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DETERMINING A MEROMORPHIC FUNCTION BY ITS PREIMAGES OF FINITE SETS

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Abstract. We give some sufficient conditions for the zero set of a polynomial to be a unique range set for meromorphic functions, in cases of ignoring multiplicity and with m-truncated multipilicity. As consequences, we obtained some previous results of Yi ([23]).

1. Introduction. Main results

In this paper, by a meromorphic function we mean a meromorphic function on the complex plane $\mathbb{C}$.

Let f be a non-constant meromorphic function on $\mathbb{C}$. For every $a \in \mathbb{C}$, we define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity } d, \end{cases}$$

and set $\nu_f^\infty = \nu_{\frac{1}{f}}^0$. Define the function $\overline{\nu}_f^a : \mathbb{C} \to \mathbb{N}$ by $\overline{\nu}_f^a(z) = \min \{\nu_f^a(z), 1\}$ and set $\overline{\nu}_f^\infty = \overline{\nu}_{\frac{1}{f}}^0$.

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Let m be a positive integer. For every $a \in \mathbb{C} \cup \{\infty\}$, we define the function $\nu^a_{f,m)}$ from $\mathbb{C} \cup \{\infty\}$ into $\mathbb{N}$ by

$$\nu^a_{f,m)}(z) = \begin{cases} 0 & \text{if } \nu^a_f(z) > m \\ \nu^a_f(z) & \text{if } \nu^a_f(z) \leq m, \end{cases}$$

and set $\nu^{\infty}_{f,m)} = \nu^0_{\frac{1}{f},m)}$, and define the function $\overline{\nu}^a_{f,m)}$ from $\mathbb{C} \cup \{\infty\}$ into $\mathbb{N}$ by $\overline{\nu}^a_{f,m)}(z) = \min\ \{\overline{\nu}^a_{f,m)}(z), 1\}$, and set $\overline{\nu}^{\infty}_{f,m)} = \overline{\nu}^0_{\frac{1}{f},m)}$.

We denote by $\mathcal{M}(\mathbb{C})$ the field of meromorphic functions in $\mathbb{C}$. For $f \in \mathcal{M}(\mathbb{C})$ and $S \subset \mathbb{C} \cup \{\infty\}$, $S \neq \emptyset$, we define *the preimage of S counting multiplicity* by

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu^a_f(z)) : z \in \mathbb{C}\},$$

and *the preimage of S ignoring multiplicity* by

$$\overline{E}_f(S) = \bigcup_{a \in S} \{(z, \overline{\nu}^a_f(z)) : z \in \mathbb{C}\}.$$

Furthermore, we define *the preimage of S counting multiplicity with m-truncated multiplicity* by

$$E_{f,m)}(S) = \bigcup_{a \in S} \{(z, \nu^a_{f,m)}(z)) : z \in \mathbb{C}\},$$

and by a similar manner,

$$\overline{E}_{f,m)}(S) = \bigcup_{a \in S} \{(z, \overline{\nu}^a_{f,m)}(z)) : z \in \mathbb{C}\}.$$

Note that $E_{f,1)}(S) = \overline{E}_{f,1)}(S)$ and $E_{f,1)}(S) \subset \overline{E}_f(S)$.

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{C})$ and let a set $S \subset \mathbb{C} \cup \{\infty\}$. Two functions f, g of $\mathcal{F}$ are said to *share S, counting multiplicity* (share S CM) if $E_f(S) = = E_g(S)$, to *share S, ignoring multiplicity* (share S IM) if $\overline{E}_f(S) = \overline{E}_g(S)$, and to *share S, counting multiplicity with m-truncated multiplicity* (share $S_{m)}$ CM) if $E_{f,m)}(S) = E_{g,m)}(S)$, and to *share S, ignoring multiplicity with m-truncated multiplicity* (share $S_{m)}$ IM) if $\overline{E}_{f,m)}(S) = \overline{E}_{g,m)}(S)$. Let f, g be two non-constant meromorphic (entire) functions. If the condition $E_f(S) = E_g(S)$ (resp., $\overline{E}_f(S) = \overline{E}_g(S)$) implies $f = g$ for any two non-constant meromorphic (entire) functions f, g, then S is called a unique range set counting multiplicity (resp., ignoring multiplicity) for meromorphic (entire) functions, or in brief, URSM (URSE) (resp., URSM-IM (URSE-IM)). $S_{m)}$ is called a unique range set counting multiplicity with m-truncated multiplicity (resp., ignoring multiplicity with m-truncated multiplicity) for meromorphic (entire) functions if the condition $E_{f,m)}(S) = E_{g,m)}(S)$ (resp., $\overline{E}_{f,m)}(S) = \overline{E}_{g,m)}(S)$) implies $f = g$ for any pair of non-constant meromorphic (entire) functions, or in brief, URSM$_{m)}$-CM (URSE$_{m)}$ -CM) (resp., URSM$_{m)}$-IM (URSE$_{m)}$ -IM)).

In 1976 F. Gross ([9]) proved that there exist three finite sets S_j ($j = 1, 2, 3$) such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$, $j = 1, 2, 3$ must be identical. In the same paper F. Gross posed the following question:

Question 1. *Can one find two (or possible even one) finite set S_j ($j = 1, 2$) such that any two entire functions f and g must be identical if $E_f(S_j) = E_g(S_j)$ ($j = 1, 2$)?*

H. X. Yi [18]–[20], [22], [24] first gave an affirmative answer to Question 1. Since then, many results have been obtained for this and related topics (see [2]–[14], [16],[18]–[24]).

Concerning Question 1, a natural question is the following.

Question 2. *What is the smallest cardinality for such a finite set S such that any two non-constant meromorphic functions f and g must be identical, if either $E_f(S) = E_g(S)$ or $\overline{E}_f(S) = \overline{E}_g(S)$?*

So far, the best answer to Question 2 for the case of URSM was obtained by Frank and Reinders ([6]). They proved the following result.

Theorem A. *The set $\{z \in \mathbb{C} | \ P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2} z^{n-2} - c = 0\}$, where $n \geq 11$ and $c \neq 0, 1$, is a unique range set for meromorphic functions counting multiplicity.*

In 1997, H. X. Yi ([21]) first gave an answer to Question 2 for the case of URSM-IM with 19 elements. He considered polynomials of the form

$$P_Y(z) \in \mathbb{C}[z] : \quad P_Y(z) = z^n + a^m + b,$$

where $(m, n) = 1$, $n > 2m + 14$ and $m \geq 2$ and proved that $S = \{z \in \mathbb{C} \mid P_Y(z) = 0\}$ is a URSM-IM. Bartels's Theorem ([3]) said that $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$ is a URSM-IM if $n \geq 17$. So far, the best answer to Question 2 for the case of URSM-IM was obtained by B. Chakraborty ([4]). He proved the following result.

Theorem B. *Let $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$. If $n \geq 15$, then S is a* URSM-IM.

In [1] the following new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements was given.

Let $n \in \mathbb{N}^*, n \geq 3$. Consider polynomial $P(z)$:

$$P_K(z) = z^n - \frac{2na}{n-1} z^{n-1} + \frac{na^2}{n-2} z^{n-2} + 1 = Q_K(z) + 1, \tag{1.1}$$

where $a \in \mathbb{C},\ a \neq 0$. Suppose that

$$Q_K(a) \neq -1, \quad -2 \tag{1.2}$$

Theorem C ([1]). *Let $P_K(z)$ be defined by* (1.1) *with condition* (1.2)*, and let $S = \{z \in \mathbb{C} \mid P(z) = 0\}$. If $n \geq 15$, then S is a* URSM-IM.

