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CONTINUATION OF THE LAUDATION TO
Professor Karl-Heinz Indlekofer on his 80th birthday

by Imre Kátai (Budapest, Hungary)

On the last five years many important events have happened with him:

- Three new grandchildren were born: Matthea Luna Emilia (in 2019; daughter of Dorothee), Niclas Karl and Romy Marie (in 2020, twin children of Thomas).

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-Irmgard and Karl-Heinz celebrated their golden wedding (in 2022).

- He obtained Golden doctoral certificate from the University of Freiburg for special scientific merits (2020).

- He continued his work in the church and in number theory.

1. Orthonormal systems in spaces of number theoretical function ([158])

For a function $f:\mathbb{N}\to\mathbb{C}$ we define $\|.\|_\alpha$ by

$$\|f\|_\alpha := \left\{\limsup_{x\to\infty}\frac{1}{x}\sum_{n\le x}|f(n)|^\alpha\right\}^{\frac{1}{\alpha}},\ 1\le\alpha<\infty.$$

Let

$$\mathcal{L}^\alpha := \{f:\mathbb{N}\to\mathbb{C} : \|f\|_\alpha<\infty\}$$

be the linear space of functions on $\mathbb{N}$ with bounded seminorm $\|f\|_\alpha$. By L^α we denote the quotient space $\mathcal{L}^\alpha$ modulo null-functions (i.e functions f with $\|f\|_\alpha=0$). For $\alpha\ge 1$, the norm space L^α is complete.

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$. Then

$$\mathcal{E}(\mathcal{A}) := \{s\in\mathcal{E}, s=\sum_{j=1}^{m}\alpha_j 1_{A_j};\ \alpha_j\in\mathbb{C},\ A_j\in\mathcal{A},\ j=1,...,m;\ m\in\mathbb{N}\}$$

denotes the space of simple functions on $\mathcal{A}$.

Definiton 1. For a given algebra $\mathcal{A}$ and for $1\le\alpha<\infty$ the space $\mathcal{L}^{*\alpha}(\mathcal{A})$ is defined as the $\|\cdot\|_\alpha$-closure of $\mathcal{E}(\mathcal{A})$. A function $f\in\mathcal{L}^{*\alpha}(\mathcal{A})$ is called *uniformly* $(\mathcal{A})-\alpha$ *summable*. By $L^\alpha(\mathcal{A})$ we denote the quotient space $\mathcal{L}^{*\alpha}(\mathcal{A})$ modulo null functions.

Remark 1. If $\mathcal{A}=\mathcal{P}(\mathbb{N})$ is the algebra of *all* subsets of $\mathbb{N}$ then

$$\mathcal{L}^{*1}(\mathcal{A}) = \|.\|_1\ \ \textit{closure of}\ \ l^\infty$$

is the space $\mathcal{L}^*$ of *uniformly summable functions* introduced by K.-H. Indlekofer (*Math. Z.*, **172** (1980), 255–271).

Here we consider algebras $\mathcal{A}$, where every $A\in\mathcal{A}$ possesses an *asymptotic density* $\delta(A)$ which is defined by

$$\delta(A) := \lim_{n\to\infty}\frac{1}{n}\sum_{\substack{m\le n\\ m\in A}}1,$$

if the limit exists. Then δ is *finitely additive* on $\mathcal{A}$, i.e. δ is a *content* on $\mathcal{A}$.

We say that an arithmetical function f possesses an (*arithmetical*) *mean value* $M(f)$ if

$$M(f) := \lim_{n\to\infty}\frac{1}{n}\sum_{m\le n}f(m)$$

exists. If every $A \in \mathcal{A}$ possesses an asymptotic density then every $f \in \mathcal{L}^{*1}(\mathcal{A})$ possesses a mean-value. Further, we define an inner-product on $L^{*2}(\mathcal{A})$ by

$$\langle f, g\rangle := M(f\bar{g}), \quad f, g \in L^{*2}(\mathcal{A}).$$

Remark 2. The above described construction of $\mathcal{L}^{*\alpha}(\mathcal{A})$ was the starting point of an integration theory by K.-H. Indlekofer (see *Probability Theory and Applications*, Math. Appl., **80** (1992), 299–308 and *Advanced Studies in Pure Mathematics*, **49** (2005), 133–170).

Embedding $\mathbb{N}$, endowed with the discrete topology, in the compact space $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, we get:

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\} \ \text{ where } \ \bar{A} := clos_{\beta\mathbb{N}} A,$$

is an algebra in $\beta\mathbb{N}$ (for details see *Probability Theory and Applications*, Math. Appl., **80** (1992), 299–308 and *Advanced Studies in Pure Mathematics*, **49** (2005), 133-–170).

