Annales Univ. Sci. Budapest., Sect. Comp. 54 (2023) 51–69

CONTINUATION OF THE LAUDATION TO Professor Bui Minh Phong on his 70-th birthday

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We congratulated him in this journal on his 60th (**38** (2012) 5–11) and 65th birthday (**47** (2018) 87–96). During the last five years many important things has happened with him. Two new grandchildren were born and so the number of grandchildren grew up to seven. His family consisted of 16 people in all, they now live on three continents: Hungary (Europe), Vietnam (Asia) and Australia (Australia).

Five years ago, he had a health-related problem, but fortunately everything was resolved quickly, giving him the opportunity to continue to work and contribute. In recent times, besides studying math, he also often writes poems, conveys his feelings and thoughts about family, friends, country and the beauty of the landscape.

https://doi.org/10.71352/ac.54.051

In the past five years Bui Minh Phong continued his research in number theory. He wrote 26 new papers in number theory, some of them with coauthors. He is now Professor Emeritus at Eötvös Loránd University (Budapest).

Here, we provide the highlights of his mathematical results and we enlarge the categories to classify his new results as follows:

1. Additive arithmetical functions with values in topological groups ([104], [107])

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, integers, real and complex numbers, respectively. Further let $\mathbb{Q}_x$ and $\mathbb{R}_x$ stand respectively for the multiplicative group of the positive rational numbers and the positive real numbers.

Let G be an Abelian group. A mapping $\varphi : \mathbb{N} \to G$ is completely additive, if

$$\varphi(nm) = \varphi(n) + \varphi(m) \quad \forall n, m \in \mathbb{N}.$$

Let $\mathcal{A}_G^*$ be the set of completely additive functions.

If G is considered as a multiplicative (commutative) group, then the mapping $V : \mathbb{N} \to G$ satisfying the relation

$$V(nm) = V(n)V(m) \quad \forall n, m \in \mathbb{N}$$

is called a completely multiplicative function. $\mathcal{M}_G^*$ denotes the set of these functions.

We can extend the domain of φ and V to $\mathbb{Q}_x$ by the relations

$$\varphi\left(\frac{m}{n}\right) = \varphi(m) - \varphi(n) \text{ and } V\left(\frac{m}{n}\right) = V(m)(V(n))^{-1}$$

uniquely. Furthermore, the relations

$$\varphi(rs) = \varphi(r) + \varphi(s) \text{ and } V(rs) = V(r)V(s) \quad \forall r, s \in \mathbb{Q}_x$$

hold.

Let now G be an Abelian topological group, $\varphi : \mathbb{Q}_x \to G$ be a homomorphism. We shall say that φ is continuous at the point 1, if $r_\nu \to 1$ implies that $\varphi(r_\nu) \to 0$.

Theorem 1. (I. Kátai and B. M. Phong [104]) *If G is an additively written Abelian topological group with the translation invariant metric ρ and*

$$\frac{1}{\log x} \sum_{n \le x} \frac{\rho(\varphi(n), \varphi(n+1))}{n} \to 0 \quad (x \to \infty),$$

where $\varphi : \mathbb{N} \to G$ *is a completely additive function, then the extension* $\varphi : \mathbb{R}_x \to$ $\to G$ *is a continuous homomorphism.*

Theorem 2. (I. Kátai and B. M. Phong [107]) *Let* G *be an additively written Abelian topological group with the translation invariant metric* ρ. *Let* $\psi, \varphi \in \mathcal{A}_G^*$ *for which*

$$\frac{1}{\log x} \sum_{n \le x} \frac{\rho(\psi(n+1), \varphi(n))}{n} \to 0 \quad (x \to \infty).$$

Then $\varphi(n) = \psi(n) \quad (n \in \mathbb{N})$, *and the extension* $\varphi : \mathbb{R}_x \to G$ *is a continuous homomorphism.*

Theorem 3. (I. Kátai and B. M. Phong [107]) *Let* G *be an additively written Abelian topological group with the translation invariant metric* ρ. *Assume that* $\psi, \varphi \in \mathcal{A}_G^*$ *and*

$$\frac{1}{\log x} \sum_{n \le x} \frac{\rho(\psi([\sqrt{2}n]), \varphi(n) + A)}{n} \to 0 \quad (x \to \infty).$$

Then $\varphi(n) = \psi(n) \quad (n \in \mathbb{N})$, *and the extension* $\varphi : \mathbb{R}_x \to G$ *is a continuous homomorphism. Furthermore* $A = \psi(\sqrt{2})$.

We note that in the proofs of the above theorems the authors used the results of O. Klurman (*Compos. Math.*, **153** (2017), no. 8, 1622-–1657) and O. Klurman, A. P. Mangerel (*Algebra Number Theory*, **14** (2020), no. 1, 155—189).

2. Arithmetical functions satisfying some relation ([105], [116], [121], [126])

An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive (multiplicative) if $(n, m) = 1$ implies that

$$f(nm) = f(n) + f(m) \quad (f(nm) = f(n)f(m))$$

and it is completely additive (multiplicative) if the above equality holds for all positive integers n and m. Let $\mathcal{A}$, $\mathcal{M}$, $\mathcal{A}^*$ and $\mathcal{M}^*$ denote the set of all additive, multiplicative, completely additive and completely multiplicative functions, respectively.

