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COMPARING THE PRIME NUMBER THEOREM AND THE SUM OF THE MOEBIUS FUNCTION

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Abstract. In classical analytic number theory, the Prime Number Theorem and the asymptotic estimate for the sum of the Moebius function can easily be deduced from one another. We show that the analogues of these relations do not necessarily imply one another in Beurling generalized number systems, and we give an explanation for this different behavior.

1. Introduction

The Prime Number Theorem (PNT) is the assertion that $\pi(x)$, the number of primes not exceeding x, is asymptotic to $x/\log x$, i.e.

$$\pi(x) \sim x/\log x\,, \quad x \to \infty\,. \tag{1.1}$$

It is a familiar result of analytic number theory that the PNT is "equivalent" to the proposition

$$\sum_{n\le x} \mu(n) = o(x)\,. \tag{1.2}$$

Here μ is the Moebius function and $f = o(g)$ means that $f(x)/g(x) \to 0$ as $x \to \infty$. Also, we write $f = O(g)$ or $f \ll g$ if $f(x)/g(x)$ is bounded. We use the word "equivalent" in the informal sense that each of these relations can be deduced from the other by quite simple real variable arguments.

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We shall survey when these relations imply one another in the universe of Beurling generalized (g-) numbers [7], [9]. A Beurling g-number system is defined as the abelian semigroup $\mathcal{N}$ generated by a sequence $\mathcal{P}$ of real numbers (g-primes) with $1 < p_1 \leq p_2 \leq p_3 \leq \ldots$ and $p_i \to \infty$. While the rational integers have both an additive and a multiplicative structure, g-integers are only multiplicative. Beurling theory seeks to draw conclusions about the distribution of $\mathcal{P}$ (resp. $\mathcal{N}$) from (weak!) hypotheses on $\mathcal{N}$ (resp. $\mathcal{P}$).

The g-number analogues of (1.1) *and* (1.2) *are in general not equivalent*: under mild conditions, (1.2) follows from (1.1), but the converse is false without further conditions. We shall study these implications and examine why one relation does or does not imply the other.

By analogy with the classical case, write $\pi(x) = \pi_{\mathcal{P}}(x)$ for the counting function of the g-primes. The counting function of $\mathcal{N}$ is

$$N(x) = N_{\mathcal{P}}(x) := \#\{n_i \leq x : n_i \in \mathcal{N}_{\mathcal{P}}\}$$

with appropriate multiplicity. For example, the g-number system created by adjoining $\sqrt{2}$ to the rational primes has g-integers

$$1,\ \sqrt{2},\ 2,\ \sqrt{2}^2,\ 2\sqrt{2},\ \sqrt{2}^3,\ 3,\ 2^2,\ 2\sqrt{2}^2,\ \sqrt{2}^4,\ 3\sqrt{2},\ 5,\ \ldots$$

and $N(5) = 12$. The PNT for g-numbers also is expressed by (1.1).

To stay reasonably close to the rational integers, *we shall always assume that $N(x) = O(x)$ and usually that $N(x) \sim cx$ for some positive number c as $x \to \infty$.* In the first case, we say $\mathcal{N}$ has *O-density* and in the second, that it has *positive density c*. (It is best not to require that $c = 1$ in the positive density case, for as simple a g-number system as the odd rational integers has $N(x) \sim x/2$, i.e. density $1/2$.) We shall make further hypotheses on $N(x) - cx$ when it is appropriate.

Unless we state otherwise, each of our statements is valid for g-numbers as well as their rational counterparts.

2. Technical matters

Here we collect some notation and several relations that we shall need later.

We shall use *multiplicative convolution*, a commutative and associative multiplication of measures that arise from right-continuous Borel functions that are of bounded variation on every interval $[1, X]$ (cf. [1, §3.4.1], [7, §2.6]). Convolution is defined on a Borel set E by

$$\int_E dF * dG := \iint_{uv \in E} dF(u)\, dG(v). \tag{2.1}$$

For $E = [1, x]$, the recipe takes the form

$$\int_{1-}^{x} dF * dG = \iint_{st \le x} dF(s)\, dG(t) = \int_{1-}^{x} F(x/t)\, dG(t) = \int_{1-}^{x} G(x/t)\, dF(t).$$

The Mellin transform of a measure dF associated with a function F of polynomial growth at infinity is defined on some halfplane of $\mathbb{C}$ by

$$\widehat{F}(s) := \int_{1-}^{\infty} x^{-s} dF(x).$$

The Mellin transform of a convolution is a product of Mellin transforms:

$$\int_{1-}^{\infty} x^{-s}(dF * dG)(x) = \int_{1-}^{\infty} x^{-s} dF(x) \cdot \int_{1-}^{\infty} x^{-s} dG(x)\,.$$

