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THE RING OF ARITHMETICAL FUNCTIONS IN THE RATIONAL DOMAIN

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Abstract. The arithmetical functions with rational argument are interpreted as sums of weighted divisors:

$$f(r) = \sum_{d|r} w(d).$$

The logarithmic density of subsets of rational numbers is introduced. It is proved, that if $w(d) \geqslant 0$, then asymptotic logarithmic density of the set $\{r : f(r) \geqslant z\}$ exists.

1. The classes of functions

The set of natural numbers will be denoted by $\mathbb{N}$. If the integers $m, n \in \mathbb{N}$ are coprime, we write $m \perp n$.

Let $\mathbb{Q}_+$ be the set of positive rational numbers represented always as reduced fractions $\frac{m}{n}$, m, $n \in \mathbb{N}$, $m \perp n$. The notation $r|t$ for the reduced fractions $r = r_1/r_2, t = t_1/t_2$ means that $r_1|t_1, r_2|t_2$.

Consider the set $\mathscr{F}$ of all functions $f : \mathbb{Q}_+ \to \mathbb{R}$. The sum of functions $f, g \in \mathscr{F}$ is defined as usual

$$(f + g)(r) = f(r) + g(r), \quad r \in \mathbb{Q}_+.$$

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We define the second operation in $\mathscr{F}$ (the Dirichlet convolution) denoted by $\circ$:

$$(f\circ g)\Big(\frac{m}{n}\Big)=\sum_{\substack{a_1a_2=m\\ b_1b_2=n}} f\Big(\frac{a_1}{b_1}\Big)g\Big(\frac{a_2}{b_2}\Big).$$

Note, that the function $e:\mathbb{Q}_+\to\mathbb{R}$ defined by $e(1)=1, e(r)=0, r\neq 1$, is the unity element corresponding to this operation:

$$f\circ e=f,\quad f\in\mathscr{F}.$$

Theorem 1.1. *The set of functions $\mathscr{F}$ with the operations $+,\circ$ is a commutative ring.*

Proof. The commutativity and distributivity properties are obvious. The associativity follows from identity valid for all $f_1, f_2, f_3\in\mathscr{F}$:

$$(f_1\circ(f_2\circ f_3))\Big(\frac{m}{n}\Big)=((f_1\circ f_2)\circ f_3)\Big(\frac{m}{n}\Big)= \\ =\sum_{\substack{a_1a_2a_3=m\\ b_1b_2b_3=n}} f_1\Big(\frac{a_1}{b_1}\Big)f_2\Big(\frac{a_2}{b_2}\Big)f_3\Big(\frac{a_3}{b_3}\Big).\qquad\blacksquare$$

Let

$$\begin{aligned}
\mathscr{F}^\circ&=\{f\in\mathscr{F}: f(1)\neq 0\},\\
\mathscr{A}^+&=\{f\in\mathscr{F}: f(m/n)=f(m)+f(1/n)\},\\
\mathscr{A}&=\{f\in\mathscr{A}^+: f(m), g(n)=f(1/n) \text{ are additive functions in } \mathbb{N}\},\\
\mathcal{M}^\circ&=\{f\in\mathscr{F}^\circ: f(m/n)=f(m)f(1/n)\},\\
\mathcal{M}&=\{f\in\mathcal{M}^\circ: f(m), g(n)=f(1/n) \text{ are multiplicative functions in } \mathbb{N}\},\\
\mathscr{B}&=\{f\in\mathscr{F}: \text{ if } r|t, \text{then } f(r)\leqslant f(t)\}.
\end{aligned}$$

Note, that if $f\in\mathscr{A}$, then for $m_1/n_1, m_2/n_2\in\mathbb{Q}_+, m_1n_1\perp m_2n_2$, we have

$$f\Big(\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}\Big)=f\Big(\frac{m_1}{n_1}\Big)+f\Big(\frac{m_2}{n_2}\Big).$$

Correspondingly for $f\in\mathcal{M}$,

$$f\Big(\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}\Big)=f\Big(\frac{m_1}{n_1}\Big)\cdot f\Big(\frac{m_2}{n_2}\Big).$$

We call the functions $f\in\mathscr{A}$ additive and $f\in\mathcal{M}$ multiplicative.