In 2000, H. Fujimoto established a sufficient condition for a finite subset S of $\mathbb{C}$ to be a uniqueness range set for meromorphic functions. Let us recall his result.

For a discrete subset $S = \{a_1, a_2, ..., a_n\} \subset \mathbb{C}$, we consider its generated polynomial with the following form

$$P(z) = (z - a_1)(z - a_2) \cdots (z - a_n). \tag{1.3}$$

Assume that the derivative of $P(z)$ has mutually distinct k zeros $d_1, d_2, \ldots, d_k$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively. We often consider polynomials satisfying the following condition introduced by Fujimoto [7]:

$$P(d_i) \neq P(d_j), 1 \leq i < j \leq k. \tag{1.4}$$

The number k is called the *derivative index* of $P(z)$.

A polynomial $P(z)$ is called a *strong uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, and a nonzero constant c, the condition $P(f) = cP(g)$ implies $f = g$ (see [2], [8], [12]). In this case we say $P(z)$ is a SUPM (SUPE).

Theorem D ([7]). *Let $P(z)$ be a polynomial of the form* (1.3) *satisfying the condition* (1.4)*. Suppose that $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$, and $P(z)$ is a strong uniqueness polynomial.*

1. *If $n > 2k + 6$ $(n > 2k + 2)$, then S is a* URSM (URSE).
2. *If $n > 2k + 12$ $(n > 2k + 5)$, then S is a* URSM-IM (URSE-IM).

Remark 1. Regarding theorems A, B, C, D, it is easy to see that, in the case of URSM (counting multiplicity), Theorem A is a consequence of Theorem D, since P_{FR} is a strong uniqueness polynomial of degree 8 [6, p. 191, Case 2]. However, in the case of URS-IM (ignoring multiplicity), Theorem B and Theorem C are not consequences of Theorem D, because with $k = 2$ we have $n \geq 17$.

From Remark 1, a natural question is the following.

Question 3. *Can one give some sufficient conditions for polynomial $P(z)$ such that $S = \{z \in \mathbb{C} \mid P(z) = 0\}$ is a* URSM-IM (URSE-IM) *for meromorphic (entire) functions, and then obtain Theorem B and Theorem C as consequences?*

Concerning Question 3, in 1997 H. X. Yi [23] considered $\mathrm{URSM}_{m)}$ for polynomial P_{FR} and proved the following

Theorem E. *Let $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$.*

1/ *If $n \geq 11$, then S is a* $\mathrm{URSM}_{3)}$-CM.

2/ *If $n \geq 12$ ($n \geq 7$), then S is a* $\mathrm{URSM}_{2)}$-CM ($\mathrm{URSE}_{2)}$-CM).

3/ If $n \geq 15$ ($n \geq 9$), then S is a $\mathrm{URSM}_{1)}$-CM ($\mathrm{URSE}_{1)}$-CM).

In [12] it is given a sufficient condition for a finite subset S of $\mathbb{C}$ to be a $\mathrm{URSM}_{m)}$-CM.

Theorem F. *Let $P(z)$ be a strong uniqueness polynomial of the form* (1.3) *satisfying the condition* (1.4) *with $P^{'}(z) = nz^{m_1}(z-d_2)^{m_2}\cdots(z-d_k)^{m_k}$ and $S = \{z \in \mathbb{C} \mid P(z) = 0\}$. Suppose that $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$, and all zeros and poles of f and g have multiplicity at least s, l, respectively.*

1. *If $n > 2k - 2 + \frac{4}{s} + \frac{4}{l}$ ($n > 2k - 2 + \frac{4}{s}$), then S is a* URSM (URSE) *and is a* $\mathrm{URSM}_{m)}$-CM ($\mathrm{URSE}_{m)}$-CM) *with $m \geq 3$.*

2. *If $n > 2k - \frac{3}{2} + \frac{4}{s} + \frac{9}{2l}$ ($n > 2k - \frac{3}{2} + \frac{4}{s}$), then S is a* $\mathrm{URSM}_{2)}$-CM ($\mathrm{URSE}_{2)}$-CM)

3. *If $n > 2k + \frac{4}{s} + \frac{6}{l}$ ($n > 2k + \frac{4}{s}$), then S is a* $\mathrm{URSM}_{1)}$-CM ($\mathrm{URSE}_{1)}$-CM).

Remark 2. Regarding Theorems E, F, it is easy to see that, in the case of $\mathrm{URSM}_{m)}$, Theorem E is a consequence of Theorem F, since P_{FR} is a strong uniqueness polynomial of degree 8 [6, p. 191, Case 2] and by taking $k = 2$, $s = 1, l = 1$ in Theorem F.

Note that $E_{f,1)}(S) = \overline{E}_{f,1)}(S)$, $E_{f,1)}(S) \subset \overline{E}_f(S)$ and $P^{'}_{FR}(z)$ and $P^{'}_{K}(z)$ both have a zero at 0 with higher multiplicities:

$$P^{'}_{FR}(z) = \frac{n(n-1)(n-2)}{2} z^{n-3}(z-1)^2, \quad P^{'}_{K}(z) = nz^{n-3}(z-a)^2.$$

These facts and Remark 1, Remark 2 suggest us to consider polynomial $P(z)$ with $P^{'}(z) = nz^{m_1}(z-d_2)^{m_2}...(z-d_k)^{m_k}$.

We give some sufficient conditions for polynomial $P(z)$ such that $S = \{z \in \in \mathbb{C} \mid P(z) = 0\}$ is a uniqueness range set for the cases of ignoring multiplicity (URSM-IM and URSE-IM) and of ignoring multiplicity with m-truncated multiplicity ($\mathrm{URSM}_{m)}$-IM and $\mathrm{URSE}_{m)}$-IM). As consequences, we obtain Theorem

D in the case of URSM-IM and construct new uniqueness range sets for the cases of URSM-IM and URSE-IM, and of URSM$_{m)}$-IM and URSE$_{m)}$-IM. In particular, we obtain again Theorem B and Theorem C.

Now let us describe main results of the paper. We first give a sufficient condition for a finite subset S of $\mathbb{C}$ to be a uniqueness range set for the cases of URSM-IM and URSE-IM and of URSM$_{m)}$-IM and URSE$_{m)}$-IM.

Theorem 1. *Let $P(z)$ be a strong uniqueness polynomial of the form* (1.3) *satisfying the condition* (1.4) *with $P'(z) = nz^{m_1}(z-d_2)^{m_2}\cdots(z-d_k)^{m_k}$ and $S = \{z \in \mathbb{C} \mid P(z) = 0\}$, and let m be a positive integer. Suppose $k \geq 3$, or $k = 2$ and* $\min\{m_1, m_2\} \geq 2$.