As usual in prime number theory, a Chebyshev function is important for our analysis. Set

$$\psi(x) = \psi_{\mathcal{P}}(x) := \sum_{p_i^{\alpha} \le x} \log p_i,$$

with $p_i \in \mathcal{P}$ (again with appropriate multiplicity) and α running over nonnegative integers. G-numbers satisfy the analogue of Chebyshev's elementary prime number identity: in measure language Chebyshev's identity can be expressed as

$$\mathrm{L}dN = d\psi * dN. \tag{2.2}$$

Here $\mathrm{L} = \mathrm{L}(t)$ is the operator of multiplying by $\log t$. It is easy to show that L is a derivation: a linear operator satisfying $\mathrm{L}(dF * dG) = (\mathrm{L}dF) * dG + dF * (\mathrm{L}dG)$.

Instead of expressing the PNT by (1.1), we shall use the (easily proved) equivalent relation

$$\psi(x) \sim x. \tag{2.3}$$

And in place of the Moebius μ function we use the measure dM, defined as the convolution inverse of dN, i.e. the solution of the equation $dM * dN = \delta$. The measure δ is point mass 1 at 1, the unity element of convolution. Instead of (1.2) we shall study whether $M(x) = o(x)$ holds.

The following is a Moebius variant of the Chebyshev identity (2.2) that will be of use presently. Since L is a derivation,

$$0 = \mathrm{L}\delta = \mathrm{L}(dN * dM) = dM * \mathrm{L}dN + dN * \mathrm{L}dM.$$

Now (2.2) implies that $dM * \mathrm{L}dN = d\psi$. Thus $dN * \mathrm{L}dM = -d\psi$, and convolving each side of the last identity with dM yields

$$\mathrm{L}dM = -d\psi * dM. \tag{2.4}$$

The Chebyshev identity and this M-variant are valid in the g-number versions as well as in the classical case.

We define the analogue of the classical weighted prime counting function by

$$\Pi(x) := \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots . \tag{2.5}$$

Note that $\mathrm{L}d\Pi = d\psi$. The next formula expresses N in terms of Π.

Lemma 2.1. *Assuming* (2.4), *we have*

$$dN = \delta + d\Pi + \frac{1}{2!}d\Pi^{*2} + \frac{1}{3!}d\Pi^{*3} + \cdots =: \exp^* d\Pi. \tag{2.6}$$

Here $d\Pi^{*n}$ *denotes the* n*-fold convolution of* $d\Pi$ *with itself, and the associated function* N *converges in total variation on every interval* $[1, X]$.

Proof. Since L is a derivation, for every positive integer n we have

$$\mathrm{L}d\Pi^{*n} = n\, d\Pi^{*n-1} * \mathrm{L}d\Pi = n\, d\Pi^{*n-1} * d\psi.$$

Applying L to $\exp^* d\Pi$ termwise we find

$$\mathrm{L}\exp^* d\Pi = d\psi * \exp^* d\Pi.$$

Let $d\phi := dM * \exp^* d\Pi$; then

$$\mathrm{L}d\phi = (\mathrm{L}dM) * \exp^* d\Pi + dM * d\psi * \exp^* d\Pi.$$

If we use (2.4) to replace $\mathrm{L}dM$, we find that $\mathrm{L}d\phi = 0$. Since L is multiplication by $\log t$, we must have $d\phi(t) = 0$ for all $t > 1$. Also, $d\phi\{1\} = 1$, so $d\phi = \delta$. Now convolve each side of the formula for $d\phi$ by dN to obtain (2.6).

The convergence claim follows immediately from the tail estimate

$$\begin{aligned}\sum_{n\ge K}\int_1^X \frac{1}{n!}\, d\Pi^{*n} &= \sum_{n\ge K}\frac{1}{n!}\int\limits_{x_1\cdots x_n\le X}\cdots\int d\Pi(x_1)\cdots d\Pi(x_n)\\ &\le \sum_{n\ge K}\frac{1}{n!}\Big\{\int_1^X d\Pi\Big\}^n = o(1), \quad K\to\infty.\end{aligned}$$

■

Similar reasoning shows that

$$dM = \exp^*\{-d\Pi\}. \tag{2.7}$$

As is shown in [7, Ch. 3], $\exp^*$ has the expected properties of an exponential, e.g. $\exp^*(dF + dG) = \exp^* dF * \exp^* dG$.