Theorem 1.2. *The sets of functions $\mathcal{M}\subset\mathcal{M}^\circ\subset\mathscr{F}^\circ$ are groups in respect to operation $\circ$.*

Proof. Recall, that the unity element in respect to $\circ$ is the function $e:\mathbb{Q}_+\to\mathbb{R}$ defined by $e(1)=1, e(r)=0$ if $r\neq 1$. Let

$$\mathbb{Q}_{\Omega\leqslant N}=\{m/n\in\mathbb{Q}_+:\Omega(mn)\leqslant N\},\quad N\geqslant 1,$$

where $\Omega(k)$ stands for the total number of primes dividing an integer k.

For $f\in\mathscr{F}^\circ$ we have to define a function $g\in\mathscr{F}^\circ$, such that $f\circ g=e$. Note, that for $(f\circ g)(r)=e(r)$ as $r\in\mathbb{Q}_{\Omega\leqslant N}$, we have to define $g(r)$ only for $r\in\mathbb{Q}_{\Omega\leqslant N}$. Hence, if we take $g(1)=1/f(1)$, then $(f\circ g)(r)=e(r)$ as $r\in\mathbb{Q}_{\Omega\leqslant 0}$.

Suppose $g(r)$ is already defined for $r\in\mathbb{Q}_{\Omega\leqslant N}$ in such a way that $(f\circ g)(r)=$ $=e(r)$ holds as $r\in\mathbb{Q}_{\Omega\leqslant N}$. We show that the definition of g can be extended to $r\in\mathbb{Q}_{\Omega\leqslant N+1}$ still preserving the condition $(f\circ g)(r)=e(r)$.

Indeed, we have to define g for the rationals $pm/n, m/(qn)$, where p,q are primes $p\perp n, q\perp m$ and $m/n\in\mathbb{Q}_{\Omega\leqslant N}$.

Because of

$$(f\circ g)\Big(\frac{pm}{n}\Big)=\sum_{\substack{a_1a_2=pm\\ b_1b_2=n\\ a_1b_1>1}} f\Big(\frac{a_1}{b_1}\Big)g\Big(\frac{a_2}{b_2}\Big)+f(1)g\Big(\frac{pm}{n}\Big)$$

and $g(a_2/b_2)$ is already defined, we will get $(f\circ g)(pm/n)=0$ taking

$$g\Big(\frac{pm}{n}\Big)=-\frac{1}{f(1)}\sum_{\substack{a_1a_2=pm\\ b_1b_2=n\\ a_1b_1>1}} f\Big(\frac{a_1}{b_1}\Big)g\Big(\frac{a_2}{b_2}\Big).$$

Hence, by induction there is a function $g\in\mathscr{F}^\circ$ such that $f\circ g=e$, and $\mathscr{F}^\circ$ is a group.

Let $f,g\in\mathcal{M}^\circ$. It is straighforward to show that

$$(f\circ g)\Big(\frac{m}{n}\Big)=\sum_{a_1a_2=n} f(a_1)g(a_2)\sum_{b_1b_2=m} f\Big(\frac{1}{b_1}\Big)g\Big(\frac{1}{b_2}\Big).$$

It follows then, that if $h=f\circ g$, then $h(m/n)=h(m)h(1/n)$, i. e., $h\in\mathcal{M}^\circ$. Note, that arithmetical function $h(n), n\in\mathbb{N}$, is the Dirichlet convolution of arithmetical functions $f(n),g(n),n\in\mathbb{N}$. Hence, if $f(n),g(n)$ are multiplicative, then $h(n)$ is multiplicative, too. This claim is true for $f(1/n),g(1/n),h(1/n)$ interpreted as arithmetical functions with natural argument. It follows from this, that $\mathcal{M}$ is closed in respect to operation $\circ$. This completes the proof of proposition. ■

