# ON SOME RESULTS OF INDLEKOFER FOR MULTIPLICATIVE FUNCTIONS II.

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**Abstract.** In this paper we describe how convolution arithmetic is be used for the investigation of the asymptotic mean behaviour of multiplicative functions.

## 1. Introduction

The average values

$$M(f,x) := x^{-1} \sum_{n \le x} f(n)$$

of multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ have long been an object of study in the theory of numbers. The problem of establishing the existence of the *mean values*

$$M(f) := \lim_{x \to \infty} M(f,x)$$

was considered by Wintner [8] in his book on Eratosthenian Averages where he, in particular, asserted that limit $M(f)$ always exists if $f$ assumes only the values $\pm 1$. The sketch of his proof, however, could not be substantiated, and, thus, the problem remained for a considerable time as a conjecture, variously ascribed to Erdős and Wintner (see [2]). In his paper [9], of 1967, Wirsing proved his celebrated mean-value theorem which asserts, in particular, that any real-valued multiplicative function $f$ of modulus $\le 1$ has a mean value (see Proposition 1.2). This solved the aforementioned conjecture of Erdős and Wintner. Two typical results are as follows.

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**Proposition 1.1.** *Let $g$ be a multiplicative function which assumes real non-negative values only. Let*

$$\sum_{p\le x}\frac{g(p)\log p}{p}\sim\alpha\log x,\quad x\to\infty,$$

*hold with a constant $\alpha>0$. Furthermore, let $g(p)=O(1)$ for all primes $p$, and let*

$$\sum_{p,k\ge 2}p^{-k}g(p^k)<\infty.$$

*Besides this, if $\alpha\le 1$, then let*

$$\sum_{p^k\le x,k\ge 2}g(p^k)=O(x(\log x)^{-1}).$$

*Then*

$$\sum_{n\le x}g(n)\sim\frac{e^{-\gamma\alpha}x}{\Gamma(\alpha)\log x}\prod_{p\le x}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right)$$

*as $x\to\infty$. Here $\gamma$ denotes Euler's constant.*

Furthermore Wirsing proved in [9]

**Proposition 1.2.** *Let $g$ satisfy the conditions of Proposition 1.1, and let $f$ be a real-valued multiplicative function which satisfies $|f(n)|\le g(n)$ for every positive integer $n$. Then*

$$\lim_{x\to\infty}\frac{\sum_{n\le x}f(n)}{\sum_{n\le x}g(n)}=\prod_p\left(1+\sum_{k=1}^{\infty}p^{-k}f(p^k)\right)\left(1+\sum_{k=1}^{\infty}p^{-k}g(p^k)\right)^{-1},$$

*where the product either converges properly to a nonzero limit, or improperly to zero.*

These results are generalizations of the case where $|g(n)|\equiv 1$, so that $\alpha=1$. In these circumstances, the general case ($f$ complex-valued and $|f|\le 1$) was handled by Halász [3], and his main result is given by the following

**Proposition 1.3.** *Let $f:\mathbb{N}\to\mathbb{C}$ be multiplicative, $|f|\le 1$. If there exists a real number $a_0$ such that the series*

$$\sum_p p^{-1}(1-Ref(p)p^{-ia})\tag{1.1}$$

*converges for* $a = a_0$*, then, as* $x \to \infty$*,*

$$x^{-1} \sum_{n \le x} f(n) = \frac{x^{ia_0}}{1+ia_0} \prod_{p \le x} (1-p^{-1}) \left(1 + \sum_{m=1}^{\infty} p^{-m(1+ia_0)} f(p^m)\right) + o(1).$$

*If the series* (1.1) *diverges for all* $a \in \mathbb{R}$*, then*

$$x^{-1} \sum_{n \le x} f(n) = o(1), \quad x \to \infty.$$

*In both cases, there are constants* $c, a_0$ *and a slowly oscillating function* $\tilde{L}(u)$ *with* $|\tilde{L}(u)| = 1$ *such that, as* $x \to \infty$*,*

$$x^{-1} \sum_{n \le x} f(n) = c x^{ia_0} \tilde{L}(\log x) + o(1).$$

Proposition 1.3 includes Wirsing's result (for $|f| \le 1$), and essentially the case where (1.1) diverges for all $a \in \mathbb{R}$ (and $f$ is complex-valued) is not covered by Proposition 1.2.

Wirsing's proof was elementary, but quite complicated, whereas Halász's proof was based upon analytic methods.

In [1], Daboussi and Indlekofer succeeded in finding an elementary proof of Halász's theorem (see also Indlekofer [4] for a simplified and shorter proof).