If $n > 2k+10$ ($n > 2k+4$), then S is a URSM-IM (URSE-IM) *and* URSM$_{m)}$-IM (URSE$_{m)}$-IM)*.*

Corollary 2. *Theorem* 1 *implies Theorem D in the case of* URSM-IM (URSE-IM)*.*

Indeed, suppose that $P(z)$ with $P'(z) = n(z-d_1)^{m_1}(z-d_2)^{m_2}\cdots(z-d_k)^{m_k}$ satisfies the conditions of Theorem D in the case of URSM-IM. Write

$$P(z) = (z-d_1)^n + b_1(z-d_1)^{n-1} + \cdots + b_{n-1}z + b_0,$$

and set

$$R(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_0, \quad t_i = d_i - d_1, \quad i = 1, \ldots, k,$$

and $T = \{z \in \mathbb{C} \mid R(z) = 0\}$. Then

$$P(z) = R(z-d_1),\ R'(z) = nz^{m_1}(z-t_2)^{m_2}\cdots(z-t_k)^{m_k}.$$

Since $P(z)$ is a strong uniqueness polynomial and $\overline{E}_f(S) = \overline{E}_g(S)$, we see that $R(z)$ is a strong uniqueness polynomial of the form (1.3) satisfying the condition (1.4) with $R'(z) = nz^{m_1}(z-t_2)^{m_2}\cdots(z-t_k)^{m_k}$ and $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$ and $\overline{E}_f(T) = \overline{E}_g(T)$. Then, applying Theorem 1, we conclude that if $n > 2k+10$ ($n > 2k+4$), then T is a URSM-IM (URSE-IM). Therefore, S is a URSM-IM (URSE-IM) if $n > 2k+10$($n > 2k+4$).

As a consequence of Theorem 1, we construct following new uniqueness range sets, which are URSM-IM (URSE-IM) and URSM$_{m)}$-IM (URSE$_{m)}$-IM).

Let l, p be positive integers, and let $a \in \mathbb{C}$ be a nonzero constant. Set

$$(1.5) \qquad P(z) = (l+p+1)\left(\sum_{i=0}^{p}\binom{p}{i}\frac{(-1)^i}{l+p+1-i}a^i z^{l+p+1-i}\right) + 1,$$

For simplicity, we set $n = l + p + 1$ and

$$Q(z) = (l + p + 1)\Big(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l + p + 1 - i} a^i z^{l+p+1-i}\Big).$$

Then $P(z) = Q(z) + 1$. Suppose that

$$Q(a) \neq -1, \neq -2. \tag{1.6}$$

Note that $P(z)$, defined by (1.5) with condition (1.6), is a polynomial of degree $n = l + p + 1$ having no multiple zeros.

So, $P^{'}(z) = nz^l(z - a)^p$ has a zero at 0 of order l.

Note that polynomials of the form (1.6) were investigated in [2] and [12].

Then we prove the following

Theorem 2. *Let $P(z)$ be defined by* (1.5) *with conditions* (1.6) *and let $S = \{z \in \mathbb{C} \mid P(z) = 0\}$. If $n \geq 15$ ($n \geq 7$), then S is a* URSM-IM (URSE-IM) *and* URSM$_{m)}$-IM (URSE$_{m)}$-IM).

Remark 3. By using Theorem 1, we can construct uniqueness range sets for case $k \geq 2$. In paticular, we obtain Yi's Theorem in [21], Bartels's Theorem in [3], Theorem B, and Theorem C, by taking P_Y, P_{FR}, P_K to be strong uniqueness polynomials, respectively.

2. Lemmas and definitions

We assume that the reader is familiar with the notations of the Nevanlinna theory (see, for example, [5], [15]). We need some lemmas.

Lemma 2.1. ([5, p. 98], [15, p. 43]) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, \ldots, a_q$ be distinct points of $\mathbb{C}$. Then*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) - N_0(r, \frac{1}{f'}) + S(r, f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of function $(f - a_1)\cdots(f - a_q)$, and $S(r, f) = o(T(r, f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.2. ([17, Lemma 3]) *For any non-constant meromorphic function f,*

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r, f) + S(r, f).$$

Definition. Let f be a non-constant meromorphic function, and k be a positive integer. For simplicity, we set $\nu_f(z) = \nu_f^0(z)$ and denote by $\overline{N}_{(k}(r, f)$ (resp., $\overline{N}_{(k}(r, \frac{1}{f}))$ the counting function of the poles (resp., zeros of f with $\nu_f(z) \geq k$) with $\nu_f^\infty \geq k$, where each pole (zero) is counted only once. We also denote by $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ the counting function of the zeros z of f' satisfying $f(z) \neq 0$, where each zero is counted only once.

Let be given two non-constant meromorphic functions f and g. For simplicity, denote by $\nu_1(z) = \nu_f(z)$ (resp., $\nu_2(z) = \nu_g(z)$), if z is a zero of f (resp., g). Let $\overline{E}_f(0) = \overline{E}_g(0)$. We denote by $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1)$ (resp., $\overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2)$) the counting function of the common zeros z, satisfying $\nu_1(z) = \nu_2(z) = 1$ (resp., $\nu_1(z) > \nu_2(z) \geq 1$), where each zero is counted only once) and by $N(r, \frac{1}{f}; \nu_1 \geq 2)$ the counting function of the zeros z of f, satisfying $\nu_1(z) \geq 2$. Similarly, we define the counting functions $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1)$, $\quad N(r, \frac{1}{g}; \nu_2 \geq 2)$.

Let m be a positive integer and $\overline{E}_{f,m)}(0) = \overline{E}_{g,m)}(0)$.

We denote by $\overline{N}(r, \frac{1}{f}; m \geq \nu_1 > \nu_2)$ (resp., $\overline{N}(r, \frac{1}{g}; m \geq \nu_2 > \nu_1)$ the counting function of the common zeros z, satisfying $m \geq \nu_1(z) > \nu_2(z) \geq 1$ (resp., $m \geq \nu_2(z) > \nu_1(z) \geq 1$), where each zero is counted only once and by $N(r, \frac{1}{f}; m \geq \nu_1 \geq 2)$ (resp., $N(r, \frac{1}{g}; \nu_2 \geq 2)$) the counting function of the zeros z of f, satisfying $m \geq \nu_1(z) \geq 2$ (resp., $m \geq \nu_2(z) \geq 2$).

Lemma 2.3. *Let f, g be two non-constant meromorphic functions. Set*

$$F = \frac{1}{f}, \quad G = \frac{1}{g}, \quad L = \frac{F''}{F'} - \frac{G''}{G'}, \quad S(r) = S(r, f) + S(r, g).$$

Suppose that $L \not\equiv 0$.

