based on the structure of related to the Hartley and Fourier cosine generalized covolutions. From equation (1.2), we choose $(a, b) = (-\infty, +\infty)$ and considering

$$\left(1-\frac{d^2}{dx^2}\right)(f * h)(x) := \left(1-\frac{d^2}{dx^2}\right)\left(\underset{3}{*}(f, k_1, k_2)\right)(x). \tag{3.9}$$

Here, the left-hand side of (3.9) is chosen to replace the right-hand side of equation (1.2) and defined by (2.8). The kernel $K(x,u) = -\frac{1}{2\sqrt{2\pi}}\big[g(x+u) + g(x-u) + g(-x+u) - g(-x-u)\big]$. Using formulas (3.9), (2.10) we can rewrite the equation (1.2) in the following form

$$\left(1 - \frac{d^2}{dx^2}\right)\left(\underset{3}{*}(f,k_1,k_2)\right)(x) + \left(f \underset{4}{*} g\right)(x) = \psi_2(x), \tag{3.10}$$

here, k_1, k_2, g, ψ_2 are known functions, $\left(\underset{3}{*}(f,k_1,k_2)\right)(x) \in L_2(\mathbb{R})$ defined by (2.8) and $(f \underset{4}{*} g) \in L_2(\mathbb{R})$ defined by formula (2.10); and f is the unknown function needed to find.

Theorem 3.3. *Suppose k_1, k_2, g, ψ_2 are given functions such that $k_1 \in L_2(\mathbb{R}_+)$, k_2, g, ψ_2 belonging to $L_2(\mathbb{R})$ space. Furthermore, suppose that the following conditions occur simultaneously*

$$(1+y^2)\big|(F_c k_1)(|y|)(H_2 k_2)(y)\big| < +\infty, \quad y \in \mathbb{R}, \tag{3.11}$$

$$\frac{(H_1\psi_2)(y)}{(1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y) + (H_1 g)(y)} \in L_2(\mathbb{R}). \tag{3.12}$$

Then the equation (3.10) has the unique solution in the $L_2(\mathbb{R})$ of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{(H_1\psi_2)(y)}{(1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y) + (H_1 g)(y)} \operatorname{cas}(xy)\, dy \; x \in \mathbb{R},$$

where F_c, $H_{\left\{\substack{1\\2}\right\}}$ are defined respectively by (2.2), (2.1).

First, we need the following Lemma.

Lemma 3.2. *Suppose that f, g are functions belonging to $L_2(\mathbb{R})$ space then $\left(f \underset{4}{*} g\right) \in L_2(\mathbb{R})$ and we get the factorization equalities (2.11)*

Proof.

$$\int_{-\infty}^{+\infty} \left|\left(f \underset{4}{*} g\right)(x)\right|^2 dx \leq$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \big| f(y)\big[g(x+y) + g(x-y) + g(-x+y) - g(-x-y)\big]\big|^2 dy \right) dx \leq$$

$$\leq \frac{4}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y)|^2 \big[|g(x+y)|^2 + |g(x-y)|^2 + $$

$$+ |g(-x+y)|^2 + |g(-x-y)|^2\big]\ dydx =$$

$$= \frac{4}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y)|^2 |g(x-y)|^2\ dxdy + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y)|^2 |g(x+y)|^2\ dxdy \right).$$

Since f, g are functions belonging to the $L_2(\mathbb{R})$ space, using Fubini's Theorem and the variable transformation formula for $t = x - y;\ t = x + y$, we obtain

$$\int_{-\infty}^{+\infty} \left|(f \underset{4}{*} g)(x)\right|^2 dx \leq 4\sqrt{\frac{2}{\pi}} \left(\int_{-\infty}^{+\infty} |f(y)|^2 dy \right) \left(\int_{-\infty}^{+\infty} |g(t)|^2 dt \right) < +\infty,$$

which implies that $\left(f \underset{4}{*} g\right)(x)$ belong to the $L_2(\mathbb{R})$ space. The proof of equality (2.11) in the $L_2(\mathbb{R})$ is the same as that in $L_1(\mathbb{R})$ space (refer [7]). ■

Proof of Theorem 3.3. Using formulas (2.9), (3.5) and condition (3.11) we have

$$\left(1 - \frac{d^2}{dx^2}\right) \left(\underset{3}{*}(f, k_1, k_2)\right)(x) =$$

(3.13)
$$= \left(1 - \frac{d^2}{dx^2}\right) H_1\Big((H_1 f)(y)(F_c k_1)(|y|)(H_2 k_2)(y)\Big)(x) =$$

$$= H_1\Big((1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y)(H_1 f)(y)\Big) \in L_2(\mathbb{R}).$$

Since the Hartley transform is unitary on $L_2(\mathbb{R})$. Such that, from (3.13) we have

(3.14)
$$H_1 \left(1 - \frac{d^2}{dx^2}\right) \left(\underset{3}{*}(f, k_1, k_2)\right)(x) =$$

$$= (1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y)(H_1 f)(y) \in L_2(\mathbb{R}).$$