3. The PNT and O-density imply $M(x) = o(x)$

Proposition 3.1. *Assume $\mathcal{N}$ is a g-number system that satisfies the O-density condition $N(x) \ll x$ and suppose the PNT holds. Then*

$$M(x) = o(x). \tag{3.1}$$

Proposition 3.1 has a couple of cousins, each concluding $M(x) = o(x)$, but with incomparable hypotheses. One of these [7, Prop. 14.10] assumes that the PNT holds and that the g-integers satisfy the logarithmic density condition

$$\int_{1-}^{x} t^{-1}\, dN(t) \sim c \log x$$

for some positive c. Another, [4, Cor. 1.2], assumes that the g-integer system has density (a more demanding condition than O-density) and, in place of the PNT, the weaker condition that the generalized Chebyshev function satisfies the upper bound $\psi(x) = O(x)$. The present result comes from [3, §2]. Its proof is based on the following approximate formula for M.

Lemma 3.1. *Under the hypotheses of Proposition 3.1,*

$$\frac{M(x)}{x} = \frac{-1}{\log x} \int_{1}^{x} \frac{M(t)}{t^2}\, dt + o(1). \tag{3.2}$$

Proof of the Lemma 3.1. We integrate each side of the identity $\mathrm{L}dM = -d\psi * dM$ and obtain

$$\int_{1-}^{x} \mathrm{L}dM = \int_{1-}^{x} -\psi\Big(\frac{x}{t}\Big)\, dM(t).$$

If we add and subtract

$$\int_{1-}^{x} \frac{x}{t}\, dM(t)$$

to the right side of this formula, we obtain

$$\int_{1-}^{x} \mathrm{L}dM = \int_{1-}^{x} \Big\{\frac{x}{t} - \psi\Big(\frac{x}{t}\Big)\Big\}\, dM(t) - \int_{1-}^{x} \frac{x}{t}\, dM(t) \tag{3.3}$$

Because

$$|dM| = |\exp^*\{-d\Pi\}| \le \exp^*\{d\Pi\} = dN \tag{3.4}$$

and $M(1) = N(1)$, we have $|M(t)| \le N(t) = O(t)$ for all $t \ge 1$. Integrating by parts the left side of (3.3) yields

$$M(x)\log x - \int_{1}^{x} \frac{M(t)}{t}\, dt = M(x)\log x + O(x).$$

By the PNT, the first term on the right side of (3.3) is $\int_{1-}^{x} o(x/t)\, dN(t) = o(x\log x)$. Also, the last term of (3.3) is, upon integrating by parts, $M(x) + x\int_1^x \frac{M(t)}{t^2}\, dt$.

Finally, we combine the estimates for (3.3), divide through by $x\log x$, and note that $M(x)/(x\log x) = o(1)$ to obtain (3.2). ■

Proof of Proposition 3.1. We can assume that $M(x)$ has an infinite number of sign changes. If not, there exists a number z beyond which $M(x)$ is of one sign. By (3.2), as $x \to \infty$,

$$\frac{M(x)}{x} = \frac{-1}{\log x}\int_{1}^{z} \frac{O(t)}{t^2}\, dt - \frac{1}{\log x}\int_{z}^{x} \frac{M(t)}{t^2}\, dt + o(1)$$

or

$$\frac{M(x)}{x} + \frac{1}{\log x}\int_{z}^{x} \frac{M(t)}{t^2}\, dt = \frac{O(\log z)}{\log x} + o(1) = o(1).$$

Since $M(x)$ and the integral are of the same sign, $M(x)/x \to 0$ as $x \to \infty$, and this case is done.

Now suppose that the right-continuous function M changes sign at x. We show that $M(x)/x = o(1)$. If $M(x)/x = 0$, the matter is finished. If, however, $M(x) > 0$, then there is a number $y \in (x-1,\ x)$ with $M(y) \le 0$. Applying

(3.2) again, along with the trivial bound $M(t) = O(t)$, we find

$$\begin{aligned}\frac{M(x)}{x} &= \frac{-\log y}{\log x}\frac{1}{\log y}\int_1^y \frac{M(t)}{t^2}\,dt - \frac{1}{\log x}\int_y^x \frac{M(t)}{t^2}\,dt + o(1)\\ &= \frac{\log y}{\log x}\frac{M(y)}{y} + \frac{1}{\log x}\int_y^x \frac{O(t)\,dt}{t^2} + o(1).\end{aligned}$$

Thus $M(x)/x - M(y)/y = o(1)$; since $M(x)/x$ and $-M(y)/y$ is each nonnegative, each is $o(1)$. The case $M(x) < 0$ is handled similarly.