1) *If $E_{f,1)}(0) = E_{g,1)}(0)$, then*

$$\text{i)} \quad \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) \leq N(r, L) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + S(r).$$

Moreover, if a is a common simple zero of f and g, then $L(a) = 0$.

$$\text{ii)} \quad N(r, L) \leq \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}_{(2}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{g}) +$$
$$+ \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0).$$

2) *If* $\overline{E}_{f,m)}(0)=\overline{E}_{g,m)}(0),\ m\geq 1,$ *then*

i) $$\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{f};m\geq\nu_1>\nu_2)+$$
$$+\overline{N}(r,\frac{1}{g};m\geq\nu_2>\nu_1)\leq N(r,L)+\frac{1}{2}(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+$$
$$+N(r,\frac{1}{f};m\geq\nu_1\geq 2)+N(r,\frac{1}{g};m\geq\nu_2\geq 2)+S(r).$$

ii) $$N(r,L)\leq\overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+\overline{N}(r,\frac{1}{f};m\geq\nu_1>\nu_2)+$$
$$+\overline{N}(r,\frac{1}{g};m\geq\nu_2>\nu_1)+\overline{N}_{(m+1}(r,\frac{1}{f})+\overline{N}_{(m+1}(r,\frac{1}{g})+$$
$$+\overline{N}(r,\frac{1}{f'};f\neq 0)+\overline{N}(r,\frac{1}{g'};g\neq 0).$$

iii) $$\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})\leq\overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+$$
$$+\frac{1}{2}(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+N(r,\frac{1}{f};\nu_1\geq 2)+N(r,\frac{1}{g};\nu_2\geq 2)+$$
$$+\overline{N}(r,\frac{1}{f'};f\neq 0)+\overline{N}(r,\frac{1}{g'};g\neq 0)+S(r).$$

3) *If* $\overline{E}_f(0)=\overline{E}_g(0),$ *then* (see: [13, Lemma 2.4], [14, Lemma 2.2], [1, Lemma 2.3])

i) $$N(r,L)\leq\overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+\overline{N}(r,\frac{1}{f};\nu_1>\nu_2)+$$
$$+\overline{N}(r,\frac{1}{g};\nu_2>\nu_1)+\overline{N}(r,\frac{1}{f'};f\neq 0)+\overline{N}(r,\frac{1}{g'};g\neq 0).$$

ii) $$\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{f};\nu_1>\nu_2)+\overline{N}(r,\frac{1}{g};\nu_2>\nu_1)\leq N(r,L)+$$
$$+\frac{1}{2}(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+N(r,\frac{1}{f};\nu_1\geq 2)+$$
$$+N(r,\frac{1}{g};\nu_2\geq 2)+S(r).$$

iii) $$\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})\leq\overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+\frac{1}{2}(N(r,\frac{1}{f})+N(r,\frac{1}{g}))+$$
$$+N(r,\frac{1}{f};\nu_1\geq 2)+N(r,\frac{1}{g};\nu_2\geq 2)+\overline{N}(r,\frac{1}{f'};f\neq 0)+$$
$$+\overline{N}(r,\frac{1}{g'};g\neq 0)+S(r).$$

Proof. We have

$$L=\frac{f''}{f'}-2\frac{f'}{f}-\frac{g''}{g'}+2\frac{g'}{g}. \tag{2.1}$$

We now consider the poles of L. By (2.1), it is clear that L has only simple poles, moreover, if a is a pole of L, then:

+ $f(a)=\infty$, or $f^{'}(a)=0$ with $f(a)\neq 0$, or $f(a)=0$, or

+ $g(a)=\infty$, or $g^{'}(a)=0$ with $g(a)\neq 0$, or $g(a)=0$.

Now let a be a pole of f with $\nu_f^{\infty}(a)=1$, set

$$F(z)=R(z)(z-a),$$

where $R(z)$ is a holomorphic function; $R(a)\neq 0$. Then, we have

$$\frac{F^{"}}{F^{'}}=\frac{R^{"}.(z-a)+2R^{'}}{R^{'}.(z-a)+R}, \tag{2.2}$$

and similarly for a pole of g. From this, it implies that L has no poles at simple poles of f and g

1) i) Note that

$$\overline{N}(r,\frac{1}{f};\nu_1\geq 2)\leq\frac{1}{2}(N(r,\frac{1}{f};\nu_1\geq 2),\quad \overline{N}(r,\frac{1}{g};\nu_2\geq 2)\leq\frac{1}{2}(N(r,\frac{1}{g};\nu_2\geq 2).$$

From this and since $E_{f,1)}(0)=E_{g,1)}(0)$, we have

$$N(r,\frac{1}{f};\nu_1=1)=N(r,\frac{1}{g};\nu_2=1),$$

$$\begin{aligned}\overline{N}(r,\frac{1}{f})&=N(r,\frac{1}{f};\nu_1=1)+\overline{N}(r,\frac{1}{f};\nu_1\geq 2)\leq\\&\leq\frac{1}{2}N(r,\frac{1}{f};\nu_1=1)+\frac{1}{2}(N(r,\frac{1}{f};\nu_1=1)+N(r,\frac{1}{f};\nu_1\geq 2)=\\&=\frac{1}{2}N(r,\frac{1}{f};\nu_1=1)+\frac{1}{2}N(r,\frac{1}{f}),\\ \overline{N}(r,\frac{1}{g})&=N(r,\frac{1}{g};\nu_2=1)+\overline{N}(r,\frac{1}{g};\nu_2\geq 2)\leq\\&\leq\frac{1}{2}N(r,\frac{1}{g};\nu_1=1)+\frac{1}{2}(N(r,\frac{1}{g};\nu_1=1)+N(r,\frac{1}{g};\nu_1\geq 2)=\\&=\frac{1}{2}N(r,\frac{1}{g};\nu_1=1)+\frac{1}{2}N(r,\frac{1}{g}),\ \overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})\leq\\&\leq N(r,\frac{1}{f};\nu_1=\nu_2=1)+\frac{1}{2}(N(r,\frac{1}{f})+N(r,\frac{1}{g})).\end{aligned} \tag{2.3}$$

Suppose a is a zero of f with multiplicity $\nu_1=1$. Since $E_{f,1)}(0)=E_{g,1)}(0)$, a is a zero of g with multiplicity $\nu_2=1$. Set

$$F=\frac{F_1}{z-a},\quad G=\frac{G_1}{z-a},$$

where F_1, G_1 are holomorphic functions; $F_1(a)\neq 0; G_1(a)\neq 0$.

By straight calculations, we have

$$L = \frac{F_1".(z-a)}{F_1'.(z-a) - F_1} - \frac{G_1".(z-a)}{G_1'.(z-a) - G_1}. \tag{2.4}$$

This shows $L(a) = 0$ if a is a common simple zero of f and g.

Moreover, by the logarithmic derivative lemma, we have

$$m(r, L) = m(r, \frac{F''}{F'} - \frac{G''}{G'}) \leq m(r, \frac{F''}{F'}) + m(r, \frac{G''}{G'}) =$$
$$= S(r, F') + S(r, G').$$

On the other hand, from Lemma 2.2 we get

$$T(r, F') \leq 2T(r, f) + S(r, f),\ T(r, G') \leq 2T(r, g) + S(r, g).$$

So

$$S(r, F') = S(r, f),\ S(r, G') = S(r, g).$$

From this and (2.4) we get

$$N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1) \leq N(r, \frac{1}{L}) \leq T(r, \frac{1}{L}) = T(r, L) + O(1) =$$
$$= N(r, L) + m(r, L) + O(1) \leq$$
$$\leq N(r, L) + S(r, f) + S(r, g).$$

From this and (2.3) we obtain conclusion i).

ii) By (2.1)- (2.4) and note that L has only simple poles, we can see that if a is a pole of L, then

+ $f(a) = \infty$ with $\nu_f^\infty(a) \geq 2$, or $f'(a) = 0$ with $f(a) \neq 0$, or $f(a) = 0$ with $\nu_f(a) \geq 2$,

or

+ $g(a) = \infty$ with $\nu_g^\infty(a) \geq 2$, or $g'(a) = 0$ with $g(a) \neq 0$, or $g(a) = 0$ with $\nu_g(a) \geq 2$.