Indlekofer, Kátai and Wagner [5] used the methods of [4] to compare $\sum_{n \le x} f(n)$ with $\sum_{n \le x} g(n)$ where $g \ge 0$ is multiplicative and $|f| \le g$. They showed

**Proposition 1.4.** *Let* $g$ *be a multiplicative function which assumes real non-negative values only. Let*

$$\sum_{p \le x} \frac{\log p}{p} g(p) \sim \tau \log x, \quad x \to \infty,$$

*hold with a constant* $\tau > 0$*. Furthermore, let* $g(p) = O(1)$ *for all primes* $p$*, and let*

$$\sum_{p, k \ge 2} p^{-k} g(p^k) < \infty.$$

*Besides this, if* $\tau \le 1$*, then let*

$$\sum_{p^k \le x, k \ge 2} g(p^k) = O\left(x (log x)^{-1}\right).$$

*Let $f$ be a complex-valued function, which satisfies $|f(n)| \leq g(n)$ for every positive integer $n$. If there exists a real number $a_0$ such that the series*

$$\sum_p p^{-1}(g(p) - Ref(p)p^{-ia}) \tag{1.2}$$

*converges for $a = a_0$, then*

$$\sum_{n\leq x} f(n) = \frac{x^{ia_0}}{1+ia_0} \prod_{p\leq x} \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{m(1+ia_0)}}\right) \left(1 + \sum_{m=1}^{\infty} \frac{g(p^m)}{p^m}\right)^{-1} \sum_{n\leq x} g(n) + o\left(\sum_{n\leq x} g(n)\right)$$

*as $x \to \infty$. If the series (1.2) diverges for all $a \in \mathbb{R}$, then*

$$\sum_{n\leq x} f(n) = o\left(\sum_{n\leq x} g(n)\right), \quad x \to \infty.$$

*In both cases, there are constants $c, a_0$ and a slowly oscillating function $\tilde{L}$ with $|\tilde{L}(u)| = 1$ such that, as $x \to \infty$,*

$$\sum_{n\leq x} f(n) = \left(cx^{ia_0}\tilde{L}(\log x) + o(1)\right) \sum_{n\leq x} g(n).$$

In [7] Indlekofer gave a new proof of Proposition 1.1. For this he used the convolution arithmetic desribed, for example, in [6]. The idea of the proof is as follows.

Let $g$ as in the Proposition 1.1. Define an *exponentially multiplicative function* $g_0$ by

$$g_0(p^k) = \frac{g(p)}{k!} \quad (p \ \textit{prime}, \quad k \in \mathbb{N}).$$

Then $g = h * g_0$ where

$$\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty.$$

Further, define the multiplicative function $\tau_\alpha$ by

$$\sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} = \zeta^\alpha(s),$$

where $\zeta(s)$ is Riemann's zeta-function. Then put

$$A(x) := H(1)e^{\alpha c} \exp\left(\sum_{p\le x} \frac{g(p)-\alpha}{p}\right),$$

where $H(1) = \sum_{n=1}^{\infty} \frac{h(n)}{n}$ and

$$c = \sum_{p} \left(\frac{1}{p} + \log(1-\frac{1}{p})\right).$$

Define, for $1 \le u \le x$,

$$\begin{aligned} M(u) &:= 1 * (g - A(x)\tau_\alpha)(u) = \\ &= \sum_{n\le u} (g(n) - A(x)\tau_\alpha(n)). \end{aligned}$$

Then by convolution arithmetic, Indlekofer shows

$$L^2 M = M * (\Lambda_{g_0} * \Lambda_{g_0} + L_0\Lambda_{g_0}) + (R_1+R_2+R_3) * \Lambda_{g_0} + L(R_1+R_2+R_3),$$

where

$$\begin{aligned} R_1 &= L * (g - A(x)\tau_\alpha), \\ R_2 &= \mathbf{1} * (L_0 h * g_0), \\ R_3 &= -\mathbf{1} * A(x)\tau_\alpha * (\Lambda_{\tau_\alpha} - \Lambda_{g_0}). \end{aligned}$$

In the first step this leads to

$$|M(x)| \ll \frac{x}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right) \quad as \;\; x \to \infty.$$

Next, define an exponentially multiplicative function $\overline{g}_0$ by

$$\overline{g}_0(p^k) = \begin{cases} g_0(p^k) & for\ p \le x,\ k \ge 1 \\ \alpha/k! & for\ p > x,\ k \ge 1 \end{cases}$$

and put $\overline{g} = h * \overline{g}_0$. Choosing

$$K_0(u) = 1 * \Lambda_{\overline{g}_0} * (\overline{g} - A(x)\tau_\alpha)(u)$$

he obtains, for $2 \le u \le x$,

$$M(u) = \frac{K_0(u)}{\log u} + o\left(\frac{u}{\log u} \exp\left(\sum_{p\le u} \frac{g(p)}{p}\right)\right). \tag{1.3}$$

Now

$$\int_1^x \frac{|M(u)|}{u^2}du \leq \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2}du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2}du =: \\ =: I_1 + I_2.$$