Finally, suppose that M changes sign at y (so that $M(y)/y = o(1)$) and $M(x)$ is of one sign for $y < x \le z$. We shall show that $M(x)/x = o(1)$ throughout this interval. Using (3.2) again, we find for any $x \in (y, z]$

$$\frac{M(x)}{x} = \frac{-\log y}{\log x}\frac{1}{\log y}\int_1^y \frac{M(t)}{t^2}\,dt - \frac{1}{\log x}\int_y^x \frac{M(t)}{t^2}\,dt + o(1)$$

or

$$\frac{M(x)}{x} + \frac{1}{\log x}\int_y^x \frac{M(t)}{t^2}\,dt = \frac{\log y}{\log x}\frac{M(y)}{y} + o(1) = o(1)$$

as $y \to \infty$. Now $M(x)/x$ and the integral have the same sign, so $M(x)/x = = o(1)$. ■

4. Density and $M(x) = o(x)$ do not imply the PNT

In this section we give an example, derived from one of Beurling [2], exhibiting the properties of the title. Our account is based on material from [7, Prop. 14.12].

Theorem 4.1. *The conditions $M(x) = o(x)$ and*

$$N(x) = Ax + O(x(\log ex)^{-3/2}) \tag{4.1}$$

do not imply the PNT.

Note that (4.1) is a more demanding condition on a g-number system than just having a density.

In addition to giving conditions that are not sufficient to yield the PNT, the analysis of this example also shows a key difference between the PNT and the estimate of M. This difference will be the topic of Section 6.

At this point, this article takes an unusual turn: in place of discrete primes, we use a *continuous prime density distribution*. One of the main reasons for using measures is that they offer a relatively clean way of treating the present example. The details are still quite technical, and we shall give only an outline. The continuous example can be converted into a discrete one (cf. [4], [5]); here too, we content ourselves with a brief indication of how this is done.

Define a "wobbly g-prime counting function" π_w on $[1, \infty)$ by setting

$$\pi_w(x) := \int_1^x \frac{1-\cos(\log t)}{\log t}\, dt. \tag{4.2}$$

The integrand is asymptotic to $(\log t)/2$ as $t \to 1+$, so the integral is convergent. We see that $\pi_w(1) = 0$ and $\pi_w \uparrow$; $\pi_w(x)$ has the desired properies for a (continuous) prime counting function, and we show that the "g-integer" system it generates satisfies (4.1) and $M_w(x) = o(x)$.

We establish the proposition via several lemmas, beginning with the punchline. The estimate we give for $\pi_w(x)$ will appear also in the next section.

Lemma 4.1. *The "prime counting" function $\pi_w(x)$ does not satisfy the PNT.*

Proof. Integrating (4.2) twice by parts, we find

$$\pi_w(x) = \frac{x}{\log x} - \frac{x}{2\log x}\{\sin(\log x) + \cos(\log x)\} + O\Big(\frac{x}{\log^2 x}\Big).$$

Then a trigonometric identity shows

$$\frac{\pi_w(x)}{x/\log x} = 1 - \frac{\sin(\pi/4 + \log x)}{\sqrt{2}} + o(1).$$

Thus the PNT does not hold for π_w.

Define Π_w, the analogue of the classical weighted prime counting function by

$$\Pi_w(x) := \pi_w(x) + \Big\{\frac{1}{2}\pi_w(x^{1/2}) + \frac{1}{3}\pi_w(x^{1/3}) + \dots\Big\} =: \pi_w(x) + \Pi_2(x). \tag{4.3}$$

By Lemma 2.1, $d\Pi_w$ is one term in $\exp^* d\Pi_w$. Since N_w satisfies the O-density condition, we have $\pi_w(x) = O(x)$. Also, a small calculation shows that $\Pi_2(x) \ll \ll \pi_w(x^{1/2}) \ll x^{1/2}$. Thus the main contribution to the estimates of N_w and M_w will come from $\pi_w(x)$, and we treat $\Pi_2(x)$ as a perturbation term.