From this we obtain conclusion ii).

2) If $m = 1$, then

$$\overline{N}(r, \frac{1}{f}; m \geq \nu_1 > \nu_2) = 0,\ \overline{N}(r, \frac{1}{g}; m \geq \nu_1 > \nu_2) = 0,\ N(r, \frac{1}{f}; m \geq \nu_1 \geq 2) = 0,$$

$$N(r, \frac{1}{g}; m \geq \nu_1 \geq 2) = 0,\ \ \overline{N}_{(m+1}(r, \frac{1}{f}) = \overline{N}_{(2}(r, \frac{1}{f}),\ \ \overline{N}_{(m+1}(r, \frac{1}{f}) = \overline{N}_{(2}(r, \frac{1}{f}).$$

From this and Part 1) it follows that inequality 2) holds with $m = 1$.

Now we prove the inequality for $m \geq 2$.

i) By the hypothesis we have $\overline{E}_{f,m)}(0) = \overline{E}_{g,m)}(0)$, $m \geq 2$.

By using properties of the Stieltjes integral (see [5, p. 14]), we get:

$$N(r, \frac{1}{f}) - n(0, \frac{1}{f}) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|},$$

where a_i are zeros of f, counting multiplicity and

$$N_{(m}(r, \frac{1}{f}) - n(0, \frac{1}{f}) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|},$$

where a_i are zeros of f, counting multiplicity with m-truncated multiplicity and

$$\overline{N}(r, \frac{1}{f}) - \overline{n}(0, \frac{1}{f}) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|},$$

where a_i are zeros of f, ignoring multiplicity and

$$\overline{N}_{(m}(r, \frac{1}{f}) - \overline{n}(0, \frac{1}{f}) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|},$$

where a_i are zeros of f, ignoring multiplicity with m-truncated multiplicity. We obtain the similar equalities for $N(r, \frac{1}{f}; m \geq \nu_1 \geq 2)$, $\overline{N}(r, \frac{1}{g})$, $N_{(m}(r, \frac{1}{g})$, $\overline{N}_{(m}(r, \frac{1}{g})$, $\overline{N}(r, \frac{1}{f}; m \geq \nu_1 > \nu_2)$, $N(r, \frac{1}{g}; m \geq \nu_2 \geq 2)$, $\overline{N}(r, \frac{1}{g}; m \geq \nu_2 > \nu_1)$. We are going to prove Part 2 by using these inequalities and the arguments in [13, Lemma 2.4], [14, Lemma 2.2], and [4, Lemma 2.6].

Set

$$M = \overline{N}_{m)}(r, \frac{1}{f}) + \overline{N}_{m)}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f}; m \geq \nu_1 > \nu_2) + \overline{N}(r, \frac{1}{g}; m \geq \nu_2 > \nu_1),$$

$$T = N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1) + \frac{1}{2}(N_{m)}(r, \frac{1}{f}) + N_{m)}(r, \frac{1}{g}))+$$
$$+ N(r, \frac{1}{f}; m \geq \nu_1 \geq 2) + N(r, \frac{1}{g}; m \geq \nu_2 \geq 2).$$

We first prove that $M \leq T$.

Let a be a zero of f with multiplicity $p \leq m$. From $\overline{E}_{f,m)}(0) = \overline{E}_{g,m)}(0)$ it follows that a is a zero of g with multiplicity $q \leq m$. We consider the following cases:

Case 1. $p = q \le m$.

If $p = q = 1$, then a is counted with $1 + 1 + 0 + 0 = 2$ times in M and it is counted with $1 + \frac{1}{2}(1 + 1) = 2$ times in T.

If $p = q \ge 2$, then a is counted with $1 + 1 + 0 + 0 = 2$ times on M and it is counted with $0 + \frac{1}{2}(p + p) + p + p = 3p > 2$ times in T.

Case 2. $q < p \le m$.

If $q = 1$, then $p \ge 2$ and a is counted with $1+1+1+0 = 3$ times in M and we see that a is counted with $0 + \frac{1}{2}(p + 1) + p + 0 = p + \frac{p+1}{2} > 3$ times in T.

If $q \ge 2$, then $p > 2$ and a is counted with $1 + 1 + 1 + 0 = 3$ times in M. we see that a is counted with $0 + \frac{1}{2}(p + q) + p + q = 0 + \frac{3(p+q)}{2} > 3$ times in T.

Case 3. $p < q \le m$.

The proof of Case 3 is completed by using the arguments similar to ones in Case 2.

So $M \le T$. Moreover,

$$\overline{N}(r, \frac{1}{f}) = \overline{N}_{m)}(r, \frac{1}{f}) + \overline{N}_{(m+1}(r, \frac{1}{f}),\ \overline{N}(r, \frac{1}{g}) = \overline{N}_{m)}(r, \frac{1}{g}) + \overline{N}_{(m+1}(r, \frac{1}{g});$$
$$N(r, \frac{1}{f}) = N_{m)}(r, \frac{1}{f}) + N_{(m+1}(r, \frac{1}{f}),\ N(r, \frac{1}{g}) = N_{m)}(r, \frac{1}{g}) + N_{(m+1}(r, \frac{1}{g});$$
$$\overline{N}_{(m+1}(r, \frac{1}{f}) \le \frac{1}{2} N_{(m+1}(r, \frac{1}{f}),\ \overline{N}_{(m+1}(r, \frac{1}{g}) \le \frac{1}{2} N_{(m+1}(r, \frac{1}{g}).$$

From this and $M \le T$ it follow that

$$\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f}; m \ge \nu_1 > \nu_2) + \overline{N}(r, \frac{1}{g}; m \ge \nu_2 > \nu_1) \le$$
$$\le N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) +$$
$$+ N(r, \frac{1}{f}; m \ge \nu_1 \ge 2) + N(r, \frac{1}{g}; m \ge \nu_2 \ge 2).$$

By the proof of i) of Part 1) we get $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1) \le N(r, L) + S(r)$. Combining above inequalities we obtain conclusion i) of 2.

ii) Suppose a is a zero of f with multiplicity $\nu_1 = p \le m$. Since $\overline{E}_{f,m)}(0) = \overline{E}_{g,m)}(0)$, a is a zero of g with multiplicity $\nu_2 = q \le m$. Set

$$F = \frac{F_1}{(z - a)^p},\ G = \frac{G_1}{(z - a)^q},$$

where F_1, G_1 are holomorphic functions; $F_1(a) \ne 0; G_1(a) \ne 0$. By straight calculations, we have

$$L = \frac{F_1^{"}.(z - a) + (1 - p)F_1^{'}}{F_1^{'}.(z - a) - pF_1} - \frac{G_1^{"}.(z - a) + (1 - q)G_1^{'}}{G_1^{'}.(z - a) - qG_1} + \frac{q - p}{z - a}. \tag{2.5}$$

This relation shows that L has a pole at a only if $p \neq q$ with $p \leq m$ and $q \leq m$, i.e. when f and g have a zero at a with different multiplicities and both multiplicities are $\leq m$. By (2.1)- (2.5) and note that L has only simple poles, we can see that if a is a pole of L, then

+ $f(a) = \infty$ with $\nu_f^\infty(a) \geq 2$, or $f'(a) = 0$ with $f(a) \neq 0$,

or

+ $f(a) = 0$ with $m \geq \nu_1(a) > \nu_2(a)$, or $g(a) = 0$ with $m \geq \nu_2(a) > \nu_1(a)$,

or

+ $f(a) = 0$ with $\nu_f(a) \geq m+1$, or $g(a) = \infty$ with $\nu_g^\infty(a) \geq 2$,

or

+ $g'(a) = 0$ with $g(a) \neq 0$, or $g(a) = 0$ with $\nu_g(a) \geq m+1$.