In the $L_2(\mathbb{R})$ space, applying H_1 transform on both sides of the equation (3.10), we have

$$H_1 \left(1 - \frac{d^2}{dx^2}\right) \left(\underset{3}{*}(f, k_1, k_2)\right)(y) + H_1 \left(f \underset{4}{*} g\right) = (H_1 \psi_2)(y).$$

In view of (3.14),(2.11), we deduce that

$$(1+y^2)(F_c k_1)(y)(H_2 k_2)(y)(H_1 f)(y) + (H_1 f)(y)(H_1 g)(y) = (H_1 \psi_2)(y).$$

This equivalent to

$$(H_1 f)(y)\Big[(1+y^2)(F_c k_1)(y)(H_2 k_2)(y) + (H_1 g)(y)\Big] = (H_1\psi_2)(y).$$

Under the condition (3.12), this yields

$$(H_1 f)(y) = \frac{(H_1\psi_2)(y)}{(1+y^2)(F_c k_1)(y)(H_2 k_2)(y) + (H_1 g)(y)} \in L_2(\mathbb{R})$$

implying that $f \in L_2(\mathbb{R})$. Therefore

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{(H_1\psi_2)(y)}{(1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y) + (H_1 g)(y)} \operatorname{cas}(xy)\, dy, \ x \in \mathbb{R}$$

based on the inverse formula of Hartley transform. ■

When comparing with the result in [17], in this theorem, the second-order differential operator must be directly applied to the polyconvolution, resulting in more complex computations through the Bessel function. The following example is an illustration of the above result.

Example 3.4. Let $\psi_2 = g = \sqrt{\frac{\pi}{2}}\, e^{-|x|} \in L_2(\mathbb{R})$, using formula 1.4.1, page 23 in [4] then $H_1\left(\sqrt{\frac{\pi}{2}}\, e^{-|x|}\right) = \frac{1}{1+y^2}$. Furthermore, using 1.2.17 in [4], we obtain $K_0(x) = \int\limits_0^{+\infty} \frac{1}{\sqrt{1+y^2}} \cos(xy) dy$, where K_0 is Bessel function of the second kind [12]. Choose $k_1(x) = \sqrt{\frac{2}{\pi}} K_0(x)$, we deduce that $\int\limits_0^{+\infty} |k_1(x)|^2 dx = \frac{\pi}{2}$, hence $k_1(x)$ belongs to the $L_2(\mathbb{R}_+)$. We obtain

$$(F_c k_1)(|y|) = F_c\left(\sqrt{\frac{2}{\pi}} K_0(x)\right)(|y|) = \frac{1}{\sqrt{1+y^2}}.$$

By putting

$$k_2(x) = \begin{cases} \sqrt{\frac{\pi}{2}} K_0(x) & \text{if } \ x \geq 0, \\ \sqrt{\frac{\pi}{2}} K_0(-x) & \text{if } \ x \leq 0. \end{cases}$$

It means $k_2(x)$ is even extension of $k_1(x)$, we have

$$(H_2 k_2)(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} k_2(x) \operatorname{cas}(xy)\, dy = \frac{2}{\sqrt{2\pi}} \int\limits_0^{+\infty} k_2(x) \cos(xy)\, dy =$$

$$= \int\limits_0^{+\infty} K_0(x) \cos(x|y|)\, dx = \frac{1}{\sqrt{1+y^2}}.$$

Thus, chosen k_1, k_2, Ψ_2, g satisfy (3.11), implies that

$$(1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y) = 1$$

is finite, and condition (3.12) becomes

$$\frac{(H_1\psi_2)(y)}{(1+y^2)(F_c k_1)(|y|)(H_2 k_2)(y) + (H_1 g)(y)} = \frac{1}{2+y^2} \in L_2(\mathbb{R}).$$

The solution is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{1}{2+y^2} \operatorname{cas}(xy) dy = \frac{2}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \frac{1}{2+y^2} \cos(xy)\, dy =$$
$$= \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}x} \in L_2(\mathbb{R}_+).$$

4. Cauchy-type initial value problem for integro–differential equations

The Theorem 4.1 in [8], Chapter 4, page 224, gives a closed-form solution of the Cauchy-type problem.