Then
(1.4)

$$I_1 \leq \sum_{n\leq x^\varepsilon} \frac{|g(n) - A(x)\tau_\alpha(n)|}{n} \ll \varepsilon \exp\left(\sum_{p\leq x}\frac{g(p)}{p}\right) + \varepsilon^\alpha \exp\left(\sum_{p\leq x}\frac{g(p)}{p}\right)$$

and

$$I_2 \leq \frac{1}{\varepsilon}\left(\frac{1}{\log x}\int_2^x \frac{|K_0(u)|^2}{u^3}du\right)^{\frac{1}{2}} + o\left(\exp\left(\sum_{p\leq x}\frac{g(p)}{p}\right)\right).$$

Put $\sigma = 1 + \frac{1}{\log x}$. Then

$$\int_2^x \frac{|K_0(u)|^2}{u^3}du \leq e^2 \int_2^x \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du.$$

Since $(s = \sigma + it)$

$$\int_0^\infty K_0(e^u)e^{-us}e^{-iut}dt = \frac{\overline{G}_0'(s)}{\overline{G}_0(s)}(\overline{G}(s) - A(x)\zeta^\alpha(s)),$$

where

$$\overline{G}_0(s) = \sum_{n=1}^\infty \frac{\overline{g}_0(n)}{n^s} \quad \text{and} \quad \overline{G}(s) = \sum_{n=1}^\infty \frac{\overline{g}(n)}{n^s},$$

he concludes, by Parseval's equation,

$$\int_1^\infty \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2}. \tag{1.5}$$

Estimating the last integral in (1.5) by $o\left(\log x \exp\left(2\sum_{p\leq x}\frac{g(p)}{p}\right)\right)$ ends the proof of Proposition 1.1.

In this paper we we give a new proof of the result of Indlekofer–Kátai–Wagner [5] (see Proposition 1.2) by using the method of Indlekofer [7].

## 2. Proof of Proposition 1.2

We first assume that (1.2) diverges for all $a \in \mathbb{R}$. Then put $f = h_1 * f_0$, where $f_0$ is exponentially multiplicative,

$$f_0(p^k) = \frac{f(p^k)}{p^k} \quad (p \;\; prime, \quad k \in \mathbb{N})$$

and

$$\sum_{n=1}^{\infty} \frac{|h_1(n)|}{n} < \infty.$$

Defining $(1 \le u \le x)$

$$K_1(u) = (\mathbf{1} * \Lambda_{f_0} * f)(u)$$

we conclude

$$\mathbf{1} * L_0 f = \mathbf{1} * \Lambda_{f_0} * f + \mathbf{1} * \Lambda_{h_1} * f$$

which implies (cf. (1.3)), for $2 \le u \le x$,

$$M(u) = \frac{K_1(u)}{\log u} + o\left(\frac{u}{\log u} \exp\left(\sum_{p \le u} \frac{g(p)}{p}\right)\right).$$

Now we argue as in [5]

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &\le \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du := \\ &:= I_1 + I_2. \end{aligned}$$

Then (cf.(1.4))

$$I_1 \ll \varepsilon^\alpha \exp\left(\sum_{p \le x} \frac{g(p)}{p}\right)$$

and

$$I_2 \ll \frac{1}{\varepsilon}\left(\frac{1}{\log x} \int_2^x \frac{|K_1(u)|^2}{u^3} du\right)^{\frac{1}{2}} + o\left(\exp\left(\sum_{p \le x} \frac{g(p)}{p}\right)\right).$$

From this we obtain, by Parseval's equality,

$$\int_2^x \frac{|K_1(u)|^2}{u^3} du \ll \int_{-\infty}^{\infty} \left|\frac{F_0'(s)^2}{F_0(s)}\right|^2 |F(s)|^2 \frac{dt}{|s|^2}.$$

Now, observe (cf. (4.5) of [7])

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+ik+it)\Big|^2 dt \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big|\frac{\zeta'}{\zeta}(1+\frac{1}{\log x}+ik+it)\Big|^2 dt \ll \log x$$

and

$$|F(s)| = o(G(\sigma)) \quad for \quad |t| \le K.$$

Then

$$\int_{-\infty}^{\infty} \Big|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+ik+it)\Big|^2 |F(1+\frac{1}{\log x}+it)|^2 \frac{dt}{|s|^2} \le$$

$$\le \sum_{\substack{k\in\mathbb{Z}\\ |k|\le K}} \frac{o(G^2(\sigma))}{k^2+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+ik+it)\Big|^2 |dt+ \tag{2.1}$$

$$+ \sum_{\substack{k\in\mathbb{Z}\\ |k|> K}} \frac{G^2(\sigma)}{k^2+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big|\frac{F_0'}{F_0}(1+\frac{1}{\log x}+ik+it)\Big|^2 |dt \le$$

$$\le \delta \log x \exp\left(2\sum_{p\le x}\frac{g(p)}{p}\right) \quad if \quad K \ge K_0(\delta) \quad for \quad every \quad \delta > 0.$$

This ends the proof, if (1.2) diverges for all $a \in \mathbb{R}$.