From this we obtain conclusion ii).

iii) Note that

$$N(r, \frac{1}{f}; m \geq \nu_1 \geq 2) + \overline{N}_{(m+1}(r, \frac{1}{f}) = N(r, \frac{1}{f}; m \geq \nu_1 \geq 2) + \\ + \overline{N}(r, \frac{1}{f}; \nu_1 \geq m+1) \leq \\ \leq N(r, \frac{1}{f}; \nu_1 \geq 2).$$

Similarly,

$$N(r, \frac{1}{g}; m \geq \nu_1 \geq 2) + \overline{N}_{(m+1}(r, \frac{1}{g}) \leq N(r, \frac{1}{g}; \nu_1 \geq 2).$$

From the above inequalities and i), ii) it follows iii).

3. iii) The proof of this part is completed by using the arguments similar to ones in 2. iii). ■

Lemma 2.4. ([8, Theorem 1.4]) *Let $P(z)$ be a polynomial of degree n satisfying the condition (1.4). Then $P(z)$ is a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{i=1}^{k} q_l,$$

where k is the derivative index of P.

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{m_1, m_2, m_3\} \geq 2$, or when $k = 2$, $\min\{m_1, m_2\} \geq 2$, and $m_1 + m_2 \geq 5$, the above inequality also holds.

H. Fujimoto [7, Proposition 7.1] proved the following:

Lemma 2.5. *Let $P(z)$ be a polynomial of degree n satisfying the condition* (1.4)*. Assume furthermore that $n \geq 5$ and there are two non-constant meromorphic function f and g such that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1$$

for two constants $c_0 \neq 0$ and c_1. If $k \geq 3$ or if $k = 2$, $\min\{m_1, m_2\} \geq 2$, then $c_1 = 0$.

Lemma 2.6. ([2, Lemma 2.2]) $\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l+p+1-i}$ *is not an integer, where $l, p \geq$ ≥ 1 are integers.*

In [2, Lemma 2.2], Banerjee proved Lemma 2.6 for $l, p \geq 3$, but it is clear that this lemma is valid for $l, p \geq 1$.

We recall that, $P(z)$ is defined by (1.5):

$$P(z) = (l + p + 1)\Big(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l+p+1-i} a^i z^{l+p+1-i}\Big) + 1 = Q(z) + 1.$$

$$Q(z) = (l + p + 1)\Big(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l+p+1-i} a^i z^{l+p+1-i}\Big),$$

with the condition (1.6) $Q(a) \neq -1, \neq -2$, and degree of $P(z)$ is $n = l+p+1$.

Lemma 2.7. *Let $P(z)$ be defined by* (1.5) *with condition* (1.6)*, and let $l \geq 3$ and $p \geq 2$. Then $P(z)$ is a strong uniqueness polynomial for meromorphic functions.*

Proof. Note that, $P^{'}(z) = nz^l(z-a)^p$, and $P^{'}(z)$ has a zero at 0 with multiplicity l and a zero at a with multiplicity p.

By Lemma 2.6, we see that $\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l+p+1-i}$ is not an integer. Set

$$A = \sum_{i=0}^{p} \tbinom{p}{i} \frac{(-1)^i}{l+p+1-i}.$$

Then $A \neq 0$. We have $P(0) = Q(0) + 1 = 1$, $P(a) = Q(a) + 1 = nAa^n + 1$. From this and $a \neq 0$, we get $P(a) \neq P(0)$. Set $F = P(f)$, $G = P(g)$. From $P(f) = cP(g), c \neq 0$, it implies

$$\text{(2.6)} \qquad F = cG,\ T(r,f) + S(r,f) = T(r,g) + S(r,g),\ S(r,f) = S(r,g).$$

Now we consider the following possible cases:

Case 1. $c \neq 1$.

If $c = P(a)$, from (2.6) we have

$$F - 1 = P(a)(G - \frac{1}{P(a)}). \tag{2.7}$$

We consider $P(z) - \frac{1}{P(a)}$. By $P(0) = 1$ and $P(a) = c \neq 1$ we obtain $P(0) - -\frac{1}{P(a)} \neq 0$. Because $P(z)$ satisfies the condition (1.6) we have $Q(a) + 1 \neq -1$. Moreover, since $P(a) = nAa^n + 1 = Q(a) + 1 \neq -1$ and $P(a) = c \neq 1$ we obtain $P(a) - \frac{1}{P(a)} \neq 0$. Therefore $P(z) - \frac{1}{P(a)}$ has only simple zeros, let they be given by $b_i^{'}, i = 1, 2, \ldots, n$. Note that $P(z) - 1$ has a zero at 0 with multiplicity $n - p = l + 1$, and p distinct simple zeros. Let $c_i^{'}, i = 1, 2, \ldots, p$ be distinct simple zeros of $P(z) - 1$. Applying Lemma 2.1 to the function g and the values $b_1^{'}, b_2^{'}, \ldots, b_n^{'}$, and by (2.6), (2.7) we get

$$\begin{aligned}(n-1)T(r,g) = (l+p)T(r,g) \leq \\ &\leq \overline{N}(r,g) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g - b_i^{'}}) + S(r,g) \leq \\ &\leq T(r,g) + \overline{N}(r, \frac{1}{f}) + \sum_{i=1}^{p} \overline{N}(r, \frac{1}{f - c_i^{'}}) + S(r,g) \leq \\ &\leq T(r,g) + T(r,f) + pT(r,f) + S(r,g) = \\ &= (p+2)T(r,g) + S(r,g)\end{aligned}$$

So $(l-2)T(r,g) \leq S(r,g)$, a contradiction to the assumption that $l \geq 3$.

If $c \neq P(a)$, then from (2.6) we have

$$F - c = c(G - 1). \tag{2.8}$$

We consider $P(z) - c$. By $P(0) = 1$ and $c \neq 1$ we have $P(0) - c = 1 - c \neq 0$. Moreover $c \neq P(a)$. So $P(a) - c \neq 0$, $P(0) - c \neq 0$. Therefore $P(z) - c$ has only simple zeros, let they be given by $e_i, i = 1, 2, \ldots, n$. Now we consider $P(z) - 1$. We see that $P(0) = 1, P(z) - P(0) = P(z) - 1$ has a zero at 0 with multiplicity $n - p = l + 1$, and p distinct simple zeros. Let $t_i, i = 1, 2, \ldots, p$ be distinct simple zeros of $P(z) - 1$. Applying Lemma 2.1 to the function f and the values $e_1, e_2, \ldots, e_n$, and by (2.8) we get

$$\begin{aligned}(n-1)T(r,f) = (l+p)T(r,f) \leq \\ &\leq \overline{N}(r,f) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g - e_i}) + S(r,f) \leq\end{aligned}$$

$$\leq T(r,f) + \overline{N}(r,\frac{1}{g}) + \sum_{i=1}^{p} \overline{N}(r,\frac{1}{f-t_i}) + S(r,f) \leq$$
$$\leq T(r,f) + T(r,g) + pT(r,g) + S(r,f) =$$
$$= (p+2)T(r,f) + S(r,f).$$

So, $(l-2)T(r,f) \leq S(r,f)$, a contradiction to the assumption that $l \geq 3$.