$$\text{(4.1)} \qquad \begin{cases} \left(D^\alpha_{a+} f\right)(x) - \lambda f(x) = \Phi(x), & (a < x \le b;\ \alpha > 0; \lambda \in \mathbb{R}) \\ \left(D^{\alpha-k}_{a+} f\right)(a+) = b_k, & (b_k \in \mathbb{R}; k = 1, \cdots, n = -[-\alpha]), \end{cases}$$

where $\Phi(x) \in C_\gamma[a,b]$, $(0 \le \gamma < 1)$ with the Riemann-Liouville fractional derivative $\left(D^\alpha_{a+} f\right)(x)$ of order $\alpha > 0$ are given by

$$\left(D^\alpha_{a+} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dx^n}\right) \left\{ \int\limits_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \right\}$$

with $(n = [\alpha]+1; x > a)$. In this part, we study the Cauchy-type initial value problem based on (4.1), by choosing $\lambda = -1$ and substituting operator $\left(D^\alpha_{a+} f\right)$ by $D(g \underset{4}{*} f)$, where D is the second-order differential operator having the form $\left(I - \frac{d^2}{dx^2}\right)$. Namely, the Cauchy-type initial value problem is stated as follows:

$$\text{(4.2)} \qquad f(x) - f''(x) + D\left(g \underset{4}{*} f\right)(x) = \left(g \underset{4}{*} \varphi\right)(x),$$

with the initial condition

$$\text{(4.3)} \qquad \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} f'(x) = 0.$$

Here g, φ are given functions, $(.\underset{4}{*}.)$ is the Hartley convolution defined by (2.10) and f is a twice-differentiable function needed to find. Following the structure of Hartley convolution and the factorization properties, we will obtain the unique explicit L_1-solution. One of the difficulties in solving problem on $L_1(\mathbb{R})$ space if compared with previous results in [16, 18] is that the transforms F_c and $H_{\{\frac{1}{2}\}} : L_2(\mathbb{R}) \leftrightarrow L_2(\mathbb{R})$ is an isometric isomorphism mapping (unitary) [5], while it is no longer true for $L_1(\mathbb{R})$. Moreover, on the $L_1(\mathbb{R})$ space, the theorem (68, page 92, in [14]) also no longer holds. To overcome that, we used the Wiener-Lévy theorem [9] and the denseness in $L_1(\mathbb{R})$ of Schwartz's function classes [19] simultaneously to solve.

Firstly, we need the following auxiliary lemma.

Lemma 4.1. *Suppose that h, k are given functions belonging to $L_1(\mathbb{R})$, then*

$$\|h \underset{4}{*} k\|_{L_1(\mathbb{R})} \leq \sqrt{\frac{2}{\pi}} \|h\|_{L_1(\mathbb{R})} \|k\|_{L_1(\mathbb{R})}. \tag{4.4}$$

Proof. Following [23], we know that if $h, k \in L_1(\mathbb{R})$ then $h \underset{4}{*} k \in L_1(\mathbb{R})$. Therefore we have

$$\|h \underset{4}{*} k\|_{L_1(\mathbb{R})} = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} h(y) \big[k(x+y) + k(-x+y) + \right.$$
$$\left. + k(x-y) - k(-x-y) \big] dy \right| dx \leq$$
$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} |h(y)| \big[|k(x+y)| + |k(-x+y)| + \right.$$
$$\left. + |k(x-y)| - |k(-x-y)| \big] dy \right\} dx.$$

Applying the Fubini's Theorem and change of variables formula, we obtain

$$\|h \underset{4}{*} k\|_{L_1(\mathbb{R})} \leq \sqrt{\frac{2}{\pi}} \left(\int_{-\infty}^{+\infty} |h(y)| dy \right) \left(\int_{-\infty}^{+\infty} |k(t)| dt \right) =$$
$$= \sqrt{\frac{2}{\pi}} \|h\|_{L_1(\mathbb{R})} \|k\|_{L_1(\mathbb{R})}. \blacksquare$$

Theorem 4.1. *Let g, φ are functions belonging to $L_1(\mathbb{R})$ space and satisfy the condition*

$$1 + \left(H_{\{\frac{1}{2}\}} g \right)(y) \neq 0, \ y \in \mathbb{R}, \tag{4.5}$$

then the equation (4.2) *has a unique solution in* $C^2(\mathbb{R})\cap L_1(\mathbb{R})$ *and represented by* $f(x)=\left(\sqrt{\frac{\pi}{2}}e^{-|t|} \underset{4}{*} \left(l \underset{4}{*} \varphi\right)\right)(x)$. *Moreover, we have*

$$\|f\|_{L_1(\mathbb{R})}\leq 4\sqrt{\frac{2}{\pi}}\|l\|_{L_1(\mathbb{R})}\|\varphi\|_{L_1(\mathbb{R})},$$

where l *is a function belonging to* $L_1(\mathbb{R})$ *space and determined by*

$$\left(H_{\{\frac{1}{2}\}}l\right)(y)=\frac{\left(H_{\{\frac{1}{2}\}}g\right)(y)}{1+\left(H_{\{\frac{1}{2}\}}g\right)(y)}.$$

Proof. For $y\in\mathbb{R}$ we have

$$\left(H_{\{\frac{1}{2}\}}f''(x)\right)(y)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}f''(x)\operatorname{cas}(\pm xy)\,dx=$$
$$=\frac{1}{\sqrt{2\pi}}\left\{f'(x)\operatorname{cas}(\pm xy)\Big|_{-\infty}^{+\infty}+\right.$$
$$\left.+y(\sin xy\mp\cos xy)f(x)\Big|_{-\infty}^{+\infty}-y^2\int\limits_{-\infty}^{+\infty}f(x)\operatorname{cas}(\pm xy)dx\right\}.$$