Let us extend the Möbius function $\mu:\mathbb{N}\to\{-1,0,1\}$ to $\mathbb{Q}_+$ taking

$$\mu\Big(\frac{m}{n}\Big)=\mu(m)\mu(n).$$

Then $\mu \in \mathcal{M}$. We denote by I the function $I(r) = 1, r \in \mathbb{Q}_+$. As the functions of natural argument μ and I are inverses to each other in respect to Dirichlet convolution. This is true also in the domain of rational numbers, i. e.,

$$(\mu \circ I)(r) = e(r), \quad r \in \mathbb{Q}_+.$$

Let $f \in \mathscr{F}$ be an arbitrary function. Then

$$f(r) = (f \circ e)(r) = (I \circ (f \circ \mu))(r) = \sum_{d|r} w(d), \quad w = f \circ \mu,$$

here for $d = d_1/d_2, r = r_1/r_2$, as agreed, we write $d|r$ if $d_1|r_1, d_2|r_2$.

Let us consider this relation as injective mapping $F_I : \mathscr{F} \to \mathscr{F}$:

$$f = F_I(w) = I \circ w.$$

The injectivity property remains, if we consider restrictions of F_I to $\mathscr{F}^\circ$, $\mathcal{M}^\circ$, $\mathcal{M}$.

For $r = m/n$ we write $\omega(r) = \omega(mn)$, where $\omega(k)$ stands for the number of distinct prime divisors of k.

Theorem 1.3. *The function $f = F_I(w)$ is in $\mathscr{A}^+$, if and only if $w(1) = 0$ and $w(d) = 0$ for all $d \in \mathbb{Q}_+$, such that $d, 1/d \notin \mathbb{N}$.*

The function $f = F_I(w)$ is in $\mathscr{A}$, if and only if $w(d) = 0$ for all $d \in \mathbb{Q}_+$, such that $\omega(d) \neq 1$.

The function $f = F_I(w)$ is in $\mathscr{B}$, if and only if $w(d) \geqslant 0$ for all $d, d \neq 1$.

Proof. If $w(1) = 0$ and $w(d_1/d_2) = 0$ for all fractions with $d_1 > 1, d_2 > 1$, then $F_I(w) \in \mathscr{A}^+$, because

$$f\Big(\frac{m}{n}\Big) = \sum_{d_1|m} w(d_1) + \sum_{d_2|n} w(1/d_2) = f(m) + f(1/n).$$

Let $f \in \mathscr{A}^+$. Obviously, $w(1) = 0$. We prove $w(d_1/d_2) = 0$ as $d_1, d_2 > 1$ by induction on $N = \Omega(d_1 d_2)$. If $N = 2$, i.e., d_1, d_2 are prime , we have

$$f\Big(\frac{d_1}{d_2}\Big) = f(d_1) + f(1/d_2) = w(d_1) + w\Big(\frac{1}{d_2}\Big) + w\Big(\frac{d_1}{d_2}\Big),$$

and $w(d_1/d_2) = 0$ follows because of $w(d_1) = f(d_1), w(d_2) = f(1/d_2)$. Suppose $w(b_1/b_2) = 0$ as $b_1, b_2 > 1$ and $\Omega(b_1 b_2) \leqslant N$. Let $\Omega(d_1 d_2) = N + 1$, where $d_1, d_2 > 1$ and $d_1 \perp d_2$. Then either $d_1/d_2 = pc_1/c_2, p \perp c_2$ or $d_1/d_2 = c_1/(pc_2), p \perp c_1$, where $c_1 \perp c_2$ and $\Omega(c_1 c_2) = N$. Let us consider the first case:

$$f\Big(\frac{pc_1}{c_2}\Big) = \sum_{b_1|pc_1} w(b_1) + \sum_{b_2|c_2} w\Big(\frac{1}{b_2}\Big) + w\Big(\frac{pc_1}{c_2}\Big) = f(pc_1) + f\Big(\frac{1}{c_2}\Big),$$

hence $w\left(\frac{pc_1}{c_2}\right) = 0$. The equality $w\left(\frac{c_1}{pc_2}\right) = 0$ follows in the same way.

Introducing some minor changes we can provide the proof of the assertion on $F_I(w) \in \mathscr{A}$.