Let δ be a content on $\mathcal{A}$, i.e. $\delta : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is finitely additive, and define $\bar{\delta}$ on $\bar{\mathcal{A}}$ by

$$\bar{\delta}(\bar{A}) = \delta(A), \quad \bar{A} \in \bar{\mathcal{A}},$$

then $\bar{\delta}$ is a pseudo-measure on $\bar{\mathcal{A}}$ and can be extended to a measure on $\sigma(\bar{\mathcal{A}})$, which we denote by $\bar{\delta}$, too. This leads to the measure space $(\beta\mathbb{N}, \sigma(\bar{\mathcal{A}}), \bar{\delta})$.

2. Some Hilbert spaces and corresponding orthonormal systems ([158])

2.1. A simple case

Let $\mathcal{A}_0$ be the algebra generated by the sets

$$A_p := \{n \in \mathbb{N} : \ p|n \ \}, \quad p \ \ prime$$

and put

$$\delta(A_p) := M(1_{A_p}) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{m \leq n \\ p|m}} 1 = \frac{1}{p}.$$

Note that the following relations of the characteristic functions

$$1_{A\cap B} = 1_A \cdot 1_B, \quad 1_{A\setminus B} = 1_A - 1_A \cdot 1_B \quad 1_{A\cup B} = 1_A + 1_B - 1_A \cdot 1_B$$

imply that the characteristic function of a set $A \in \mathcal{A}$ is a finite linear combination of products of $1_{A_{p_1}} \cdots 1_{A_{p_r}}$. Thus the asymptotic density $\delta(A)$ exists for all $A \in \mathcal{A}_0$.

For every prime p put
$$h_p := p1_{A_p} - 1$$
and define $h_n : \mathbb{N} \to \mathbb{Z}$ by $h_n = 1$ for $n = 1$ and
$$h_n := \prod_{p|n} h_p \quad for \;\; every \;\; square \;\; free \;\; n \in \mathbb{N}.$$
Then obviously, for every prime p,
$$M(h_p) = 0 \;\text{ and }\; M(h_p^2) = p - 1.$$

Now, if $f : \mathbb{N} \to \mathbb{C}$ is such that $M(f)$ exists and $f(pm) = f(m)$ for all $m \in \mathbb{N}$, we conclude
$$\sum_{m \le x} h_p(m) f(m) = p \sum_{pm \le x} f(pm) - \sum_{m \le x} f(m) = p \sum_{m \le \frac{x}{p}} f(m) - \sum_{m \le x} f(m)$$
and $M(h_p f) = 0$, i.e.
$$M(h_n) = 0 \;\; if \;\; \mu^2(n) = 1, \;\; n > 1$$
and
$$M(h_n h_{n'}) = 0 \;\; if \;\; \mu^2(n) = \mu^2(n') = 1 \;\; and \;\; n \ne n'.$$
In the same way we obtain
$$M(h_p^2 f) = (p-1)M(f).$$
By induction, this leads to
$$M(h_n^2) = \varphi(n) \;\; if \;\; \mu^2(n) = 1.$$
Putting $h_n^* := \frac{1}{(\varphi(n))^{1/2}} h_n \quad (\mu^2(n) = 1$, we have shown the following:

Theorem 1. *The set*
$$\{h_n^* \; : \; n \;\text{ squarefree}\}$$
is a complete orthonormal system for $L^{*2}(\mathcal{A}_0)$.

2.2. Almost even functions

For primes p and $k = 0, 1, 2, \ldots$ let
$$A_{p^k} := \{n \in \mathbb{N} : \; p^k | n\}$$

be the set of natural numbers divisible by p^k. Let $\mathcal{A}_1$ be the algebra generated by the sets $\{A_{p^k}\}$. Then, for all A_{p^k} the asymptotic density $\delta(A_{p^k})$ exists and equals $\frac{1}{p^k}$, and, as above, the asymptotic density $\delta(A)$ exists for all $A \in \mathcal{A}_1$.

Define, $h_n = 1$for $n = 1$ and

$$h_n := \prod_{p^k || n} h_{p^k} \quad for \quad n > 1,$$

Putting

$$h_n^* = \frac{1}{(\varphi(n))^{1/2}} h_n$$

where φ is Euler's function, it is easy to show (see above) that $\{h_n^*\}$ is an orthonormal system. We conclude

Theorem 2. *The set*

$$\{h_n^* : n \in \mathbb{N}\}$$

is a complete orthonormal system for $L^{*2}(\mathcal{A}_1)$.