By using the condition (4.3), we obtain

$$\left(H_{\{\frac{1}{2}\}}f''(x)\right)(y)=-y^2\left(H_{\{\frac{1}{2}\}}f(x)\right)(y). \tag{4.6}$$

Thus f,g are functions belonging to the $L_1(\mathbb{R})$, then $\left(g\underset{4}{*}f\right)\in L_1(\mathbb{R})$ (see [7]). On the other hand, according to the assumption $f',f''\in L_1(\mathbb{R})$, then $\left(g\underset{4}{*}f'\right)$ and $\left(g\underset{4}{*}f''\right)$ also belong to the $L_1(\mathbb{R})$ space. Following the formula (2.10) for $x\in\mathbb{R}$ we have

$$\left(g\underset{4}{*}f\right)(x)=$$
$$=\frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}g(y)\Big\{f(x+y)+f(-x+y)+f(x-y)-f(-x-y)\Big\}dy. \tag{4.7}$$

Notation $\mathcal{S}$ is the space of functions $f(x)\in C^\infty(\mathbb{R})$ and whose derivative decreases rapidly when $|x|$ tends to ∞, it is also known as Schwartz space. Then, the closure of $\mathcal{S} = L_1(\mathbb{R})$, which means that $\mathcal{S}$ is the dense set in the $L_1(\mathbb{R})$ space. Thus, for any f, f', f'' belong to the $L_1(\mathbb{R})$ then there exists a sequence of functions $\{f_n\} \in S$ such that $\{f_n\} \to f$, $\{f'_n\} \to f'$ and $\{f''_n\} \to f''$ when n tends to ∞. With the above function classes, together with the assumption that g is a function belonging to the $L_1(\mathbb{R})$ space, then the integral in the formula (4.7) is convergent. We continue to change order of the integration and the differentiation as follow

$$\frac{d^2}{dx^2}\left(g \underset{4}{*} f_n\right)(x) = \frac{1}{2\sqrt{2\pi}}\times \tag{4.8}$$

$$\times \int_{-\infty}^{+\infty} g(y)\frac{d^2}{dx^2}\Big\{f_n(x+y) + f_n(-x+y) + f_n(x-y) - f_n(-x-y)\Big\}dy =$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(y)\Big\{f''_n(x+y) + f''_n(-x+y) + f''_n(x-y) - f''_n(-x-y)\Big\}dy =$$

$$= \left(g \underset{4}{*} f''_n\right)(x) \in L_1(\mathbb{R}).$$

Letting (4.8) through the limit sign, we obtain

$$\lim_{n\to\infty}\frac{d^2}{dx^2}\left(g \underset{4}{*} f_n\right)(x) = \lim_{n\to\infty}\left(g \underset{4}{*} f''_n\right)(x),$$

this is equivalent to

$$\frac{d^2}{dx^2}\left(g \underset{4}{*} f\right)(x) = \left(g \underset{4}{*} f''\right)(x). \tag{4.9}$$

By using the formulas (2.11), (4.6) and (4.9), we obtain

$$\begin{aligned} H_{\{\frac{1}{2}\}}\left(\frac{d^2}{dx^2}\left(g \underset{4}{*} f\right)(x)\right)(y) &= H_{\{\frac{1}{2}\}}\left(g \underset{4}{*} f''\right)(y) = \\ &= \left(H_{\{\frac{1}{2}\}}g\right)(y)\left(H_{\{\frac{1}{2}\}}f''\right)(y) = \\ &= -y^2\left(H_{\{\frac{1}{2}\}}g\right)(y)\left(H_{\{\frac{1}{2}\}}f\right)(y) = -y^2 H_{\{\frac{1}{2}\}}\left(g \underset{4}{*} f\right)(y). \end{aligned} \tag{4.10}$$

Form (4.10), we deduce that

$$H_{\{\frac{1}{2}\}}\left(\left(1 - \frac{d^2}{dx^2}\right)\left(g \underset{4}{*} f\right)(x)\right)(y) = (1+y^2)H_{\{\frac{1}{2}\}}\left(g \underset{4}{*} f\right)(y). \tag{4.11}$$

Applying the Hartley $H_{\{\frac{1}{2}\}}$ transformation on both sides of the equation (4.2) and concurrent using consecutively the formulas (4.6), (4.11), (2.11), we obtain

$$\left(H_{\{\frac{1}{2}\}}f\right)(y) - \left(H_{\{\frac{1}{2}\}}f''\right)(y) + H_{\{\frac{1}{2}\}}\left(\left(1-\frac{d^2}{dx^2}\right)\left(g \underset{4}{*} f\right)(x)\right)(y) = \\ = H_{\{\frac{1}{2}\}}\left(g \underset{4}{*} \varphi\right)(y),$$

equivalently

$$(1+y^2)\left(H_{\{\frac{1}{2}\}}f\right)(y) + (1+y^2)\left(H_{\{\frac{1}{2}\}}g\right)(y)\left(H_{\{\frac{1}{2}\}}f\right)(y) = \\ = \left(H_{\{\frac{1}{2}\}}g\right)(y)\left(H_{\{\frac{1}{2}\}}\varphi\right)(y).$$