The same simple reasoning by induction on $\Omega(r)$ gives the proof of the claim on $F_I(w) \in \mathscr{B}$. ■

2. The densities and multiples

For a subset A of natural numbers $\mathbb{N}$ and natural number $x > 1$ denote

$$\nu_x^0(A) = \frac{1}{x} \sum_{n \in A \cap [1;x]} 1, \quad \nu_x^1(A) = \frac{1}{\log x} \sum_{n \in A \cap [1;x]} \frac{1}{n}.$$

The lower and upper limits as $x \to \infty$ will be denoted by $\underline{\nu}^r(A), \overline{\nu}^r(A)$ the value of the limit, if it exists, by $\nu^r(A)$, respectively, $r = 0, 1$.

It follows from the chain of inequalities

$$\underline{\nu}^0(A) \leqslant \underline{\nu}^1(A) \leqslant \overline{\nu}^1(A) \leqslant \overline{\nu}^0(A),$$

that the existence of $\nu^0(A)$ implies the existence of $\nu^1(A)$. If $\nu^0(A)$ exists, we say that A possesses asymptotic density, and if $\nu^1(A)$ exists, A possesses asymptotic logarithmic density. Even the subsets A of apparently simple structure may not possess asymptotic density.

For $A \subset \mathbb{N}$ the set of natural numbers divisible by some $a \in A$ will be denoted by $\mathcal{M}(A)$, i.e., $\mathcal{M}(A)$ is the set of multiples of $a \in A$.

A.S. Besicovitch gave an example of A such that $\mathcal{M}(A)$ does not possess asymptotic density, see [1]. In 1937 H. Davenport and P. Erdős proved that every set of multiples have logarithmic density. Their original proof in [2] is based on Tauberian theorems, see also [6], Theorem 02. The direct and elementary proof of this theorem was provided by the authors in [3], it can be found also in the monograph of H. Halberstam and K.F. Roth, [5]. We will use the Erdős–Davenport theorem in the form, which results from the arguments in [5].

Lemma 2.1 (Erdős–Davenport)**.** *Let $A \subset \mathbb{N}$ and $A_N = A \cap [1; N]$ for $N \in \mathbb{N}$. Then $\nu^1(\mathcal{M}(A_N))$, $\nu^1(\mathcal{M}(A))$ exist, and*

$$\nu^1(\mathcal{M}(A)) = \lim_{N \to \infty} \nu^1(\mathcal{M}(A_N)). \tag{2.1}$$

Let $0 < \lambda_1 < \lambda_2$ some fixed numbers, $J = (\lambda_1; \lambda_2)$ and $x > 1$. We introduce the sets

$$\mathbb{Q}_{x,J} = \left\{ \frac{m}{n} \in \mathbb{Q}_+ : n \leqslant x \right\} \cap J.$$

Let $R \subset \mathbb{Q}_+$ and $r_1, r_2 \in \{0,1\}$. Then if $\mathbb{Q}_{x,J} \neq \emptyset$, we denote

$$S_{x,J}^{r_1 r_2}(R) = \sum_{m/n \in \mathbb{Q}_{x,J} \cap R} m^{-r_1} n^{-r_2}, \quad \nu_x^{r_1 r_2}(R) = \frac{S_{x,J}^{r_1 r_2}(R)}{S_{x,J}^{r_1 r_2}(\mathbb{Q}_{x,J})}.$$

Let q_1, q_2 be some coprime natural numbers and

$$\mathbb{Q}^{q_1,q_2} = \left\{ \frac{m}{n} \in \mathbb{Q}_+ : mq_1 \perp nq_2 \right\}. \tag{2.2}$$

Note, that taking $q_1 = q_2 = 1$, we get $\mathbb{Q}^{q_1,q_2} = \mathbb{Q}_+$. We use the asymptotics for $S_{x,J}^{r_1 r_2}(\mathbb{Q}^{q_1,q_2})$ established in [7].