The convolution exponential has the same homomorphic property as the usual exponential [7, Prop. 3.10]. Thus we have

$$dN_w = \exp^*(d\pi_w + d\Pi_2) = \exp^* d\pi_w * \exp^* d\Pi_2 =: dN_1 * dN_2,$$

a convolution of a main term, dN_1, and a secondary term, dN_2. We separately analyze the two parts of this formula. Then we shall establish (4.1) by showing that $N_w(x)$ keeps the form of its bigger convolution factor, $N_1(x)$. Finally, we shall show $M(x) = o(x)$. ■

Lemma 4.2. *With a suitable constant c_0, we have*

$$N_1(x) := \int_{1-}^{x} \exp^* d\pi_w = x + \frac{c_0 x \cos(\log x)}{\log^{3/2} x} + O\Big(\frac{x}{\log^{5/2} x}\Big).$$

Lemma 4.3.

$$N_2(x) := \int_{1-}^{x} \exp^* d\Pi_2 \ll x^{1/2}.$$

Lemma 4.4. *Let π_w be defined by (4.2) and let Π_w be the weighted g-prime function defined by (4.3). The associated g-integer counting function*

$$N_w(x) := \int_{1-}^{x} \exp^* d\Pi_w$$

satisfies (4.1)

Proof of Lemma 4.2. Let

$$\widehat{\pi_w}(s) := \int_1^\infty x^{-s} d\pi_w(x) = \int_1^\infty x^{-s}\Big\{\frac{1-\cos(\log x)}{\log x}\Big\}\,dx =$$
$$= \int_1^\infty x^{-s}\Big\{\frac{1-(x^i + x^{-i})/2}{\log x}\Big\}\,dx\,.$$

Then

$$\widehat{\pi_w}'(s) = -\int_1^\infty x^{-s}\Big\{1 - \frac{x^i + x^{-i}}{2}\Big\}\,dx = -\frac{1}{s-1} + \frac{1/2}{s-1-i} + \frac{1/2}{s-1+i}\,,$$

and since $\widehat{\pi_w}(s)$ and the log expression below both vanish as $s \to +\infty$, we have

$$\widehat{\pi_w}(s) = \frac{\log(s-1-i) + \log(s-1+i)}{2} - \log(s-1) =$$
$$= \log\left\{\frac{\sqrt{(s-1-i)(s-1+i)}}{s-1}\right\} = \log\left\{\frac{\sqrt{(s-1)^2+1}}{s-1}\right\}.$$

Using the branch of the square root that is positive for s real and $s > 1$, we find

$$\widehat{N_1}(s) = \exp \int_1^\infty x^{-s} d\pi_w(x) = \frac{\sqrt{(s-1)^2+1}}{s-1}\,. \tag{4.4}$$

By the Mellin inversion formula, for any $x > 1$ and $a > 1$,

$$N_1(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s\, \widehat{N_1}(s)\, \frac{ds}{s}\,.$$

To simplify convergence considerations, introduce

$$\phi(s) := \widehat{N_1}(s) - 1 = \frac{\sqrt{(s-1)^2+1} - (s-1)}{s-1}\,.$$

Rationalizing the numerator, we obtain

$$\phi(s) = \frac{1}{(s-1)\{\sqrt{(s-1)^2+1} + (s-1)\}} \ll \frac{1}{|s|^2}\,, \quad |s| \text{ large.} \tag{4.5}$$

Now write the Mellin formula as $N_1(x) = I + 1$, say, with

$$I := \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s\, \phi(s)\, \frac{ds}{s}\,.$$

The integrand of I has singularities at 0, 1, and $1 \pm i$. We deform the contour to the vertical line $\{s : \Re s = 1/2\}$, with loops taken to avoid crossing the rays $(-\infty - i,\, 1 - i)$ and $(-\infty + i,\, 1 + i)$. By the residue theorem, the contribution to I from the pole of $\phi(s)$ at $s = 1$ is x. By the bound $\phi(s) \ll 1/|s|^2$ from (4.5) and the fact that $|x^s| = x^{1/2}$ on the vertical line, the contribution to I from the vertical portion of the new contour is $O(x^{1/2})$.

It remains to treat the loop integrals about $1 \pm i$; we sketch the calculation near $1 + i$. Write

$$\widehat{N_1}(s)/s = (s - 1 - i)^{1/2} f(s)$$

where $f(s)$ is analytic for $|s - 1 - i| \le 1/2$. We have on this disc

$$\phi(s)/s = (\widehat{N_1}(s) - 1)/s = -1/s + c_0'(s - 1 - i)^{1/2} + O(|s - 1 - i|^{3/2})$$

for c_0' a suitable constant.

The integral of $-1/s$ about the loop has the value 0 by analyticity.

The horizontal portion of the loop stretches along the "top" and "bottom" of the line segment $[1/2+i,\, 1-\epsilon+i]$, and there is a circle of radius ϵ about $1+i$. On the circle, ϕ is bounded and so the contribution here goes to 0 with ϵ.