Assume that (1.4) converges for $a = a_0$. Define $g_{a_0}$ by

$$g_{a_0} = g(n)n^{ia_0} \quad for \quad n \in \mathbb{N}$$

and put

$$M = \mathbf{1} * (f - A_x g_{a_0}),$$

where

$$A_x = \frac{H_1(1+ia_0)}{H(1)} \exp\left(\sum_{p\le x} \frac{f(p)p^{-ia_0} - g(p)}{p}\right).$$

Then

$$LM = \mathbf{1} * L_0(f - A_x g_{a_0}) + R_1',$$

where $R_1' = L * (f - A_x g_{a_0})$ and

$$R_1'(x) \ll \frac{x}{\log x} \exp\left(\sum_{p\le x}\frac{g(p)}{p}\right).$$

Further

$$\mathbf{1} * L_0(f - A_x g_{a_0}) = \mathbf{1} * \Lambda_{f_0} * (f - A_x g_{a_0}) + R_2' + R_3',$$

where

$$R_2' = \mathbf{1} * \Lambda_{h_1} * (f - A_x g_{a_0}),$$
$$R_3' = \mathbf{1} * A_x g_{a_0} * (\Lambda_f - \Lambda_{g_{a_0}}).$$

The convergence of $\sum_p (g(p) - Ref(p)p^{-ia_0})p^{-1}$ implies

$$\sum_p \frac{|g(p) - Ref(p)p^{-ia_0}|^2}{p} < \infty$$

and

$$R_3'(x) = O\left(\varepsilon^\alpha x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right) + o\left(x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).$$

In any case

$$R_2'(x) = o\left(x \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).$$

Now, in the same way as above we obtain

$$\int_{x^\varepsilon}^{x} \frac{|M(u)|}{u^2} du \ll \frac{1}{\varepsilon}\left(\frac{1}{\log x}\int_{-\infty}^{\infty}\left|\frac{F_0'(s)}{F_0(s)}\right|^2 |F(s) - A_x G(s - ia_0)|^2 \frac{dt}{|s|^2}\right)^{\frac{1}{2}} + \tag{2.2}$$
$$+ o\left(\exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right).$$

The integral on the right side of (2.2) can be estimated by

$$\int_{-\infty}^{\infty}\left|\frac{F_0'(s+ia_0)}{F_0(s+ia_0)}\right|^2 |F(s+ia_0) - A_x G(s)|^2 \frac{dt}{|s|^2}.$$

Observe

$$|F(s+ia_0) - A_x G(s)| = o(G(\sigma)) = o\left(\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right) \quad for \ \ t \in I_1(\varepsilon)$$

and

$$\max\{|F(s+ia_0),|G(s)|\}\ll\varepsilon^{\alpha\beta}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\quad for\quad t\in I_2(\varepsilon).$$

Arguing as in (2.1) shows

$$\sum_{n\le x}f(n)=A_x\sum_{n\le x}g(n)n^{ia_0}+o\left(\sum_{n\le x}g(n)\right).\tag{2.3}$$

Now the assertion of Proposition 1.1 can be written as ([5], (2.1))

$$M^*(x):=\sum_{n\le x}g(n)\sim\alpha x(\log x)^{\alpha-1}L^*(\log x)$$

where the function $L^*$ is slowly oscillating. An integration by parts gives a representation

$$\sum_{n\le x}g(n)n^{ia_0}=x^{ia_0}M^*(x)-ia_0\int_2^x u^{ia_0-1}M^*(u)du=$$

$$=\alpha x^{1+ia_0}(\log x)^{\alpha-1}L^*(\log x)-\alpha ia_0\int_2^x u^{ia_0}(\log u)^{\alpha-1}L^*(\log u)du$$

provided $x$ is not integer. Then, within an acceptable error, $L^*(\log u)$, for $x^\varepsilon\le u\le x$, may be replaced by $L^*(\log x)$ and factored out of the representation:

$$\begin{aligned}\sum_{n\le x}g(n)n^{ia_0}&=\alpha L^*(\log x)\{x^{1+ia_0}(\log x)^{\alpha-1}-ia_0\int_2^x u^{ia_0}(\log u)^{\alpha-1}du\}\\&=\alpha L^*(\log x)\frac{x^{ia_0}}{1+ia_0}x(\log x)^{\alpha-1}+o(M^*(x))\\&=\frac{x^{ia_0}}{1+ia_0}M^*(x)+o(M^*(x))\end{aligned}$$

which, by (2.3), ends the proof of Proposition 1.2. ■

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