Lemma 2.2. *Let for the coprime integers* q_1, q_2

$$\Pi(q_1, q_2) = \prod_{p | q_1 q_2} \left(1 - \frac{1}{p+1}\right).$$

Then the following asymptotics hold

$$\begin{aligned}
\frac{S_{x,J}^{00}(\mathbb{Q}^{q_1,q_2})}{\Pi(q_1,q_2)} &= \frac{3}{\pi^2}(\lambda_2 - \lambda_1)x^2 \left\{1 + O\left(\frac{\log x}{x}\right)\right\}, \\
\frac{S_{x,J}^{01}(\mathbb{Q}^{q_1,q_2})}{\Pi(q_1,q_2)} &= \frac{6}{\pi^2}(\lambda_2 - \lambda_1)x \left\{1 + O\left(\frac{\log^2 x}{x}\right)\right\}, \\
\frac{S_{x,J}^{10}(\mathbb{Q}^{q_1,q_2})}{\Pi(q_1,q_2)} &= \frac{6}{\pi^2}\log\left(\frac{\lambda_2}{\lambda_1}\right)x \left\{1 + O\left(\frac{\log^2 x}{x}\right)\right\}, \\
\frac{S_{x,J}^{11}(\mathbb{Q}^{q_1,q_2})}{\Pi(q_1,q_2)} &= \frac{6}{\pi^2}\log\left(\frac{\lambda_2}{\lambda_1}\right)\log x \left\{1 + O\left(\frac{1}{\log x}\right)\right\}.
\end{aligned}$$

The constants in O-signs depend on q_1, q_2 *and* λ_1, λ_2.

As a Corrollary we have, that for all $q_1 \perp q_2$

$$\lim_{x\to\infty} \nu_x^{r_1 r_2}(\mathbb{Q}^{q_1,q_2}) = \Pi(q_1, q_2).$$

Let $t = t_1/t_2 \in \mathbb{Q}_+$; we define the set of multiples of t by

$$\mathcal{M}(t) = \left\{ \frac{m}{n} : m \perp n, t_1 | m, t_2 | n \right\}.$$

Note, that

$$S_{x,J}^{r_1 r_2}(\mathcal{M}(t)) = t_1^{r_1} t_2^{r_2} S_{x^*,J^*}^{r_1,r_2}(\mathbb{Q}^{t_2,t_1}), \quad x^* = x/t_2, \quad J^* = (t_2\lambda_1/t_1; t_2\lambda_2/t_1).$$

It follows now from the Lemma 2.2, that

$$\lim_{x\to\infty} \nu_x^{r_1r_2}(\mathcal{M}(t)) = \frac{1}{t_1t_2}\prod_{p|t_1t_2}\Big(1-\frac{1}{p+1}\Big).$$

For the subset $T\subset\mathbb{Q}_+$ define

$$\mathcal{M}(T) = \bigcup_{t\in T}\mathcal{M}(t).$$

Remark 2.1. If $t=t_1/t_2$, $s=s_1/s_2$, then either $\mathcal{M}(t)\cap\mathcal{M}(s)=\emptyset$, or

$$\mathcal{M}(t)\cap\mathcal{M}(s)=\mathcal{M}([t_1,s_1]/[t_2,s_2]),$$

where $[a,b]$ stands for the smallest common multiple. Hence, if T is finite, then due to the sieving procedure the densities

$$\nu^{r_1r_2}(\mathcal{M}(T)) = \lim_{x\to\infty}\nu_x^{r_1r_2}(\mathcal{M}(T)),\quad r_1,r_2\in\{0,1\}.$$

exist and are equal.

Theorem 2.1. *For arbitrary subset $T\subset\mathbb{Q}_+$ the logarithmic density*

$$\nu^{11}(\mathcal{M}(T)) = \lim_{x\to\infty}\nu_x^{11}(\mathcal{M}(T))$$

exists.