For the upper and lower line segments, use the appropriate branches of the square root and combine the contributions of the two segments. The resulting integral yields

$$(c/2)\, x^{1+i} \log^{-3/2} x + O(x^{1/2})\,,$$

with the Euler Gamma function evaluated at 3/2 contributing to the factor $c/2$. Integrating the O-term around the loop yields an error term $O(x \log^{-5/2} x)$.

The corresponding calculation near $1-i$ yields the conjugate result: the exponent of x changes to $1-i$. Together, the two loop integrals yield the cosine term and the error term in the formula for $N_1(x)$. ■

Proof of Lemma 4.3.
We begin by estimating the contribution of $(1/2)\pi_w(x^{1/2})$, the first term of Π_2. By a weak version of Lemma 4.2,

$$N_3(x) := \int_{1-}^{x} \exp^*\Big\{\frac{1}{2} d\pi_w(t^{1/2})\Big\} = \int_{1-}^{\sqrt{x}} \exp^*\Big\{\frac{1}{2} d\pi_w\Big\} \le \int_{1-}^{\sqrt{x}} \exp^* d\pi_w = O(\sqrt{x}).$$

Now

$$N_2(x) := \int_{1-}^{x} dN_3 * \exp^*\Big\{\frac{1}{3} d\pi_w(t^{1/3}) + \frac{1}{4} d\pi_w(t^{1/4}) + \cdots\Big\} \ll$$
$$\ll \int_{1-}^{x} \sqrt{\frac{x}{t}} \exp^*\Big\{d\pi_w(t^{1/3}) + d\pi_w(t^{1/4}) + \cdots\Big\}.$$

Extend the integration to infinity and use the fact that for an absolutely convergent Mellin transform exp and $\int$ "commute": we have

$$\int_{1-}^{\infty} t^{-s} \exp^* dF(t) = \int_{1-}^{\infty} t^{-s} \sum_{n=0}^{\infty} \frac{1}{n!}\, dF^{*n}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{1-}^{\infty} t^{-s} dF^{*n}(t) =$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ \int_{1-}^{\infty} t^{-s} dF(t) \Big\}^n = \exp\Big\{ \int_{1-}^{\infty} t^{-s} dF(t) \Big\}.$$

We find that

$$N_2(x) \ll \sqrt{x} \int_{1-}^{\infty} t^{-1/2} \exp^*\Big\{d\pi_w(t^{1/3}) + d\pi_w(t^{1/4}) + \cdots\Big\} =$$

$$= \sqrt{x} \exp \int_{1}^{\infty} t^{-1/2}\Big\{d\pi_w(t^{1/3}) + d\pi_w(t^{1/4}) + \cdots\Big\} =$$

$$= \sqrt{x} \exp\left\{ \int_{1}^{\infty} t^{-3/2} d\pi_w(t) \right\} \cdot \exp\left\{ \int_{1}^{\infty} t^{-2} d\pi_w(t) \right\} \cdots$$

$$= \sqrt{x} \exp\left\{ \int_{1}^{\infty} \big(t^{-3/2} + t^{-2} + t^{-5/2} + \cdots\big)\, d\pi_w(t) \right\} =$$

$$= \sqrt{x} \exp\left\{ \int_{1}^{\infty} \frac{t^{-3/2}}{1 - t^{-1/2}} \, \frac{1 - \cos(\log t)}{\log t} \, dt \right\}.$$

A small calculation shows the last integrand is bounded near $t = 1$ and is $o(t^{-3/2})$ for large t. Thus the integral is finite, and so $N_2(x) = O(\sqrt{x})$. ■

We pause here to establish a formula for the zeta function of the wobbly g-number system generated by π_w. We have

$$\zeta_w(s) := \exp \int_{1}^{\infty} x^{-s}\{d\pi_w(x) + d\Pi_2(x)\} = \widehat{N_1}(s) \cdot \exp \int_{1}^{\infty} x^{-s} d\Pi_2(x).$$

Since $\Pi_2(x) \ll x^{1/2}$, the last integral converges for $\Re s > 1/2$ and defines an analytic function $g(s)$ there. Using g and (4.4), we find for $\Re s > 1$,

$$\zeta_w(s) = \frac{\sqrt{(s-1)^2 + 1}}{s - 1} \exp\{g(s)\}. \tag{4.6}$$

Proof of Lemma 4.4.