P. Erdős proved in 1946 (*Ann. Math.*, **47** (1946), 1–20) that if $f : \mathbb{N} \to \mathbb{R}$ is an additive function such that

$$\Delta f(n) := f(n+1) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then $f(n)$ is a constant multiple of $\log n$. This assertion has been generalized in several directions. Some results were surveyed and some new problems were formulated in the above work of P. Erdős. The characterization of multiplicative function $f : \mathbb{N} \to \mathbb{C}$ under suitable "regularity condition" even in the simplest case $\Delta f(n) = o(1)$ as $n \to \infty$ is much harder. I. Kátai formulated the conjecture that if f is a multiplicative function and $\Delta f(n) \to 0 \quad (n \to \infty)$, then either $f(n) \to 0$ or $f(n) = n^{\delta+it}$, where $0 \le \delta < 1$ and $t \in \mathbb{R}$. This has been proved by E. Wirsing in a private letter, and later together with Tang Yuansheng and Shao Pintsung (see *J. Number Theory*, **56** (1996), 391–395).

Naturally, it would be nice to characterize those multiplicative functions f, g for which $g(An + B) - Ef(Cn + D) \to 0$ $(n \to \infty)$, where $AD \ne BC, A, C \in$ $\in \mathbb{N}, B, D \in \mathbb{Z}$.

Doing the first step N. L. Bassily and I. Kátai investigated those multiplicative functions f, g for which $g(2n + 1) - Ef(n) \to 0$ $(n \to \infty)$ (*Aequationes Math.*, **55** (1998), 1–14). B. M. Phong [43] obtained the result of the above problem in the case $B = C = 1, D = 0$ and $A \in \mathbb{N}$ is arbitrary.

I formulated the conjecture that if $f \in \mathcal{M}$ and

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n + 1) - f(n)| = 0$$

then either

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)| = 0$$

or

$$f(n) = n^{\delta+it} \quad (n \in \mathbb{N}), \quad 0 \le \delta < 0, t \in \mathbb{R}.$$

This has been proved by L. Klurman (*Compos. Math.*, **153** (2017), no. 8, 1622–1657). He proved that if $f \in \mathcal{M}^*, |f(n)| = 1 \quad (n \in \mathbb{N})$ and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n + 1) - f(n)|}{n} = 0,$$

then $f(n) = n^{it} \quad (n \in \mathbb{N}), \quad t \in \mathbb{R}$.

Theorem 4. (I. Kátai and B. M. Phong [105]) *Let $f, g \in \mathcal{M}^*, f(n)g(n) \ne 0$ $(n \in \mathbb{N})$, $A \ne 0$, and that*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(2n + 1) - Af(n)|}{n} = 0 \quad \textit{and} \quad \overline{\lim} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n)|}{n} > 0.$$

Then $f(n) = g(n)$ for all odd n and $A = f(2)$, $f(n) = n^{\delta+it}$ for every $n \in \mathbb{N}$, where $0 \le \delta < 1, t \in \mathbb{R}$.

Let $\|x\|$ be the distance of the real number x from a nearest integer. We proved the following results.

Theorem 5. (I. Kátai and B. M. Phong [122]) *Assume that* $f \in \mathcal{A}$, $\Delta f(n) = f(n+1) - f(n)$, *and that for every* $\varepsilon > 0$ *either*

$$\frac{1}{x}\sharp\{n \le x|\ \|\Delta f(n)\| > \varepsilon\} \to 0 \quad as \quad x \to \infty$$

or

$$\frac{1}{\log x}\sum_{\substack{n \le x \\ \|\Delta f(n)\| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty.$$

Then $f(n) = \lambda \log n + E(n)$, *where* $\lambda \in \mathbb{R}, E \in \mathcal{A}, E(n) \in \mathbb{Z}$ *for every* $n \in \mathbb{N}$.

Theorem 6. (I. Kátai and B. M. Phong [122]) *Assume that* $f \in \mathcal{A}$ *and that for every* $\varepsilon > 0$ *either*

$$\frac{1}{x}\sharp\{n \le x \mid |\Delta f(n)| > \varepsilon\} \to 0 \quad as \quad x \to \infty$$

or

$$\frac{1}{\log x}\sum_{\substack{n \le x \\ |\Delta f(n)| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty.$$

Then $f(n) = c \log n$ *for some real number* c.

Now let g be a completely multiplicative function, $g(n) \in \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{A} = \{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{T}$. Assume that

$$\lim_{x\to\infty}\frac{1}{x}\sharp\{n \le x \mid |g(n+1)\overline{g}(n) - \alpha_j| < \epsilon\} = \rho(\alpha_j) > 0$$

for each $\epsilon > 0$ which is small enough. Assume furthermore that if $\delta \in \mathbb{T} \setminus \mathbb{A}$, then

$$\lim_{x\to\infty}\frac{1}{x}\sharp\{n \le x \mid |g(n+1)\overline{g}(n) - \delta| < \epsilon\} = 0.$$

We formulated a conjecture on the possible $\mathbb{A}$ and g.

Conjecture 1. (I. Kátai and B. M. Phong [116]) *Let*

$$\mathbb{A} = \{\alpha_1, \cdots, \alpha_k\} \subseteq \mathbb{T}.$$

Let $g \in \mathcal{M}_1^*$ *and* $C(n) = g(n+1)\overline{g}(n)$ $(n \in \mathbb{N})$. *Assume that*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\min_{\beta\in\mathbb{A}}|C(n) - \beta| = 0$$

and that for every $\gamma \in \mathbb{A}$,

$$\lim_{x\to\infty} \frac{1}{x}\sharp\{n \le x \mid |C(n)-\gamma| = \min_{\beta\in\mathbb{A}} |C(n)-\beta|\} = \rho(\gamma)$$

exists, and $\rho(\gamma) > 0$. *Then there exists such an* $\ell \in \mathbb{N}$ *for which* $\mathbb{A}^\ell = \{1\}$. *Consequently*

$$g^\ell(n) = n^{i\tau}, \quad g(n) = n^{i\tau/\ell}G(n), \quad G \in \mathcal{M}_1^*, G^\ell(n) = 1.$$

Theorem 7. (I. Kátai and B. M. Phong [116]) *Conjecture* 1 *is true if*

(*A*1) $k \in \{1, 2\}$

and

(*A*2) $k = 3$, *possibly except the case when* $\mathbb{A} = \{\alpha_1, \alpha_2, \alpha_3\} = \{1, \beta, \overline{\beta}\}$.