Under the condition (4.5), we obtain

$$\left(H_{\{\frac{1}{2}\}}f\right)(y)\left\{(1+y^2)\left[1+\left(H_{\{\frac{1}{2}\}}g\right)(y)\right]\right\} = \left(H_{\{\frac{1}{2}\}}g\right)(y)\left(H_{\{\frac{1}{2}\}}\varphi\right)(y),$$

which implies that

$$\left(H_{\{\frac{1}{2}\}}f\right)(y) = \frac{1}{1+y^2}\left[\frac{\left(H_{\{\frac{1}{2}\}}g\right)(y)}{1+\left(H_{\{\frac{1}{2}\}}g\right)(y)}\right]\left(H_{\{\frac{1}{2}\}}\varphi\right)(y). \tag{4.12}$$

The Wiener-Levy theorem (refer [9]) for the Fourier transform said that if $g \in L_1(\mathbb{R})$, then $1 + (F\xi)(y) \neq 0$ for any $y \in \mathbb{R}$ is a necessary and sufficient condition for the existence of a function k belonging to the $L_1(\mathbb{R})$ such that $(Fk)(y) = \frac{(F\xi)(y)}{1+(F\xi)(y)}$, where $\xi \in L_1(\mathbb{R})$. This theorem still holds true for Hartley transform, by putting $g(x) := \frac{1+i}{2}\xi(x) + \frac{1-i}{2}\xi(-x)$, then $(F\xi)(y) = \left(H_{\{\frac{1}{2}\}}g\right)(y)$. If g belongs to the $L_1(\mathbb{R})$, then $1+\left(H_{\{\frac{1}{2}\}}g\right)(y) \neq 0$ for any $y \in \mathbb{R}$ is a necessary and sufficient condition for the existence of a function $l \in L_1(\mathbb{R})$ such that $\left(H_{\{\frac{1}{2}\}}l\right)(y) = \frac{\left(H_{\{\frac{1}{2}\}}g\right)(y)}{1+\left(H_{\{\frac{1}{2}\}}g\right)(y)}$. On the other hand $H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|t|}\right) = \frac{1}{1+y^2}$, then the formula (4.12) can be rewritten as follows

$$\left(H_{\{\frac{1}{2}\}}f\right)(y) = H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|t|}\right)(y)\left(H_{\{\frac{1}{2}\}}l\right)(y)\left(H_{\{\frac{1}{2}\}}\varphi\right)(y),$$

by using the formula (2.11), we deduce that

$$\left(H_{\{\frac{1}{2}\}}f\right)(y) = H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|t|} \underset{4}{*} \left(l \underset{4}{*} \varphi\right)\right)(y)$$

and the solution $f(x) = \left(\sqrt{\frac{\pi}{2}}e^{-|t|} \underset{4}{*} \left(l \underset{4}{*} \varphi\right)\right)(x)$ for all x almost everywhere on $\mathbb{R}$ (refer[5, 19]). Thus $\sqrt{\frac{\pi}{2}}e^{-|t|} \in L_1(\mathbb{R})$ and l, φ are functions belonging to $L_1(\mathbb{R})$ then $(.\underset{4}{*}.) \in L_1(\mathbb{R})$, which implies that $f(x)$ also belongs to $L_1(\mathbb{R})$ space. Now, we prove solution $f(x)$ of problem actually satisfies the initial condition (4.3). Indeed, by setting $G(x) = \left(l \underset{4}{*} \varphi\right)(x)$ belongs to $C_0^1(\mathbb{R}) \cap L_1(\mathbb{R})$ space, then for $x \in \mathbb{R}$ we have

$$f(x) = \left(\sqrt{\frac{\pi}{2}}e^{-|y|} \underset{4}{*} G\right)(x) =$$

$$= \frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \sqrt{\frac{\pi}{2}}e^{-|y|} \Big[G(x+y) + G(-x+y) + G(x-y) - G(-x-y)\Big] dy =$$

$$= \frac{1}{4}\Bigg\{ \int\limits_{-\infty}^{+\infty} e^{-|y|}G(x+y)dy + \int\limits_{-\infty}^{+\infty} e^{-|y|}G(-x-y)dy + \int_{-\infty}^{+\infty} e^{-|y|}G(x-y)dy -$$

$$- \int\limits_{-\infty}^{+\infty} e^{-|y|}G(-x-y)dy\Bigg\} = \frac{1}{4}\{I_1 + I_2 + I_3 - I_4\},$$

where I_1, I_2, I_3 and I_4 be the respective integrals in above expression. It's obvious that we have

$$I_1 = \int\limits_{-\infty}^{+\infty} e^{-|y|}G(x+y)dy = \int\limits_{-\infty}^{0} e^{-|y|}G(x+y)dy \int\limits_{0}^{+\infty} e^{-|y|}G(x+y)dy =$$
$$= I_1^- + I_1^+.$$