Proof. Let $N>1$ be an integer. We define

$$T_N = \Big\{\frac{t_1}{t_2}\in T : t_1,t_2\leqslant N\Big\}.$$

Note, that $\mathcal{M}(T_N)$ is the finite union of the sets having the asymptotic densities; moreover, the itersection of these sets also have the asymptotic densities. We conclude, that $\nu^{r_1r_2}(\mathcal{M}(T_N))$ exists due to the inclusion-exclusion identity for the measure of finite union of sets. Hence, it is sufficient to show, that

$$\overline{\nu}^{11}(\mathcal{M}(T)\backslash\mathcal{M}(T_N))\leqslant\epsilon$$

for an arbitrary $\epsilon>0$ as $N>N(\epsilon)$. Let us define

$$\begin{aligned} T^1 &= \Big\{t_1 : \text{there exists } t_2, \frac{t_1}{t_2}\in T\Big\},\quad T_N^1 = T^1\cap[N,+\infty),\\ T^2 &= \Big\{t_2 : \text{there exists } t_1, \frac{t_1}{t_2}\in T\Big\},\quad T_N^2 = T^2\cap[N,+\infty).\end{aligned}$$

Let us agree, that if $A \subset \mathbb{N}$, then $\mathcal{M}(A)$ stands for the subset of $\mathbb{N}$, i.e.

$$\mathcal{M}(A) = \bigcup_{a \in A} \{ak : k = 1, 2, \ldots\}.$$

From the Erdős–Davenport theorem we have, that $\nu^1(\mathcal{M}(T^i)), i = 1, 2$, exist. Moreover, $\nu^1(\mathcal{M}(T_N^i)) \leqslant \epsilon_N$, where $\epsilon_N \to 0$ as $N \to \infty$. We shall use this is the form

$$\sum_{n \in \mathcal{M}(T_N^i) \cap [1;x]} \frac{1}{n} \ll \epsilon_N \cdot \log x, \quad x \to \infty. \tag{2.3}$$

Start with the observation

$$\mathcal{M}(T) \backslash \mathcal{M}(T_N) \subset \mathcal{M}_1(T_N) \cup \mathcal{M}_2(T_N),$$

where

$$\begin{aligned} \mathcal{M}_1(T_N) &= \{m/n \in \mathbb{Q}_+ : m \in \mathcal{M}(T_N^1)\} \cap J, \\ \mathcal{M}_2(T_N) &= \{m/n \in \mathbb{Q}_+ : n \in \mathcal{M}(T_N^2)\} \cap J. \end{aligned}$$

It is sufficient to show, that $\overline{\nu}^{11}(\mathcal{M}_i(T_N)) \to 0$ as $i = 1, 2$ and $N \to \infty$. Using (2.3) we get

$$\begin{aligned} S_{x,J}^{11}\big(\mathcal{M}_2(T_N)\big) &\leqslant \sum_{n \in \mathcal{M}(T_N^2) \cap [1;x]} \frac{1}{n} \sum_{\lambda_1 n < m < \lambda_2 n} \frac{1}{m} \ll \\ &\ll \sum_{n \in \mathcal{M}(T_N^2) \cap [1;x]} \frac{1}{n} \Big\{ \log\Big(\frac{\lambda_2}{\lambda_1}\Big) + \frac{1}{\lambda_1 n} \Big\} \ll \\ &\ll \log\Big(\frac{\lambda_2}{\lambda_1}\Big) \log x \Big(\epsilon_N + \frac{1}{\lambda_1 \log x}\Big). \end{aligned}$$

Hence, $\overline{\nu}^{11}(\mathcal{M}_2(T_N)) \to 0$ as $N \to \infty$ follows because of asymptotics

$$S_{x,J}^{11}(\mathbb{Q}_+) \sim \frac{6}{\pi^2} \log\Big(\frac{\lambda_2}{\lambda_1}\Big) \log x \quad \text{as } x \to \infty.$$

For $S_{x,J}^{11}\big(\mathcal{M}_1(T_N)\big)$ we proceed as follows:

$$S_{x,I}^{11}\big(\mathcal{M}_1(T_N)\big) \leqslant \sum_{\substack{m \leqslant \lambda_2 \\ m \in \mathcal{M}(T_N^1)}} \frac{1}{m} \sum_{n < m/\lambda_1} \frac{1}{n} + \sum_{\substack{\lambda_2 < m < \lambda_2 x \\ m \in \mathcal{M}(T_N^1)}} \frac{1}{m} \sum_{m/\lambda_2 < n < m/\lambda_1} \frac{1}{n}. \tag{2.4}$$