We stated the following

Conjecture 2. (I. Kátai and B. M. Phong [116]) *Let* $0 \le u_1 < u_2 < \cdots < < u_k < 1$; $p_1, \cdots, p_k > 0$ *with* $\sum_{i=1}^k p_i = 1$. *Let* $\mathcal{H}$ *be the class of the distribution functions of the random variables* ξ *satisfying the conditions* $P(\xi = u_i) = = p_i$. *Let* $h(n)$ *be an additive function,* $\delta(n) = h(n+1) - h(n) \pmod 1$. *Assume that it has a limit distribution, and that its distribution* $F \in \mathcal{H}$,

$$F(y) = P(\xi \le y), P(\xi = u_i) = p_i \quad (i = 1, \cdots k).$$

Then $h(n) \equiv \tau \log n + E(n)$ *and*

$$\frac{1}{x}\sharp\{n \le x \mid E(n+1) - E(n) \pmod 1 = u_i\} = p_i \quad (i = 1, \ldots, k),$$

furthermore $\ell E(n) \equiv 0 \pmod 1$ *for a suitable* $\ell \in \mathbb{N}$.

Conjecture 2 is a variant of Conjecture 1.

Conjecture 3. (I. Kátai and B. M. Phong [116]) *Let* $h(n)$, $\delta(n)$ *be as above. Let* $I = [\alpha, \beta) \subseteq [0, 1)$ *and assume that*

$$\lim_{x\to\infty} \frac{1}{x}\sharp\{n \le x \mid \delta(n) \in I\} = 0.$$

Then $h(n) \equiv \tau \log n + E(n) \pmod 1$ *and* $\ell E(n) \equiv 0 \pmod 1$ *for a suitable* $\ell \in \mathbb{N}$.

Let α, β be positive real numbers such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$. In papers [84]–[87] I. Kátai and B. M. Phong formulated the following conjecture:

Conjecture 4. (I. Kátai and B. M. Phong [126]) *Let*

$$\mathcal{U} := \{z \in \mathbb{C} \mid |z| \le 1\}.$$

If $f \in \mathcal{M}, f(n) \in \mathcal{U}$, *and there exists some* C *for which either*

$$f([\beta n]) - Cf([\alpha n]) \to 0 \quad as \quad n \to \infty$$

or

$$\frac{1}{\log x} \sum_{n \le x} \frac{f([\beta n]) - Cf([\alpha n])}{n} \to 0,$$

then $f(n) = n^{i\tau}$.

This conjecture was proved in the special case, when $\alpha = 1, \beta = \sqrt{2}$.

Recently in [126] we proved the following result.

Theorem 8. (I. Kátai and B. M. Phong [126]) *Let* $f, g \in \mathcal{M}_1^*$,

$$\frac{1}{\log x} \sum_{n \le x} \frac{|g([\sqrt{2}n]) - Cf(n)|}{n} \to 0 \quad (x \to \infty),$$

then $f(n) = g(n) = n^{i\tau} \quad (\tau \in \mathbb{R})$, *where* $C = (\sqrt{2})^{i\tau}$.

3. On q-multiplicative functions with special properties ([112])

Let $\mathbb{N}_0$ be the set of all non-negative integers. For some integer $q \ge 2$ let

$$\mathbb{A}_q := \{0, 1, \cdots, q-1\}.$$

Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \mathbb{A}_q$$

and $a_r(n) = 0$ if $q^r > n$. Let $\mathcal{A}_q$ and $\mathcal{M}_q$ be the sets of q-additive and q-multiplicative functions, respectively. We say that $f \in \mathcal{A}_q$ ($g \in \mathcal{M}_q$), if $f : \mathbb{N}_0 \to \mathbb{C}$ ($g : \mathbb{N} \to \mathbb{C}$) satisfy the following conditions

$$f(0) = 0 \;\text{ and }\; f(n) = \sum_{r=0}^{\infty} f\Big(a_r(n) q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}$$

and

$$g(0) = 1 \;\text{ and }\; g(n) = \prod_{r=0}^{\infty} g\Big(a_r(n) q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}.$$

It is easy to see that $f \in \mathcal{A}_q$ $(g \in \mathcal{M}_q)$ if and only if $f(aq^r + b) = f(aq^r) + f(b)$ $(\ g(aq^r + b) = g(aq^r)g(b))$ for all $a \in \mathbb{N}_0, r \in \mathbb{N}, 0 \le b < q^r$.

In the following let I_q be the set defined as:

$$I_q := \mathcal{P} \cup \{n \in \mathbb{N} |\ \ (n, q) = 1\}.$$

I. Kátai and B. M. Phong stated the following conjecture:

Conjecture 5. *Let $q \ge 2$ be an integer and $g \in \mathcal{M}_q$. If $g(p) = 1$ for every $p \in \mathcal{P}$, then $g(qm) = 1$ for every $m \in \mathbb{N}$ and $g(qm + r) = g(r)$ for every $m \in \mathbb{N}$, $r \in \{1, \cdots, q-1\}$, furthermore $g(n) = 1$ for every $n \in I_q$.*

In [112] B. M. Phong and R. B. Szeidl proved the following

Theorem 9. (B. M. Phong and R. B. Szeidl [112]) *Conjecture 5 is true for every $q \in \{2, \ldots, 50\}$.*

We note from Theorem 9 that if $q \in \{2, \cdots, 50\}$, $q \in \mathcal{P}$ and $g \in \mathcal{M}_q$, then the condition $g(p) = 1$ for every $p \in \mathcal{P}$ implies that $g(n) = 1$ for every $n \in \mathbb{N}$.