A change of variable $t = x+y$ for the integrals I_1^- and I_1^+, we obtain $I_1^- + I_1^+ = = \int\limits_{-\infty}^{x} e^{-x+t}G(t)dt + \int\limits_{x}^{+\infty} e^{x-t}G(t)dt$. When $x \to +\infty$ then I_1^- tends to 0, and when $x \to -\infty$ then I_1^+ also tends to 0. Furthermore

$$|I_1^-| \leqslant \int\limits_{-\infty}^{x} |e^{-(x-t)}G(t)|dt \leqslant \int\limits_{-\infty}^{x} |G(t)|dt \to 0,$$

when x tends to $-\infty$. The same goes for integrals

$$|I_1^+| \leqslant \int\limits_{x}^{+\infty} |e^{-(t-x)}G(t)|dt \leqslant \int\limits_{x}^{+\infty} |G(t)|dt \to 0,$$

when $x \to +\infty$. Thus, the integral I_1 is converges to 0 when x tends to $\pm\infty$. Do exactly the same as above for the integrals I_2, I_3 and I_4 then also converge to 0 when x tends to $\pm\infty$, which implies that $\lim\limits_{x\to\pm\infty} f(x) = 0$ and $f \in C_0(\mathbb{R})$. On the other hand, $I_1 = I_1^- + I_1^+ = e^{-x} \int\limits_{-\infty}^{x} e^t G(t)dt + e^x \int\limits_{x}^{+\infty} e^{-t}G(t)dt$, considering the derivative of the integral, we obtain

$$\begin{aligned} I_1' &= -e^{-x} \int\limits_{-\infty}^{x} e^t G(t)dt + e^x \int\limits_{x}^{+\infty} e^{-t} G(t)dt = \\ &= - \int\limits_{-\infty}^{+\infty} e^{-x+t} G(t)dt + \int\limits_{x}^{+\infty} e^{x-t} G(t)dt \\ &= -I_1^- + I_1^+ \to 0 \text{ when } x \text{ tends to } \pm\infty. \end{aligned}$$

The same is still true for derivatives of the integrals I_2, I_3, and I_4, which means $(I_2' + I_3'$ - $I_4') \to 0$ when x tends to $\pm\infty$. So, we conclude that $\lim\limits_{x\to\pm\infty} f'(x) = 0$ and $f \in C^2(\mathbb{R})$. By using inequality (4.4), we have L_1-norm estimation solution as follows

$$\begin{aligned} \|f\|_{L_1(\mathbb{R})} &\leq 2\sqrt{\frac{2}{\pi}} \left\| \sqrt{\frac{\pi}{2}} e^{-|t|} \right\|_{L_1(\mathbb{R})} \|l \underset{4}{*} \varphi\|_{L_1(\mathbb{R})} \\ &\leq \frac{8}{\pi} \left\| \sqrt{\frac{\pi}{2}} e^{-|t|} \right\|_{L_1(\mathbb{R})} \|l\|_{L_1(\mathbb{R})} \|\varphi\|_{L_1(\mathbb{R})} = 4\sqrt{\frac{2}{\pi}} \|l\|_{L_1(\mathbb{R})} \|\varphi\|_{L_1(\mathbb{R})}. \blacksquare \end{aligned}$$

Remark 4.1. The integrals I_1, I_2, I_3, and I_4 in the solution of the problem (4.2) can be represented via Fourier convolutions, so we can also check the initial condition (4.3) through Wiener-Tauberian theorem [24].

(Refer [23]) Assume that $h \in L_1(\mathbb{R}+)$ and $l \in L_1(\mathbb{R})$ then $\left(h \underset{5}{*} l\right)$ belongs to the $L_1(\mathbb{R})$ space. Moreover, we get the factorization equality

$$H_{\{\frac{1}{2}\}} \left(h \underset{5}{*} l\right)(y) = (F_c h)(|y|) \left(H_{\{\frac{1}{2}\}} l\right)(y), \tag{4.13}$$

and we have L_1-norm estimation as follows

$$\|h \underset{5}{*} l\|_{L_1(\mathbb{R})} \leq \sqrt{\frac{2}{\pi}} \|h\|_{L_1(\mathbb{R}_+)} \|l\|_{L_1(\mathbb{R})} \tag{4.14}$$

Here, the convolution $(h \underset{5}{*} l)(x)$ is defined

$$\left(h \underset{5}{*} l\right)(x) := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \Big[h(x+y) + h(x-y)\Big] l(y)\, dy,\ x \in \mathbb{R}. \tag{4.15}$$

and $(F_c h)(y)$ is defined by the formula (2.2).