Case 2. $c = 1$. Then

$$P(f) = P(g).$$

Applying Lemma 2.4 to (2.8) we obtain $f = g$. ■

3. Proof of Theorems

3.1. Proof of Theorem 1.

Recall that $P(z)$ is a strong uniqueness polynomial of the form (1.3), $P(z) = = (z-a_1)\cdots(z-a_n)$ with $P^{'}(z) = nz^{m_1}(z-d_2)^{m_2}\cdots(z-d_k)^{m_k}$ and $P(z)$ satisfies the condition (1.4), $P(d_i) \neq P(d_j), 1 \leq i < j \leq k$, where k is the derivative index of $P(z)$.

Let $S = \{z \in \mathbb{C} | \ P(z) = 0\}$ and m be a positive integer. Suppose that $k \geq 3$, or $k = 2$ and $\min\{m_1, m_2\} \geq 2$. We are going to prove that, if $n > 2k + 10$ ($n > 2k + 4$), then S is a URSM-IM (URSE-IM) and URSM$_{m)}$-IM (URSE$_{m)}$-IM).

3.1.1. $n > 2k + 10$. In this case we will prove that S is a URSM-IM. Set

$$F = \frac{1}{P(f)},\ G = \frac{1}{P(g)}, L = \frac{F^{''}}{F^{'}} - \frac{G^{''}}{G^{'}},$$

$$T(r) = T(r,f) + T(r,g), S(r) = S(r,f) + S(r,g).$$

Then $T(r, P(f)) = nT(r,f) + S(r,f)$ and $T(r,P(g)) = nT(r,g) + S(r,g)$, and hence $S(r,P(f)) = S(r,f)$ and $S(r,P(g)) = S(r,g)$.

We consider two following cases:

Case 1. $L \equiv 0$. Then we have $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$ for some constants $c \neq 0$ and c_1. By Lemma 2.5 we obtain $c_1 = 0$.

Therefore, there is a constant $C \neq 0$ such that $P(f) = CP(g)$. Because $P(z)$ is a strong uniqueness polynomial, we obtain $f = g$.

Case 2. $L \not\equiv 0$.

Claim 1. We show that

$$(3.1) \quad (n-2)T(r) \leq \overline{N}(r,\frac{1}{P(f)}) + \overline{N}(r,\frac{1}{P(g)}) - N_0(r,\frac{1}{f'}) - N_0(r,\frac{1}{g'}) + S(r),$$

where $N_0(r, \frac{1}{f'})$ ($N_0(r, \frac{1}{g'})$) is the counting function of those zeros of f', which are not zeros of function

$$(f-a_1)...(f-a_n)f(f-d_2)...(f-d_k)((g-a_1)...(g-a_n)g(g-d_2)...(g-d_k)).$$

Indeed, applying Lemma 2.1 to the functions f, g and the values $a_1, \dots, a_n$, $0, d_2, ..., d_k$, and noting that

$$\sum_{i=1}^{n} \overline{N}(r, \frac{1}{f-a_i}) = \overline{N}(r, \frac{1}{P(f)}),\ \sum_{i=1}^{n} \overline{N}(r, \frac{1}{g-a_i}) = \overline{N}(r, \frac{1}{P(g)}),$$

we obtain

$$\begin{aligned}(n+k-1)T(r) \leq & \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) + \\ & + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \sum_{i=2}^{k} \overline{N}(r, \frac{1}{f-d_i}) + \\ & + \sum_{i=2}^{k} \overline{N}(r, \frac{1}{g-d_i}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r). \qquad (3.2)\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{N}(r, f) + \overline{N}(r, g) &\leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r),\\ \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) &\leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r),\\ \overline{N}(r, \frac{1}{f-d_i}) + \overline{N}(r, \frac{1}{g-d_i}) &\leq (k-1)(T(r, f) + T(r, g)) + S(r)\\ &= (k-1)T(r) + S(r), \quad i = 2, \dots, k.\end{aligned}$$

From this and (3.2) we obtain (3.1).

Claim 2. We show that

$$\begin{aligned}\overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \leq & (\frac{n}{2} + 3)T(r) + \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \\ & + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r).\end{aligned}$$

Indeed, by $\overline{E}_f(S) = \overline{E}_g(S)$ we get $(P(f))^{-1}(0) = (P(g))^{-1}(0)$. For simplicity, we set $\nu_1 = \nu_1(z)$, $\nu_2 = \nu_2(z)$, where $\nu_1(z) = \nu_{P(f)}(z)$, $\nu_2(z) = \nu_{P(g)}(z)$. Note that

$$\overline{N}_{(2}(r, P(f)) = \overline{N}(r, f),\ \overline{N}_{(2}(r, P(g)) = \overline{N}(r, g),$$
$$S(r, P(f)) = S(r, f),\ S(r, P(g)) = S(r, g), S(r) = S(r, f) + S(r, g).$$

Applying Part 3).iii) of Lemma 2.3 to the functions $P(f), P(g)$, we obtain

$$\overline{N}(r, \frac{1}{P(f)})+\overline{N}(r, \frac{1}{P(g)}) \leq \overline{N}(r, f) + \overline{N}(r, g) + \frac{1}{2}(N(r, \frac{1}{P(f)}) + N(r, \frac{1}{P(g)}))+$$
$$+ N(r, \frac{1}{P(f)}; \nu_1 \geq 2) + N(r, \frac{1}{P(g)}; \nu_2 \geq 2))+$$
$$+ \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r). \tag{3.3}$$

Moreover,

$$\overline{N}(r, f) + \overline{N}(r, g) \leq T(r) + S(r). \tag{3.4}$$

Obviously,

$$N(r, \frac{1}{P(f)}) \leq nT(r, f) + S(r, f);$$
$$N(r, \frac{1}{P(g)}) \leq nT(r, g) + S(r, g),$$
$$N(r, \frac{1}{P(f)}) + N(r, \frac{1}{P(g)}) \leq nT(r) + S(r). \tag{3.5}$$

On the other hand, from $P(f) = (f - a_1) \cdots (f - a_n)$ it follows that, if z_0 is a zero of $P(f)$ with multiplicity ≥ 2, then z_0 is a zero of $f - a_i$ with multiplicity ≥ 2 for some $i \in \{1, 2, \ldots, n\}$, and therefore, it is a zero of $f^{'}$, so we have

$$N(r, \frac{1}{P(f)}; \nu_1 \geq 2) \leq N(r, f^{'}).$$

From this and Lemma 2.2 we obtain

$$N(r, \frac{1}{P(f)}; \nu_1 \geq 2) \leq N(r, f^{'}) \leq N(r, \frac{1}{f})+\overline{N}(r, f)+S(r, f) \leq 2T(r, f)+S(r, f).$$

Similarly, we have

$$N(r, \frac{1}{P(g)}; \nu_2 \geq 2) \leq N(r, g^{'}) \leq N(r, \frac{1}{g})+\overline{N}(r, g)+S(r, g) \leq 2T(r, g)+S(r, g).$$

Therefore,

$$N(r, \frac{1}{P(f)}; \nu_1 \geq 2) + N(r, \frac{1}{P(g)}; \nu_2 \geq 2) \leq 2T(r) + S(r). \tag{3.6}$$

Combining (3.1)–(3.6) we get

$$\overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \leq (\frac{n}{2} + 3)T(r) + \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0)+$$
$$+ \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r).$$

Claim 2 is proved.