4. The identity function with equations of functions ([117], [120])

In the following, let $\mathbb{E}(n) = 1$, $\mathbb{I}(n) = n$, $0(n) = 0$ for every $n \in \mathbb{N}$ and $\mathbb{O}(1) = 1, \mathbb{O}(n) = 0$ if $n \ge 2$. Thus $\mathbb{I}$ is the identity function and 0 is the zero function.

A characterization of the identity function $\mathbb{I}$ was studied by C. Spiro (*Journal of Number Theory*, **42** (1992), 232-246.), J.-M. De Koninck, I. Kátai and B. M. Phong [38], B. M. Phong [92] and by others.

In 1992, C. Spiro proved that if $f \in \mathcal{M}$ satisfies

$$f(p + q) = f(p) + f(q) \quad (\forall p, q \in \mathcal{P}) \quad \text{and} \quad f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P},$$

then $f = \mathbb{I}$.

In 1997, J.-M. De Koninck, I. Kátai and B. M. Phong [38] proved that if a function $f \in \mathcal{M}$ satisfies the condition

$$f(p + m^2) = f(p) + f(m^2) \quad \text{for every} \quad p \in \mathcal{P}, m \in \mathbb{N},$$

then $f = \mathbb{I}$.

Let $D \in \mathbb{Z}$ be fixed. We consider the following equation

$$(1)\quad f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

In the case $D = 0$, the equation (1) was studied by B. Bašić (*Acta Mathematica Sinica, English Series*, **30** (2014), Issue 4, 689–695). Recently P.-S. Park (*Bull. Korean Math. Soc.*, **60** (2023), no. 1, 75–81) proved that if $f \in \mathcal{M}$ satisfies (1), then $f = \mathbb{I}$. He also obtained a complete solutions f of (1) in the case $D = -1$.

For some generalizations of above results we refer the works of B. M. M. Khanh (*Annales Univ. Sci. Budapest. Sect. Comp.*, **52** (2021), 195–216), B. M. Phong and R. B. Szeidl [117]. They prove that if $D \in \{1, 2, 3\}$ and an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the conditions $f(1) = 1$ and (1), then $f = \mathbb{I}$.

Theorem 10. (B. M. Phong and R. B. Szeidl [117]) *Assume that* $f : \mathbb{N} \to \mathbb{C}$ *satisfies*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}.$$

Then the following assertions holds:

($B1$) *If* $f(1) = 0$, *then* $f(n) = 0$ *for every* $n \in \mathbb{N}$,

($B2$) *If* $f(1) = \frac{1}{3}$, *then* $f(n) = \frac{1}{3}$ *for every* $n \in \mathbb{N}$,

($B3$) *If* $f(1) = 1$, *then* $f = \mathbb{I}$.

Theorem 11. (B. M. Phong and R. B. Szeidl [120]) *An arithmetical function* $f : \mathbb{N} \to \mathbb{C}$ *satisfies*

$$(2) \qquad f(n^2 - nm + m^2) = f^2(n) - f(n)f(m) + f^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}$$

if and only if

$$f \in \{0,\ \mathbb{O},\ \mathbb{E},\ \mathbb{I},\ \Theta_M\},$$

where

$$\Theta_M(n) = \begin{cases} 0, & \textit{if} \quad M \mid n \\ 1, & \textit{if} \quad M \nmid n \quad \textit{for every} \quad n \in \mathbb{N} \end{cases}$$

and $M = 2$ *or* $M = q_1 \cdots q_s \geq 5$ *is a square-free number,* $q_i \equiv 2 \pmod 3$ $(i = 1, \ldots, s)$.

Theorem 12. (B. M. Phong and R. B. Szeidl [120]) *An arithmetical function* $f : \mathbb{N} \to \mathbb{C}$ *satisfies*

$$f(n^2 - 2nm + m^2) = f^2(n) - 2f(n)f(m) + f^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}$$

if and only if if and only if

$$f \in \{0,\ \mathbb{I},\ \chi_2\},$$

where $\chi_2(n)$ *is a Dirichlet character* $\pmod 2$.

It follows easily from Theorem 11 that a function $f \in \mathcal{M}$ satisfies (2) if and only if $f \in \{\mathbb{E}, \mathbb{I}, \chi_q\}$, where χ_q is the Dirichlet principal character $\pmod q$, $q \in \mathcal{P}$, $q \equiv 2 \pmod 3$.

5. Arithmetical functions commutable with sums of squares ([114], [118], [119])

In [118] we prove the following results.

Theorem 13. (I. Kátai and B. M. Phong [118]) *Assume that the arithmetical functions $f, g : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(a^2 + b^2 + c^2 + d^2) = g(a^2) + g(b^2) + g(c^2) + g(d^2)$$

for every $a, b, c, d \in \mathbb{N}_0$. Then there are numbers A and B such that

$$f(n) = An + 4B \quad \textit{and} \quad g(m^2) = Am^2 + B$$

hold for every $n, m \in \mathbb{N}_0$.

Theorem 14. (I. Kátai and B. M. Phong [118]) *Assume that the arithmetical functions $F, G : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$F(a^2 + b^2 + c^2 + d^2) = G(a^2 + b^2) + G(c^2 + d^2)$$

for every $a, b, c, d \in \mathbb{N}_0$. Then there are numbers C and D such that

$$F(n) = Cn + 2D \quad \textit{and} \quad G(n^2 + m^2) = C(n^2 + m^2) + D.$$

It follows easily from the above results that if $f = g$ in Theorem 13 or $F = G$ in Theorem 14, then $f(n) = f(1)n$ or $F(n) = F(1)n$ holds for every $n \in \mathbb{N}_0$.