2.3. Limit periodic functions

Let $\mathcal{A}_2$ be the algebra generated by all residue classes

$$A_{a,r} := \{n \in \mathbb{N} : n \equiv a \bmod r\}, \quad 1 \le a \le r, \quad r \in \mathbb{N}.$$

Here again the asymptotic density δ is a finite additive function on $\mathcal{A}_2$. Then the following lemma holds.

Lemma 1. $\mathcal{E}(\mathcal{A}_2)$ *is the space of all periodic functions on* $\mathbb{N}$.

The space $L^{*\alpha}(\mathcal{A}_2)$ is the space of α*-limit-periodic functions.*

Defining $e_{a/r} : \mathbb{N} \to \mathbb{C}$ by

$$e_{a/r}(n) := \exp\left(2\pi i \frac{a}{r} n\right)$$

we have (see *Arithmetical Functions*, Cambridge University Press, Cambridge, 1994, p.207)

Theorem 3. *The set*

$$\{e_{a/r} : 1 \le a \le r, gcd(a,r) = 1, r = 1, 2, ...\}$$

is a complete orthonormal system in $L^{*2}(\mathcal{A}_2)$.

2.4. Almost periodic functions

For $\beta \in \mathbb{R}$ the function $e_\beta : \mathbb{N} \to \mathbb{C}$ defined by

$$e_\beta(n) = \exp(2\pi i \beta n), \quad n \in \mathbb{N}$$

possesses a mean-value $M(e_\beta)$.

Let $\mathcal{C}$ be the family of all half-open subsets of $[0,1]$ and denote by $\mathcal{A}_3$ the algebra generated by the sets $A(\beta, E) := \{n \in \mathbb{N} : \{\beta n\} \in E\}$, where $\beta \in [0,1)$, $E \in \mathcal{C}$ and $\beta n = [\beta n] + \{\beta n\}$ $(0 \leq \beta n < 1)$. Then (see *Arithmetical Functions*, Cambridge University Press, Cambridge, 1994, p.207) we have the following:

Theorem 4. *The set*

$$\{e_\beta : \beta \in [0,1]\}$$

is a complete orthonormal system in $\mathcal{L}^{*2}(\mathcal{A}_3)$.

2.5. Almost multiplicative functions

Let f be a multiplicative function which assumes only the values $\{-1, 0, 1\}$ and define the sets

$$A_f^+ = \{n : f(n) = 1\}, \quad A_f^0 = \{n : f(n) = 0\} \quad \text{and} \quad A_f^- = \{n : f(n) = -1\}$$

with characteristic functions f^+, f^0, and f^-, respectively. Obviously

$$f^+ = \frac{1}{2}(|f| + f), \quad f^0 = 1 - f^+ - f^-, \quad f^- = \frac{1}{2}(|f| - f).$$

We define the algebra $\mathcal{A}_4$ to be the algebra generated by the sets A_f^+, A_f^0, A_f^- for all multiplicative f with $f(\mathbb{N}) \subset \{-1, 0, 1\}$. Every $A \in \mathcal{A}_4$ possesses an asymptotic density by Wirsing's theorem. An arbitrary element A of $\mathcal{A}_4$ has a characteristic function which is a linear combination of such multiplicative functions. Thus, the asymptotic density $\delta(A)$ exists. Let $\mathcal{E}(\mathcal{A}_4)$ be the vector space of simple functions on $\mathcal{A}_4$. Let $\mathcal{L}^{*\alpha}(\mathcal{A}_4)$ be the $\|.\|_\alpha$ -closure of $\mathcal{E}(\mathcal{A}_4)$.

Definition 2. A function $f \in \mathcal{L}^{*\alpha}(\mathcal{A}_4)$ is called an *α-almost multiplicative* function.

First we show $\mathcal{A}_1 \subset \mathcal{A}_4$. For the proof consider

$$f^*(n) := (1 - 1_{A_{p^k}})(n) = \begin{cases} 0, & p^k | n; \\ 1, & \text{otherwise.} \end{cases}$$

Then f^* is multiplicative. Since $1_{\mathbb{N} \setminus A_{p^k}} = 1_\mathbb{N} - 1_{A_{p^k}} = 1 - 1_{A_{p^k}} \in \mathcal{E}(\mathcal{A}_4)$ we have $\mathbb{N} \setminus A_{p^k} \in \mathcal{A}_4$. This implies that $A_{p^k} \in \mathcal{A}_4$.

Since $h_n \in \mathcal{E}(\mathcal{A}_1)$ we have $h_n \in \mathcal{E}(\mathcal{A}_4)$. Every h_n can be written as a finite linear combination of $1_{A_{p_1^{\alpha_1}}} \cdots 1_{A_{p_m^{\alpha_m}}}$, where $m \in \mathbb{N}$.