Because λ_2 is fixed, then the first sum in (2.4) is zero for N sufficiently large. For the second sum in (2.4) we obtain

$$\sum_{\substack{\lambda_2<m<\lambda_2 x\\ m\in\mathcal{M}(T_N^1)}} \frac{1}{m} \sum_{m/\lambda_2<n<m/\lambda_1} \frac{1}{n} \ll \sum_{\substack{\lambda_2<m<\lambda_2 x\\ m\in\mathcal{M}(T_N^1)}} \frac{1}{m}\Big\{\log\Big(\frac{\lambda_2}{\lambda_1}\Big)+\frac{\lambda_2}{m}\Big\} \ll$$

$$\ll \epsilon_N\Big(\log\Big(\frac{\lambda_2}{\lambda_1}\Big)+1\Big)\log(\lambda_2 x).$$

This is sufficient to conclude that $\overline{\nu}^{11}(\mathcal{M}_1(T_N)) \to 0$ as $N \to \infty$. The Theorem is proved. ∎

3. The class $\mathscr{B}$

Recall the definition of the class $\mathscr{B}$:

$$\mathscr{B} = \{f \in \mathscr{F} :\ \text{if } r|t, \text{then } f(r) \leqslant f(t)\}.$$

Note, that an additive function f belongs to $\mathscr{B}$, if and only if it satisfies the condition: for all primes p

$$0 \leqslant f(p) \leqslant f(p^2) \leqslant \cdots, \quad 0 \leqslant f(1/p) \leqslant f(1/p^2) \leqslant \cdots.$$

Correspondingly, a multiplicative function g belongs to $\mathscr{B}$, if and only if it satisfies the condition: for all primes p

$$1 \leqslant g(p) \leqslant g(p^2) \leqslant \cdots, \quad 1 \leqslant g(1/p) \leqslant g(1/p^2) \leqslant \cdots.$$

Theorem 3.1. *For every $f \in \mathscr{B}$ and $z \in \mathbb{R}$ the density*

$$\nu^{11}(r \in \mathbb{Q}_+ : f(r) \geqslant z)$$

exists.

Let $0 < \delta < 1$. There exist functions $f \in \mathscr{B}$ such that

$$\overline{\nu}^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z) - \underline{\nu}^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z) > \delta \tag{3.1}$$

for all $z \geqslant z_0$.

We will use in the proof the following result of Erdős.

Lemma 3.1 (see, [4]). *Let $[T;2T]$ denotes the set of integers satisfying the inequalities $T \leqslant m \leqslant 2T$. Then*

$$\nu^0(\mathcal{M}([T;2T]) \to 0, \quad T \to \infty.$$

Note, that the existence of $\nu^0(\mathcal{M}([T;2T])$ follows from the representation of $\mathcal{M}([T;2T])$ as finite union of arithmetical progressions.

Proof. Let $f \in \mathscr{B}, z \in \mathbb{R}$ and $A(f,z) = \{r \in \mathbb{Q}_+ : f(r) \geqslant z\}$. If $r \in$ $\in A(f,z)$, then $\mathcal{M}(r) \subset A(f,z)$. Hence, $\mathcal{M}(A(f,z)) = A(f,z)$ and the existence of $\nu^{11}(A(f,z))$ follows from the Theorem 3.1.

We construct now a function $f \in \mathscr{B}, f = I \circ w$, satisfying (3.1). We have to define $w(d) \geqslant 0$ for all $d \in \mathbb{Q}_+$. Let us take $w(u/v) = 0$ if $u > 1, u \perp v$ and define $w(1/v)$ for $v \in \mathbb{N}$.

The construction is based on the result of Erdős given in Lemma 3.1.