Theorem 15. (I. Kátai and B. M. Phong [114]) *Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, h_1, h_2, h_3, h_4 : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = h_1(x_1) + h_2(x_2) + h_3(x_3) + h_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$. Then there are numbers $A, B_1, B_2, B_3, B_4 \in \mathbb{C}$ such that

$$h_i(m) = Am^2 + B_i \quad (i = 1, \ldots, 4)$$

and

$$f(n + k) = An + B_1 + B_2 + B_3 + B_4 + K$$

hold for every $n, m \in \mathbb{N}_0$.

Theorem 16. (I. Kátai and B. M. Phong [114]) *Let* $k \in \mathbb{N}$ *and* $K \in \mathbb{C}$. *Assume that the arithmetical functions* $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ *satisfy the relation*

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$. *Then there are numbers* $C, D_1, D_2, D_3, D_4 \in \mathbb{C}$ *such that*

$$H_i(m) = Am^2 + D_i \quad (i = 1, \dots, 4)$$

and

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = C(x_1^2 + x_2^2 + x_3^2 + x_4^2) + D_1 + D_2 + D_3 + D_4 + K$$

hold for every $m, x_1, x_2, x_3, x_4 \in \mathbb{N}$.

Some corollaries follow from Theorem 15 and Theorem 16, for example it follows from Theorem 16 that if $\ell \geq 4$ and the arithmetical functions $F : \mathbb{N} \to \mathbb{C}$ satisfy the relation

$$F(x_1^2 + \cdots + x_\ell^2) = F^2(x_1) + \cdots + F^2(x_\ell)$$

for every $x_1, \dots, x_\ell \in \mathbb{N}$, then

$$F \in \left\{0, \quad \mathbb{I}, \quad \frac{\varepsilon}{\ell}\right\},$$

where $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(x_1^2 + \cdots + x_\ell^2) = 1$.

We define $\mathcal{S}$ and $\mathcal{A}$ as follows:

$$\mathcal{S} := \{4^s(8t + 7) \mid \ s, t \in \mathbb{N}_0\}$$

and

$$\mathcal{A} := \{r \in \mathbb{N} \mid \ r^2 = k^2 + h^2 + t^2 \ \text{ and } \ r^2 = u^2 + v^2 \quad \text{for some } \ k, h, t, u, v \in \mathbb{N}\}.$$

For some $k \in \mathbb{N}$ let $\overline{k} \equiv k \pmod 4$ such that $\overline{k} \in \{0, 1, 2, 3\}$.

Theorem 17. (I. Kátai and B. M. Phong [119]) *Let* $k \in \mathbb{N}_0$ *and* $K \in \mathbb{C}$. *Assume that the arithmetical functions* $f, h : \mathbb{N} \to \mathbb{C}$ *satisfy the equation*

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K$$

for every $a, b, c \in \mathbb{N}$. *Then there are numbers* $A, B, C \in \mathbb{C}$ *such that*

$$h(m) = Am^2 + B\chi_2(m) + C$$

and

$$f(a^2 + b^2 + c^2 + k) = A\Big(a^2 + b^2 + c^2\Big) + B\Big(\chi_2(a) + \chi_2(b) + \chi_2(c)\Big) + D_3$$

hold for every $m, a, b, c \in \mathbb{N}$*, where* $D_3 = 3C + K$ *and* $\chi_2(r)$ *is the Dirichlet character* $\pmod 2$*, that is*

$$\chi_2(r) = \begin{cases} 1 & \text{if} \quad (r,2) = 1 \\ 0 & \text{if} \quad 2|r. \end{cases}$$

Furthermore

$$f(r^2 n + k) = Ar^2 n + \overline{n} B \chi_2(r) + D_3$$

holds for every $n \in \mathbb{N}$*,* $n \notin \mathcal{S}$ *and for every* $r \in \mathcal{A}$*.*

In the following let

$$\mathbb{M} := \{a^2 + b^2 + c^2 | a, b, c \in \mathbb{N}\}$$

and for each $k \in \mathbb{N}_0$ let

$$\mathcal{H}_k = \begin{cases} \{1, \dots, e+2\} & \text{if} \quad k = 2^e k_1, (k_1, 2) = 1, 2|e \quad \text{and} \quad k_1 \equiv 1 \pmod 8, \\ \mathbb{N} \setminus \{e+2\} & \text{if} \quad k = 2^e k_1, (k_1, 2) = 1, 2|e \quad \text{and} \quad k_1 \equiv 5 \pmod 8, \\ \mathbb{N} \ \text{ in any other cases.} \end{cases}$$

Theorem 18. (I. Kátai and B. M. Phong [119]) *Let* $k \in \mathbb{N}_0$ *and* $K \in \mathbb{C}$*. Assume that* $F, H \in \mathcal{M}$ *satisfy the relation*

$$F(a^2 + b^2 + c^2 + k) = H(a) + H(b) + H(c) + K$$

for every $a, b, c \in \mathbb{N}$*. Then one of the following assertions holds:*

$(C1)$ $H = \mathbb{E}$ *and* $F(\eta + k) = 0 \ (\forall \eta \in \mathbb{M})$ *if* $K = -3$,

$(C2)$ $H = \mathbb{E}$ *and* $F(2n+1) = 1, F(2^\alpha) = K+3 \ (\forall \alpha \in \mathcal{H}_k)$ *if* $K \neq -3$

$(C3)$ $H = \chi_2$ *and* $F(2n+1) = 1 \ (\forall n \in \mathbb{N})$ *if* $(\overline{k}, K) \neq (3, -1)$,

$(C4)$ $H = \chi_2$ *and* $F(2n+1) = (-1)^n, F(2) = (-1)^{\frac{k+1}{4}} 2, F(2^\alpha) = 0,$
$(\forall n \in \mathbb{N}, \ \forall \alpha \in \mathcal{H}_k, \alpha \geq 2)$ *if* $(\overline{k}, K) = (3, -1)$,

$(C5)$ $H = \mathbb{I}^2, F(2n+1) = 2n+1$ *and* $F(2^\alpha) = 2^\alpha \ (\forall n \in \mathbb{N}, \ \forall \alpha \in \mathcal{H}_k)$.