Claim 3. We have

$$\overline{N}(r, \frac{1}{[P(f)]^{'}}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]^{'}}; P(g) \neq 0) \leq$$
$$\leq kT(r) + N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r).$$

Indeed,

$$\overline{N}(r, \frac{1}{[P(f)]^{'}}; P(f) \neq 0) = \overline{N}(r, \frac{1}{f^{m_1}(f-d_2)^{m_2}\cdots(f-d_k)^{m_k} f'}; P(f) \neq 0) \leq$$
$$\leq \overline{N}(r, \frac{1}{f}) + \sum_{i=2}^{k} \overline{N}(r, \frac{1}{f-d_i}) + \overline{N}_0(r, \frac{1}{f'}) \leq$$
$$\leq kT(r,f) + \overline{N}_0(r, \frac{1}{f'}) + S(r,f). \tag{3.7}$$

Similarly,

$$\overline{N}(r, \frac{1}{[P(g)]^{'}}; P(g) \neq 0) \leq kT(r,g) + \overline{N}_0(r, \frac{1}{g'}) + S(r,g). \tag{3.8}$$

Inequalities (3.7) and (3.8) give us

$$\overline{N}(r, \frac{1}{[P(f)]^{'}}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]^{'}}; P(g) \neq 0) \leq$$
$$\leq kT(r) + \overline{N}_0(r, \frac{1}{f'}) + \overline{N}_0(r, \frac{1}{g'}) + S(r).$$

Claim 3 is proved.

Claim 1, 2, 3 give us:

$$(n-2)T(r) \leq (\frac{n}{2} + 3 + k)T(r) + S(r).$$

Therefore, $(n - 2k - 10)T(r) \leq S(r)$, this is a contradiction to the assumption that $n > 2k + 10$. So $L \equiv 0$. Therefore $f = g$.

3.1.2. $\boldsymbol{n > 2k + 4}$**.** We will prove that S is a URSE-IM. Note that, if f, g are entire functions, then

$$N(r,f) = 0, \quad N(r,g) = 0,$$
$$N(r,f^{'}) \leq N(r, \frac{1}{f}) + \overline{N}(r,f) + S(r,f) \leq$$
$$\leq T(r,f) + S(r,f),$$
$$N(r,g^{'}) \leq N(r, \frac{1}{g}) + \overline{N}(r,g) + S(r,g) \leq$$
$$\leq T(r,g) + S(r,g).$$

From this and by similar arguments as in the case of URSM-IM we see that, if $L \not\equiv 0$, then

$$(n-1)T(r) \le \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r),$$

$$\overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) \le (\frac{n}{2}+1)T(r) + \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \ne 0) +$$
$$+\overline{N}(r, \frac{1}{[P(g)]'}; P(g) \ne 0) + S(r),$$

$$\overline{N}(r, \frac{1}{[P(f)]'}; P(f) \ne 0) = \overline{N}(r, \frac{1}{f^{m_1}(f-d_2)^{m_2}\cdots(f-d_k)^{m_k}f'}; P(f) \ne 0) \le$$
$$\le \overline{N}(r, \frac{1}{f}) + \sum_{i=2}^{k} \overline{N}(r, \frac{1}{f-d_i}) + \overline{N}_0(r, \frac{1}{f'}) \le$$
$$\le kT(r,f) + \overline{N}_0(r, \frac{1}{f'}) + S(r,f),$$

$$\overline{N}(r, \frac{1}{[P(g)]'}; P(g) \ne 0) \le kT(r,g) + \overline{N}_0(r, \frac{1}{g'}) + S(r,g).$$

Combining above inequalities we obtain:

$$(n-1)T(r) \le (\frac{n}{2} + 1 + k)T(r) + S(r).$$

So, $(n-2k-4)T(r) \le S(r)$, a contradiction to the assumption that $n > 2k+4$. Thus we have $L \equiv 0$. The proof of this case is completed by using the arguments similar to ones in the case of URSM-IM.

3.1.3. $n > 2k+10$. We prove that S is a URSM$_{m)}$. By the similar arguments as in the case of URSM-IM we can see that if $L \not\equiv 0$, then

$$(n-2)T(r) \le \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r).$$

Note that $E_{f,m)}(S) = \overline{E}_{f,m)}(S)$ and

$$\overline{N}_{(2}(r, P(f)) = \overline{N}(r,f) \le T(r,f) + S(r,f),$$
$$\overline{N}_{(2}(r, P(g)) = \overline{N}(r,g) \le T(r,g) + S(r,g),$$
$$S(r, P(f)) = S(r,f),\ S(r,P(g)) = S(r,g), S(r) = S(r,f) + S(r,g).$$

Then applying Part 2).iii) of Lemma 2.3 to the functions $P(f), P(g)$ we get

$$\overline{N}(r, \frac{1}{P(f)})+\overline{N}(r, \frac{1}{P(g)}) \leq \overline{N}(r, f) + \overline{N}(r, g) + \frac{1}{2}(N(r, \frac{1}{P(f)})+ + N(r, \frac{1}{P(g)})) + N(r, \frac{1}{P(f)}; \nu_1 \geq 2) + N(r, \frac{1}{P(g)}; \nu_2 \geq 2))+ + \overline{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \overline{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r).$$

By the similar argument as in Claims 2, 3 of the case URSM-IM we get a contradiction to the assumption that $n > 2k + 10$. So $L \equiv 0$. The proof of this case is completed by using the arguments similar to ones in the case of URSM-IM.

3.1.4. $n > 2k + 4$. By using the arguments similar to ones in the cases of URSE-IM and $\text{URSM}_{m)}$ we can prove that S is a $\text{URSE}_{m)}$.

Theorem 1 is proved. ■

3.2. Proof of Theorem 2.

We recall that, $P(z)$ is defined by (1.5),

$$P(z) = (l + p + 1)\Big(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l + p + 1 - i} a^i z^{l+p+1-i}\Big) + 1 = Q(z) + 1.$$

$$Q(z) = (l + p + 1)\Big(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{l + p + 1 - i} a^i z^{l+p+1-i}\Big),$$

with the condition (1.6), $Q(a) \neq -1, \quad Q(a) \neq -2$, and the degree of $P(z)$ is $n = l + p + 1$.

Then applying Lemma 2.7 we see that $P(z)$ is a strong uniqueness polynomial for meromorphic functions of degree $n \geq 6$, if $l \geq 3$ and $p \geq 2$. Polynomial $P(z)$ satisfies the conditions of Theorem 1 with $k = 2$. Then applying Theorem 1 to polynomial $P(z)$ we conclude that S is a URSM-IM (URSE-IM) and $\text{URSM}_{m)}$-IM ($\text{URSE}_{m)}$-IM) if $n \geq 15$($n \geq 7$).

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