First note that since $N_2(x) \ll \sqrt{x}$, integration by parts gives

$$\int_{1-}^{y} \frac{dN_2(t)}{t} = \int_{1-}^{\infty} \frac{dN_2(t)}{t} - \int_{y}^{\infty} \frac{dN_2(t)}{t} = A + O(y^{-1/2}), \quad y \to \infty,$$

for some constant A.

Apply the Dirichlet hyperbola method, writing

$$N_w(x) = \int_{1-}^{x} dN_1 * dN_2 = I + II$$

with

$$I := \int_{1-}^{\sqrt{x}} N_1(x/t)\, dN_2(t), \qquad II := \int_{1-}^{\sqrt{x}} \{N_2(x/t) - N_2(\sqrt{x})\}\, dN_1(t).$$

Now

$$I = \int_{1-}^{\sqrt{x}} \left\{ \frac{x}{t} + O\left(\frac{x/t}{\log^{3/2} x/t} \right) \right\} dN_2(t) =$$

$$= Ax + O(x^{3/4}) + O\left(\frac{x}{\log^{3/2} x} \int_{1-}^{\sqrt{x}} t^{-1}\, dN_2(t) \right) = Ax + O\left(\frac{x}{\log^{3/2} x} \right)$$

with $A = \int_{1-}^{\infty} t^{-1}\, dN_2(t)$. Next, by another integration by parts,

$$II = \int_{1-}^{\sqrt{x}} O\big(\sqrt{x/t}\,\big)\, dN_1(t) = O\big(x^{3/4}\big)\,.$$

The two calculations together give the claimed formula for $N_w(x)$. ■

To finish proving Theorem 4.1, we show that $M(x) = o(x)$ holds for our example.

Lemma 4.5. *Suppose a g-number system has counting function*

(4.1 bis) $$N(x) = Ax + O(x(\log ex)^{-3/2})\,.$$

Then its associated Moebius sum function satisfies $M(x) = o(x)$.

Proof. On $\{s : \Re s > 1\}$ we have, by (4.6),

$$\widehat{M}(s) = \frac{1}{\zeta(s)} = \frac{s-1}{\sqrt{(s-1)^2+1}} \exp\{-g(s)\},$$

with g analytic. This function can be extended analytically to the closed half plane, except at $s = 1 \pm i$, where it has half order poles. It follows that $\widehat{M}(\sigma + it)$ has an L^1 limit on any bounded t interval as $\sigma \to 1+$.

Thus M would be amenable to estimation by the Wiener-Ikehara tauberian theorem, except that it is not monotonic. Recalling that $|dM| \le dN$ by (3.4), apply W-I to $M(x) + N(x)$, which is a nondecreasing function. The Mellin transform of this function is analytic in the closed half plane except for the order 1 pole at 1 and the half order zeros and poles at $1 \pm i$. We find $M(x)+N(x) \sim Ax$ as $x \to \infty$, and since $N(x) \sim Ax$, we conclude that $M(x) = o(x)$. ■

5. Conversion of N_w to a discrete example.

One method to create a discrete example is to define, for each positive integer r, the r^{th} g-prime by $p_r = \pi_w^{-1}(r)$, where $\pi_w(x)$ is the prime counting function of (4.2) (see [5]). Then the prime counting function satisfies $\pi(x) = \lfloor \pi_w(x) \rfloor$, and define the weighted companion, $\Pi(x)$, by (2.5).

Proposition 5.1. *There exists a* discrete *g-number system whose integer counting function satisfies*

$$N(x) = cx + O(x(\log ex)^{-3/2})$$

for which the PNT does not hold.

Remarks on the proposition. We note that $c < A$, where A is the integer density of Theorem 4.1, because $\pi(x) < \pi_w(x)$ a.e. Also, since $\pi(x) = \pi_w(x) + O(1)$, the PNT does not hold for the discrete g-prime system.

The proof of the asymptotic formula for $N(x)$ can be carried out much like that of $N_w(x)$, with the latter function serving as the main convolution factor and $\exp^*\{d\Pi - d\Pi_w\}$ as a perturbation factor.

Another method of creating a discrete g-number system from a continuous one is to choose the g-primes probabilistically. The probability of a g-prime being selected in any small interval is taken to be proportional to the size of $\int d\pi_w$ over that interval (see [6]). This method is, of course, not constructive, but it enables one to use the zeta function of the wobbly g-number system to accurately approximate the new probabilistic zeta function.

6. Why the difference?

For the rational integers, it is well-known that the truth of the PNT formula $\psi(x) \sim x$ implies that of the Moebius sum relation $M(x) = o(x)$ and vice versa. However, as we have shown, for Beurling g-integers with O-density, validity of the PNT implies $M(x) = o(x)$, but the converse need not be true.