One can deduced from the above results that if $\ell \in \mathbb{N}$, $\ell \geq 4$ and the arithmetical function $F : \mathbb{N} \to \mathbb{C}$ satisfy

$$F(n_1^2 + \cdots + n_\ell^2) = F(n_1^2) + \cdots + F(n_\ell^2) \quad \text{for every} \quad n_1, \cdots, n_\ell \in \mathbb{N},$$

then there is a complex A such that $F(m^2) = Am^2$ and $F(n_1^2 + \cdots + n_\ell^2) = A(n_1^2 + \cdots + n_\ell^2)$ hold for every $m, n_1, \dots, n_\ell \in \mathbb{N}$. Consequently, $F(n) = An$ for every n which can be written as $n = n_1^2 + \cdots + n_\ell^2$ for $n_1, \dots, n_\ell \in \mathbb{N}$.

6. Characterizations of arithmetical functions with special relations ([121], [124], [125], [129])

6.1. The equation $F(n^2+m^2+k)=H(n)+H(m)+K$

Theorem 19. (I. Kátai, B. M. M. Khanh and B. M. Phong [121]) *Let $\mathcal{E}$ be the set of positive integers, which are the sum of two squares. The numbers $k\in\mathcal{E}, K\in\mathbb{C}$ and the functions $F,H\in\mathcal{M}^*$ satisfy the equation*

$$F(n^2+m^2+k)=H(n)+H(m)+K \quad \textit{for every} \quad n,m\in\mathbb{N}$$

if and only if one of the following assertions holds:

$(D1)$ $K=k,\quad H=\mathbb{I}^2, F=\mathbb{I}$,

$(D2)$ $K=-1,\quad H=F=\mathbb{E}$,

$(D3)$ $K=-2,\quad H=\mathbb{E},\quad F(n^2+m^2+k)=0$ *for every* $n,m\in\mathbb{N}$,

$(D4)$ $K=-1,\quad k\equiv 2 \pmod 3,\quad H=\chi_3^*,\quad F=\chi_3$, *where* $\chi_3^*(m) \pmod 3$ *is the principal Dirichlet character and* $\chi_3(m) \pmod 3$ *is the non-principal Dirichlet character, i.e* $\chi_3^*(0)=0$, $\chi_3^*(1)=1$, $\chi_3^*(2)=1$, $\chi_3(0)=0$, $\chi_3(1)=1$, $\chi_3(2)=-1$.

It follows from Theorem 19 that if the numbers $k\in\mathcal{E}, K\in\mathbb{C}\setminus\{-1,-2\}$ and the functions $F,H\in\mathcal{M}^*$ satisfy the equation in Theorem 19, then (D1) holds.

6.2. The equation $Ag(n+1)=Bf(n)+C$

Theorem 20. (I. Kátai and B. M. Phong [125]) *If $A,B,C\in\mathbb{C}, AB\neq 0$ and the functions $g,f\in\mathcal{M}^*$ satisfy the relation*

$$Ag(n+1)=Bf(n)+C \quad \textit{for every} \quad n\in\mathbb{N},$$

then:

$$C\in\Big\{0,\ -B,\ A-B,\ Bf(2)^2-2Bf(2)+A\Big\}$$

and the following assertions hold:

$(E1)$ *If $C=0$, then $A=B$ and $g=f=\mathbb{E}$.*

$(E2)$ *If $C=-B$, then*

$$(g,f)=\begin{cases}(\mathbb{O},\mathbb{E}) & \textit{if} \quad f(2)=1\\ (\chi_2,\chi_2) & \textit{if} \quad f(2)\neq 1.\end{cases}$$

$(E3)$ *If $A=B+C$, then $g=f=\mathbb{E}$.*

(*E*4) *If* $C = Bf(2)^2 - 2Bf(2) + A$, *then* $f(2) \in \left\{0,\ 1,\ \frac{A+B}{B}\right\}$.

(*E*4.1) *If* $f(2) = 0$, *then* $g(2) = 0$ *and* $g = f = \chi_2$.

(*E*4.2) *If* $f(2) = 1$, *then* $f = g = \mathbb{E}$.

(*E*4.3) *If* $f(2) = \frac{A+B}{B} \notin \{0, 1\}$, *then* $A = B = C$ *and* $f = g = \mathbb{I}$.

No other solutions exist.

6.3. On the equation $G(p+1) = F(p-1) + D$

In [125] we stated the following

Conjecture 6. *If* $F \in \mathcal{M}^*$ *satisfies the relation*

$$F(p+1) = F(p-1) + 2 \quad \text{for every} \quad p \in \mathcal{P},$$

then $F(n) = n$ *for every* $n \in \mathbb{N}$.

The classical theorem of Dirichlet states that any arithmetic progression $a \pmod q$ in which a and q are relatively prime contains infinitely many prime numbers. A natural question to ask is then, how big is the first such prime, $P(a, q)$ say? The theorem is named after Yuri Vladimirovich Linnik, who proved in 1944 that there are constants c and L such that $P(a, q) < cq^L$. The constant L is called Linnik's constant and the best value of L is 5. It is also conjectured by R. Heath-Brown that $P(a, q) < q^2$.

In the following for each positive integer $n > 1$ let $P(n)$ be the largest prime divisor of n. We think that the following assertion holds.

Conjecture 7. *Let* $q \in \mathcal{P}$, $q > 2$. *Then there exists such a* $p_1 \in \mathcal{P}$, *for which* $q|(p_1 + 1)$ *and* $P\left(\frac{(p_1-1)(p_1+1)}{q}\right) < q$, *and such a* $p_2 \in \mathcal{P}$, *for which* $q|(p_2 - 1)$ *and* $P\left(\frac{(p_2-1)(p_2+1)}{q}\right) < q$.