Remark 4.2. If we choose $h = \sqrt{\frac{\pi}{2}}e^{-t}$ belongs to $L_1(\mathbb{R}_+)$, then

$$F_c\left(\sqrt{\frac{\pi}{2}}e^{-t}\right) = \left(H_{\{\frac{1}{2}\}}\sqrt{\frac{\pi}{2}}e^{|t|}\right)(x) = \frac{1}{1+y^2}.$$

Thus, we can be rewritten formula (4.12) as follows

$$\left(H_{\{\frac{1}{2}\}}f\right)(x) = F_c\left(\sqrt{\frac{\pi}{2}}e^{-t}\right)(|y|)\left(H_{\{\frac{1}{2}\}}l\right)(y)\left(H_{\{\frac{1}{2}\}}\varphi\right)(y).$$

By using the formulas (4.13) and (2.10), we obtain

$$\left(H_{\{\frac{1}{2}\}}f\right)(x) = H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-t} \underset{5}{*} \left(l \underset{4}{*} \varphi\right)\right)(y) \in L_1(\mathbb{R}).$$

Thus, $f(x) = \left(\sqrt{\frac{\pi}{2}}e^{-t} \underset{5}{*} (l \underset{4}{*} \varphi)\right)(x)$ for all x almost everywhere on $\mathbb{R}$ (refer [5, 19]) and $f(x) \in C^2(\mathbb{R}) \cap L_1(\mathbb{R})$. By using (4.14), (4.4), we also get the boundedness of solution as in the Theorem 4.1.

To check that the above solution satisfies the initial condition (4.3) in problem (4.2), we do the same as in the proof that the solution satisfies the initial conditions of the Theorem 4.1, by using (4.15) we rewrite the solution in the following form $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty}\sqrt{\frac{\pi}{2}}e^{-y}\big[G(x+y)+G(x-y)\big]dy$ with $x \in \mathbb{R}$, where $G(x) = \left(l \underset{4}{*} \varphi\right)(x) \in C_0^1(\mathbb{R}) \cap L_1(\mathbb{R})$.

We will end the article with an example illustrating the above result.

Example 4.2. Now, we choose $g(x) = \sqrt{\frac{\pi}{2}}e^{-|x|} \in L_1(\mathbb{R})$ space, then

$$H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|x|}\right) = \frac{1}{1+y^2}.$$

Under the condition (4.5), we obtain $1+\left(H_{\{\frac{1}{2}\}}g\right)(y) = 1+\frac{1}{1+y^2} \neq 0$ with $y \in \mathbb{R}$ and $\frac{\left(H_{\{\frac{1}{2}\}}g\right)(y)}{1+\left(H_{\{\frac{1}{2}\}}f\right)(y)} = \frac{1}{2+y^2} \in L_1(\mathbb{R})$, so there exists a function $l \in L_1(\mathbb{R}_+)$ such that

$$l(x) = H_{\{\frac{1}{2}\}}\left(\frac{1}{2+y^2}\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{1}{2+y^2}\,\mathrm{cas}(\pm xy)\,dy = \frac{\sqrt{\pi}}{2}e^{-\sqrt{2}|x|} \in L_1(\mathbb{R}).$$

On the other hand, according to the formula (3, p.615, see [6]) then

$$\varphi(x) = H_{\{\frac{1}{2}\}}\left(\frac{1}{\sqrt{2}}e^{-\frac{y^2}{4}}\right) == \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{1}{\sqrt{2}}e^{-\frac{y^2}{4}}\,\mathrm{cas}(xy)\,dy$$

$$= \sqrt{\frac{2}{\pi}}\int\limits_{0}^{+\infty}\frac{1}{\sqrt{2}}e^{-\frac{y^2}{4}}\cos(xy)\,dy = e^{-x^2} \in L_1(\mathbb{R}).$$

Then f belongs to the $C^2(\mathbb{R}) \cap L_1(\mathbb{R})$ space and has formula for the solution as follow $f(x) = \left(\sqrt{\frac{\pi}{2}}e^{-|t|} \underset{4}{*} \left(\frac{\sqrt{\pi}}{2}e^{-\sqrt{2}|t|} \underset{4}{*} e^{-t^2}\right)\right)(x)$, or we can completely represent the solution formula in the following form

$$f(x) = \left(\sqrt{\frac{\pi}{2}}e^{-t} \underset{5}{*} \left(\frac{\sqrt{\pi}}{2}e^{-\sqrt{2}|t|} \underset{4}{*} e^{-t^2}\right)\right)(x) \in C^2(\mathbb{R}) \cap L_1(\mathbb{R}).$$

And get L_1-norm estimation $\|f\|_{L_1(\mathbb{R})} \leq 4\sqrt{\frac{2}{\pi}}\|\frac{\sqrt{\pi}}{2}e^{-\sqrt{2}|t|}\|_{L_1(\mathbb{R})}\|e^{-t^2}\|_{L_1(\mathbb{R})} \leq 8.$

References

[1] **Appell, J., A. Kalitvin and P. Zabrejko,** *Partial Integral Operators and Integro-Differential Equations: pure and applied mathematics.* Marcel Dekker, New York (2000).