Let $k \geqslant 1$ be an integer to be specified later. Consider the sequence T_n of integers and introduce the sets of integers

$$I_n = [T_n; 2^k T_n] \cap \mathbb{N}, \quad \text{where } 2^k T_n < T_{n+1}.$$

Let $\epsilon > 0$ be an arbitrary number. Due to Lemma 3.1 it is possible to choose the sequence T_n such that

$$\sum_{n \geqslant 1} \nu^0(\mathcal{M}(I_n)) < \epsilon,$$

$$\nu_x^0(\mathcal{M}(\bigcup_{m \leqslant n} I_m) < 2 \sum_{m \leqslant n} \nu^0(\mathcal{M}(I_m)) < 2\epsilon \quad \text{as } x \geqslant T_{n+1}.$$

Take an arbitrary sequence $0 < z_1 < z_2 < \cdots$ and define $w(1/v) = z_n$ if $v \in I_n$ and $w(1/v) = 0$, if $v \notin I_n$ for all $n \geqslant 1$. Denote for brevity $\mathcal{M} = \mathcal{M}(\cup_n I_n)$.

Then for the function $f = I \circ w$ we have

$$\#(\{r : r = u/v, v \leqslant x, f(r) \geqslant z_1\} \cap J) \leqslant (\lambda_2 - \lambda_1) \cdot x \cdot \#\{v : v \leqslant x, v \in \mathcal{M}\}.$$

If $x = T_{n+1}$

$$\#\{v : v \leqslant x, v \in \mathcal{M}\} = \#\{v : v \leqslant x, v \in \mathcal{M}(\cup_{m \leqslant n} I_m)\} \leqslant 2\epsilon x.$$

Hence,

$$\#(\{r : r = u/v, v \leqslant x, f(r) \geqslant z_1\} \cap J) \ll \epsilon(\lambda_2 - \lambda_1)x^2,$$

and

$$\underline{\nu}^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z) \ll \epsilon. \tag{3.2}$$

For $z \geqslant z_1$ let $z_{m-1} \leqslant z < z_m$. Then

$$\nu_x^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z) \geqslant \nu_x^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z_m) \geqslant$$
$$\geqslant \nu_x^{00}(u/v : u/v \in \mathbb{Q}_+, v \in [T_m; 2^k T_m]).$$

If $x \geqslant 2^k T_m$

$$\nu_x^{00}(u/v : u/v \in \mathbb{Q}_+, v \in [T_m; 2^k T_m] = \frac{S^{00}_{2^k T_m, J} - S^{00}_{T_m, J}}{S^{00}_{x,J}}.$$

Taking $x = 2^k T_m$ and using asymptotics from Lemma 2.2 we get

$$\nu_x^{00}(r \in \mathbb{Q}_+ : f(r) \geqslant z) \geqslant 1 - \frac{S^{00}_{T_m, J}}{S^{00}_{2^k T_m, J}} = 1 - 2^{-2k}(1 + o(1)). \tag{3.3}$$

Obviously due to (3.2) and (3.3) the choice of k and ϵ can be combined to get the inequality (3.1). The proof is complete. ■

Remark 3.1. If $f \in \mathscr{B}$ and the set of $d \in \mathbb{Q}_+$, such that $w(d) > 0$, is finite, then $\{r \in \mathbb{Q}_+ : f(r) \geqslant z\} = \mathcal{M}(T)$ for some finite set T. Then due to the Remark 2.1 all densities $\nu^{r_1 r_2}(r \in \mathbb{Q}_+ : f(r) \geqslant z), r_1, r_2 = 0, 1$, exist and are equal.

Let $Q \subset \mathbb{Q}_+$. Define $w : \mathbb{Q}_+ \to \{0, 1\}$ taking $w(r) = 1$ if $r \in Q$ and $w(r) = 0$ otherwise.

Then $f_Q = I \circ w$ is the counting function of divisors, i.e.,

$$f_Q(r) = \#\{d : d \in Q, d|r\},$$

where the notation $d|r$ for rational numbers has the same meaning as above. Then it follows from the Theorem 3.1, that for every $Q \subset \mathbb{Q}_+$ and $m \geqslant 0$ the density

$$\nu^{11}(r \in \mathbb{Q}_+ : f_Q(r) = m)$$

exists.

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