It proved that from Conjecture 7 implies Conjecture 6.

Theorem 21. (I. Kátai and B. M. Phong [125]) *Assume that Conjecture* 7 *is true. Let* $G, F \in \mathcal{M}^*$, $D \in \mathbb{C}$ *and*

$$G(p+1) = F(p-1) + D \quad \text{for every} \quad p \in \mathcal{P}.$$

Then we have

$$(G(2), F(2), D) = \{(0,0,0), (0,1,-1); (1,1,0), (2,2,2)\},$$

furthermore

$(F1)$ *If* $(G(2), F(2), D) = (0,0,0)$, *then* $G(3) = 1$ *and* $F(n)$ *is arbitrary if* $(n,2) = 1$, $G(n)$ *is arbitrary if* $(n,6) = 1$.

$(F2)$ *If* $(G(2), F(2), D) = (0,1,-1)$, *then* $G(n)$ *is arbitrary for every odd* $n > 1$, *and* $F(p-1) = 1$ *for every* $p \in \mathcal{P}$, *consequently* $F(n)^\tau = 1$ $(n \in \mathbb{N})$ *for some* $\tau \in \{1,2,3\}$.

$(F3)$ *If* $(G(2), F(2), D) = (1,1,0)$, *then* $G = F = \mathbb{E}$.

$(F4)$ *If* $(G(2), F(2), D) = (2,2,2)$, *then* $G = F = \mathbb{I}$.

6.4. The equation $G(n) = F(n^2 - 1) + D$

For each $\omega \in \mathbb{C}$ with $\omega^3 = -1$ we define the function $\Psi_\omega : \mathbb{N} \to \mathbb{C}$ such that

$$\Psi_\omega(n) = \begin{cases} 0 & \text{if} \quad 3|n \\ \omega^{\frac{2(n-1)}{3}} & \text{if} \quad n \equiv 1 \pmod 3 \\ \omega^{\frac{4(n-2)}{3}+1} & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

One can check that $\Psi_\omega \in \mathcal{M}^*$ and $\Psi_{-1} = \chi_3^*$.

Theorem 22. (I. Kátai, B. M. M. Khanh and B. M. Phong [124]) *Let* $\mathbb{N}_1 := \mathbb{N} \setminus \{1\}$. *Assume that* $D \in \mathbb{C}$ *and the functions* $G, F \in \mathcal{M}^*$ *satisfy the relation*

$$G(n) = F(n^2 - 1) + D \quad \textit{for every} \quad n \in \mathbb{N}_1.$$

Then the following assertions hold.

$(G1)$ *If* $D = 0$ *and* $F(3) \neq 0$, *then* $G = F = \mathbb{E}$,

$(G2)$ *If* $D = 0$ *and* $F(3) = 0$, *then* $G = \mathbb{O}$ *and* $F(2) = F(3) = F(5) = F(7) = 0$, $F(n^2-1) = 0$ *for every* $n \in \mathbb{N}_1$,

$(G3)$ *If* $D \neq 0$, $F(2) = F(3) = 0$, *then* $D = 1$, $G = \mathbb{E}$ *and* $F(n^2 - 1) = 0$ *for every* $n \in \mathbb{N}_1$,

$(G4)$ *If* $D \neq 0, F(2) = 0$ *and* $F(3) \neq 0$, *then* $(D, G, F) = (1, \chi_2, \chi_4^*)$,

$(G5)$ *If* $D \neq 0, \omega = F(2) \neq 0$ *and* $F(3) = 0$, *then* $\omega^3 = -1$ *and* $(D, G, F) = (1, \chi_3, \Psi_\omega)\}$,

$(G6)$ *If* $D \neq 0$ *and* $F(2)F(3) \neq 0$, *then* $(D, G, F) = \{(-1, \mathbb{O}, \mathbb{E}), (1, \mathbb{I}^2, \mathbb{I})\}$.

We note that if $\mathcal{F}$ is the set of all $F \in \mathcal{M}^*$, for which $F(n^2 - 1) = 0$ holds for every $n \in \mathbb{N}_1$, then $|\mathcal{F}| = \infty$ (see [124]).

6.5. The equation $F(n^3) = F(n^3 - 1) + D$

Theorem 23. (I. Kátai, B. M. M. Khanh and B. M. Phong [129]) *Assume that* $D \in \mathbb{C}$ *and the function* $F \in \mathcal{M}^*$ *satisfy the relation*

$$F(n^3) = F(n^3 - 1) + D \quad \textit{for every} \quad n \in \mathbb{N}_1.$$

Then

$$(D, F) \in \{\ (0, \mathbb{O}),\ (0, E),\ (1, \mathbb{I})\}.$$

It follows from Theorem 23 that if the number $D \in \mathbb{C} \setminus \{0\}$ and the function $F \in \mathcal{M}^*$ satisfy $F(n^3) = F(n^3 - 1) + D$ for every $n \in \mathbb{N}_1$, then $(D, F) = (1, \mathbb{I})$.

We are unable to find the solutions of

$$F(n^3) = AF(n^3 - 1) + B \quad \text{for every} \quad n \in \mathbb{N}_1$$

if $A, B \in \mathbb{C}$ and $F \in \mathcal{M}^*$. We think that the following assertions hold:

Conjecture 8. *If* p *is a prime,* C *is a nonzero complex number and the function* $F \in \mathcal{M}^*$ *satisfy*

$$F(p) = 0, \quad F(pm + 1)^3 = 1 \quad \textit{and} \quad F(p^3m^3 - 1) = C \neq 0 \quad \textit{for every} \quad m \in \mathbb{N},$$

then $C = 1$ *and* $F = \chi_p$*, where* χ_p *is a Dirichlet character* (mod p).