[2] **Barbashin, E.A.,** Conditions for invariance of stability of solutions of integro-differential equations. (in Russian). *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, **1** (1957), 25–34.

[3] **Bracewell, R.N.,** *The Hartley Transform*, The Clarendon Press, Oxford University Press, New York (1986).

[4] **Bateman, H.,** *Tables of Integral Transforms*, volume 1. New York (NY), London, McGraw-Hill Book Company, Inc (1954).

[5] **Christopher, D. Sogge,** *Fourier Integrals in Classical Analysis*, Cambridge University Press, Cambridge (1993).

[6] **Debnath, L. and D. Bhatta,** *Integral Transforms and Their Applications*, 2nd edition, Chapman & Hall/CRC Press, New York (2007).

[7] **Giang, B.T, N.V. Mau, and N.M. Tuan,** Operational properties of two integral transforms of Fourier type and their convolutions, *Integral Equations and Operator Theory*, **65(3)** (2009), 363–386.

[8] **Kilbas, A., H.M. Srivastava and J.J. Trujillo,** *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science, Amsterdam (2006).

[9] **Naimark, M.A.,** *Normed Algebras*, Wolters-Noordhoff Publishing, Groningen (1972).

[10] **Poularikas, A.D. and A.M. Grigoryan.** *The Transform and Applications Handbook*, 3rd edition. The Electrical Engineering Handbook Series, CRC Press & IEEE Press, Boca Raton, Florida (2010). https://doi.org/10.1201/9781315218915

[11] **Sneddon, I.N.,** *The Use of Integral Transform*, McGraw-Hill Book Company, Inc, New York (NY), London (1972).

[12] **Srivastava, H.M. and R.G. Buschman,** *Theory and Applications of Convolution Integral Equations*, Kluwer Series on Mathematics and Its Applications, Vol. 79. Kluwer Academic Publishers, Dordrecht, Boston, and London (1992).

[13] **Thao, N.X., V.K. Tuan, L.X. Huy and N.T. Hong,** On the Fourier–Laplace convolution transforms. *Integral Transforms and Special Functions.* **26(4)** (2015), 303–313.

[14] **Titchmarsh, E.C.,** *Introduction to the Theory of Fourier Integrals*, 3rd edition, Chelsea Publishing Co. New York (1986).

[15] **Tuan, T.,** On the Fourier-sine and Kontorovich–Lebedev generalized convolution transforms and their applications, *Ukrainian Math. J.*, **72(2)** (2020), 302–316. https://doi.org/10.1007/s11253-020-01782-1

[16] **Tuan, T.,** Some classes of integral equations of convolutions–pair generated by the Kontorovich–Lebedev, Laplace and Fourier transforms, *Communications of the Korean Mathematical Society*, **36(3)**, (2021), 485–494. https://doi.org/10.4134/CKMS.C200183

[17] **Tuan, T.,** Operational Properties of the Hartley Convolution and Its Applications, *Mediterranean Journal of Mathematics*, **19** (2022), 266. (2022). https://doi.org/10.1007/s00009-022-02173-5

[18] **Tuan, T.,** Some results of Watson and Plancherel–type integral transforms related to the Hartley, Fourier convolutions and applications, *Mathematical Methods in the Applied Sciences*, **45(17)** (2022), 11158–11180. https://doi.org/10.1002/mma.8442

[19] **Rudin, W.,** *Real and Complex Analysis*, 3rd edition, McGraw-Hill Book Company, New York (1987).

[20] **Tuan, V.K.,** Integral transforms of Fourier cosine convolution type, *Journal of Mathematical Analysis and Applications*, **229(2)** (1999), 519–529. https://doi.org/10.1006/jmaa.1998.6177

[21] **Tuan, V.K. and T. Tuan,** A real-variable inverse formula for the Laplace transform, *Integral Transforms and Special Functions*, **23(8)** (2012), 551–555. https://doi.org/10.1080/10652469.2011.609817

[22] **VanAnh, P.T. and N.X. Thao,** Integral transforms of Hartley–Fourier cosine polyconvolution type, *Applicable Analysis*, **94(9)** (2015), 1749–1765.

[23] **VanAnh, H.T. and N.X. Thao,** Hartley–Fourier cosine generalized convolution inequalities, *Mathematical Inequalities & Applications*, **18(4)** (2015), 1393–1408.

[24] **Wiener, N.,** Tauberian theorems, *Annals of Mathematics*, **33(2)** (1932), 1–100.

T. Tuan
Department of Mathematics,
Electric Power University
235-Hoang Quoc Viet Rd, Hanoi
Vietnam
tuantrinhpsac@yahoo.com