What lies behind this difference? We give an answer to this question in terms of properties of the associated Mellin transforms. The Mellin transforms of N, M, and ψ are, respectively, in both the classical and g-number cases,

$$\zeta(s) = \int_{1-}^{\infty} x^{-s} dN(x), \quad \frac{1}{\zeta(s)} = \int_{1-}^{\infty} x^{-s} dM(x), \quad -\frac{\zeta'(s)}{\zeta(s)} = \int_{1-}^{\infty} x^{-s} d\psi(x).$$

As we noted in the proof of Lemma 4.5, we can establish the desired asymptotic relations by the Wiener-Ikehara theorem in the classical or g-number case (using the $M + N$ trick for $M(x)$), provided that the associated Mellin transform has a local L^1 limit, aside from a possible pole at $s = 1$. This observation leads to an answer to the question of this section:

A key difference between the ψ and the M cases lies in the nature of possible singularities of their associated Mellin transforms on their abscissa of convergence.

In the example of Theorem 4.1, $\zeta(s)$ has fractional order zeros at the points $s = 1 \pm i$. (This follows from (4.4); the contribution arising from the perturbation term $d\Pi_2$ does not change the behavior of the zeta function on $\{s : \Re s \geq 1\}$.) Thus $1/\zeta(s)$ has fractional order poles at $s = 1 \pm i$, and these do not bar a local L^1 limit. Therefore we can apply the W-I theorem to establish the M estimate. On the other hand, $(-\zeta'/\zeta)(s)$ has poles of order 1 at $1 \pm i$, and these induce an oscillatory term in the formula for $\psi(x)$, and so the PNT fails.

7. Other possible equivalences

One can also consider possible equivalences between the PNT, $M(x) = o(x)$, and such additional relations as

$$m(x) := \int_{1-}^{x} \frac{1}{u}\, dM(u) = o(1), \tag{7.1}$$

and the so-called sharp Mertens relation

$$\int_{1-}^{x} \frac{1}{u}\, d\psi(u) = \log x + c' + o(1), \tag{7.2}$$

for some number c'.

In the rational numbers, the Riemann zeta function has no zeros or singularities on $\{s : \Re s = 1\}$, and these relations can be shown mutually equivalent (in our informal sense). Implications between some of these in the g-number case are discussed in [7] and [8], and we make only a few further remarks here.

In the g-number context, relation (7.2) is the strongest of the four, in that it implies each of the others with no further assumptions. For example, relation (7.1) implies $M(x) = o(x)$, as one can see immediately by writing $M(x) = \int_{1-}^{x} t\, dm(t)$ and integrating by parts.

The converse implication, however, is false in the absence of further assumptions. For example, take $\mathcal{N}_e$ to be the g-number system with counting function

$$N_e(x) := \sum_{n \le x} \frac{1}{n}\,.$$

(We have $N_e(x) \ll \log x$, which is is *much* smaller than x.)

The arithmetic function $1/n$ has the convolution inverse $\mu(n)/n$, with μ the usual Moebius function. Then

$$M_e(x) = \sum_{n \le x} \frac{1}{n}\, \mu(n) = o(x)$$

(with much to spare!) but

$$m_e(x) = \sum_{n \le x} \frac{1}{n^2}\, \mu(n) = \frac{6}{\pi^2} + o(1) \ne o(1).$$

One can deduce $m(x) = o(1)$ from $M(x) = o(x)$ if $\Delta := N(x) - cx$ is suitably small, e.g. $\Delta \ll x \log^{-\gamma} x$, with $\gamma > 1$ suffices.

We conclude this article by asking whether $M(x) = o(x)$ and the weaker condition $N(x) \sim cx$ are sufficient to show $m(x) = o(1)$. The fact that M and m have similar Mellin transforms, $1/\zeta(s)$ and $1/\zeta(s+1)$, suggests that the asymptotic conditions for M and m might be connected. But there does not seem to be an easy path from the one relation to the other.

Here is one remark of possible use. The condition $N(x) \sim cx$ implies that the zeta function of N satisfies $\zeta(s) \sim c/(s-1)$ as $s \to 1+$, and thus

$$\lim_{s \to 0+} \int_{1-}^{\infty} x^{-s} dm(x) = \lim_{s \to 0+} 1/\zeta(s+1) = 0.$$

Thus, if one can prove that $m(x)$ has *some* limit as $x \to \infty$, then that limit is zero.

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