Conjecture 9. *If the function* $F \in \mathcal{M}^*$ *satisfies*

$$F(n^4 - 1) = 1 \quad \textit{for every} \quad n \in \mathbb{N}_1,$$

then $F = \mathbb{E}$.

Conjecture 10. *If the function* $F \in \mathcal{M}^*$ *satisfies*

$$F(n^3 + 1) = 1 \quad \textit{for every} \quad n \in \mathbb{N},$$

then $F = \mathbb{E}$.

7. Other results

J.-M. De Koninck, I. Kátai and B. M. Phong recently published six joint papers ([106], [108], [109], [113], [115], [123]) regarding families of multiplicative functions which are characterised by their behaviour at consecutive arguments. The results of these papers can find in Section 6 of Laudation to Professor Jean-Marie De Koninck.

Similarly K.-H. Indlekofer, I. Kátai and B. M. Phong has two joint papers ([110], [111]), for which we refer to Section 3 of Laudation to Professor K-H. Indlekofer on his 80th birthday.

CONTINUATION OF LIST OF PUBLICATIONS

Bui Minh Phong

The first part appeared in this journal **38** (2012) 13–17, the second part appeared in **47** (2018) 97–99.

[104] On additive arithmetical functions with values in topological groups. III, *Publ. Math. Debrecen*, **94(1-2)** (2019), 49–54. (with I. Kátai)

[105] On the pairs of completely multiplicative functions satisfying some relation, *Acta Sci. Math. Szeged*, **85(1-2)** (2019), 139–145. (with I. Kátai)

[106] On some consequences of recently proved conjectures, *Annales Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 123–128. (with J.-M. De Koninck and I. Kátai)

[107] On additive arithmetical functions with values in topological groups IV., *Annales Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 279–284. (with I. Kátai)

[108] Three new conjectures related to the values of arithmetic functions at consecutive integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 425–427. (with J.-M. De Koninck and I. Kátai)

[109] Characterising some triplets of completely multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 101–112. (with J.-M. De Koninck and I. Kátai)

[110] A remark on uniformly distributed functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 167–172. (with K.-H. Indlekofer, I. Kátai and O. Klesov)

[111] On some uniformly distributed functions on the set of shifted primes, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 173–181. (with K.-H. Indlekofer and I. Kátai)

[112] On q-multiplicative functions with special properties, *Annales Univ. Sci. Budapest., Sect. Comp.*, **51** (2020), 179–190. (with R. B. Szeidl)

[113] On the variations of completely multiplicative functions at consecutive arguments, *Publ. Math. Debrecen*, **98(1-2)** (2021), 219–230. (with J.- M. De Koninck and I. Kátai)

[114] Arithmetical functions commutable with sums of squares, *Notes on Number Theory and Discrete Mathematics*, **27(3)** (2021), 143–154. (with I. Kátai)

[115] Almost-additive and almost multiplicative functions with regularity properties, *Publ. Math. Debrecen*, **99** (2021), 151–160. (with J.-M. De Koninck and I. Kátai)

[116] On a conjecture regarding multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **52** (2021), 187–194. (with I. Kátai)

[117] On the equation $f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m)$, *Annales Univ. Sci. Budapest., Sect. Comp.*, **52** (2021), 255–278. (with R. B. Szeidl)

[118] A characterization of functions using Lagrange's Four-Square Theorem, *Annales Univ. Sci. Budapest., Sect. Comp.*, **52** (2021), 177–185. (with I. Kátai)

[119] Arithmetical functions commutable with sums of squares II., *Mathematica Pannonica New Series*, **27/NS1**, Issue 2, (2021), 179–190. (with I. Kátai)

[120] On the equation $f(n^2 - Dnm + m^2) = f^2(n) - Df(n)f(m) + f^2(m)$, *Notes on Number Theory and Discrete Mathematics*, **28(2)** (2022), 240–251. (with R. B. Szeidl)

[121] On the equation $F(n^2 + m^2 + k) = H(n) + H(m) + K$, *J. Math. Math. Sci.*, **1(4)** (2022), 135–151. (with I. Kátai and B. M. M. Khanh) https://science.thanglong.edu.vn/index.php/volc

[122] In memoriam for Professor Eduard Wirsing, *Annales Univ. Sci. Budapest., Sect. Comp.*, **53** (2022), 123–135. (with I. Kátai)

[123] Expanding on results of Wirsing and Klurman, *Annales Univ. Sci. Budapest., Sect. Comp.*, **53** (2022), 137–143. (with J.-M. De Koninck and I. Kátai)

[124] On the equation $G(n^2) = F(n^2 - 1) + D$, *Annales Univ. Sci. Budapest., Sect. Comp.*, **53** (2022), 159–173. (with I. Kátai and B. M. M. Khanh)

[125] Completely multiplicative functions with special properties, *Mathematica Pannonica New Series*, **29/NS3**, Issue 1, (2023), 68–76. (with I. Kátai)

[126] On the multiplicative group generated $\left\{\frac{[\sqrt{2}n]}{n} \mid n \in \mathbb{N}\right\}$ V., *Notes on Number Theory and Discrete Mathematics*, **29(2)** (2023), 348–353. (with I. Kátai)

[127] On a characterisation of completely additive functions, *Publ. Math. Debrecen*, accepted. (with J.-M. De Koninck and I. Kátai)

[128] Results and conjectures relates to the works of M. V. Subbarao, In: *Proceedings of the Centenary Symposium in honor of Professor M.V. Subbarao*, accepted. (with J.-M. De Koninck and I. Kátai)

[129] On the equation $F(n^3) = F(n^3 - 1) + D$ and some conjectures, *Publ. Math. Debrecen*, **103(1-2)** (2023), 257–267. (with I. Kátai and B. M. M. Khanh)