Theorem 5. *Let $f : \mathbb{N} \to \mathbb{R}$ be multiplicative with $|f| \leq 1$. Then $f \in \mathcal{L}^{*\alpha}(\mathcal{A}_4)$ for all $\alpha \geq 1$.*

Next, we construct an orthonormal system for the space $L^{*2}(\mathcal{A}_4)$.

Let $\mathcal{R}_0$ be the set of all multiplicative functions with $f(\mathbb{N}) \subset \{-1, 0, 1\}$ and $M(|f|) \neq 0$. Define the relation $\sim$ on $\mathcal{R}_0$ by

$$f \sim g \ \text{ if and only if } \sum_{\substack{p \\ f(p) \neq g(p)}} \frac{1}{p} < \infty.$$

Observe, that in this case $\sum_{\substack{p \\ f(p)=0}} \frac{1}{p} < \infty$. Obviously $\sim$ is an equivalence relation on $\mathcal{R}_0$.

Now, choose a representative from each residue class which assumes only the values ± 1, and denote this set by $\mathcal{F}_1$. Then $\mathcal{F}_1$ forms an orthonormal system. For this, let $f, g \in \mathcal{F}_1$ and observe that $\sum_{\substack{p \\ f(p) \neq g(p)}} \frac{1}{p} = \infty$. Then $M(f\bar{g}) = \langle f, g \rangle = 0$. Furthermore, for $f \in \mathcal{F}_1$ we have $f^2 = 1$ and $\langle f, f \rangle = M(f^2) = 1$.

This shows that $\mathcal{F}_1$ is an orthonormal system in $L^{*2}(\mathcal{A}_4)$. Consider, for $f \in \mathcal{F}_1$, the system

$$\mathcal{F}_2 := \{ f h_n^* : f \in \mathcal{F}_1, \ n \in \mathbb{N} \}$$

where h_n^* is the normalized function of Theorem 2.

Theorem 6. *$\mathcal{F}_2$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_4)$.*

2.6. q-ary almost even functions

First, we introduce *q-multiplicative* functions. Let $q \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \ldots, q-1\}$. The *$q$-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \ldots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in \mathbb{A}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \ldots$ are called the *digits* in the q-ary expansion of n. In fact, $\varepsilon_r(n) = 0$ if $r > \log n / \log q$. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called *q-multiplicative* if

$f(0) = 1$ and for every $n \in \mathbb{N}_0$

$$f(n) = \prod_{j=0}^{\infty} f(\varepsilon_j(n)q^j).$$

Let the algebra $\mathcal{A}_5$ be generated by the sets

$$A_{j,a} := \{n \in \mathbb{N} : \varepsilon_j(n) = a\}$$

where $j \in \mathbb{N}_0, a \in \mathbb{A}$. Every $A \in \mathcal{A}_5$ possesses an asymptotic density $\delta(A)$.

Let $\mathcal{L}^{*1}(\mathcal{A}_5)$ be the $\|.\|_1$-closure of $\mathcal{E}(\mathcal{A}_5)$. Here $\mathcal{E}(\mathcal{A}_5)$ is called the space of *q-ary even functions*. Then $\mathcal{L}^{*1}(\mathcal{A}_5)$ is denoted as the space of *q-ary almost even* functions.

Remark 3. Let f be a real-valued q- multiplicative function of modulus ≤ 1. Then the mean-values $M(|f|)$ and $M(f)$ always exist (see *Annales Univ. Sci. Budapest., Sect. Comp.*, **25** (2005), 171–294). Especially we have

(i) If $\|f\|_1 = M(|f|) > 0$ then

$$\sum_{j=0}^{\infty} \sum_{a \in \mathbb{A}} (1 - |f(aq^j)|) < \infty.$$

(ii) If

$$\sum_{a \in \mathbb{A}} f(aq^j) \neq 0 \; for \; all \; j \in \mathbb{N}_0 \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{a \in \mathbb{A}} (1 - f(aq^j)) < \infty,$$

then $M(f) \neq 0$.

As an immediate consequence we have

Corallary 1. *Let f be a real-valued q- multiplicative function of modulus ≤ 1. If*

$$\sum_{j=0}^{\infty} \sum_{a \in \mathbb{A}} (1 - f(aq^j)) < \infty$$

then $f \in \mathcal{L}^{*1}(\mathcal{A}_5)$.

This ends Remark 3.

Let $\mathcal{L}^{*2}(\mathcal{A}_5)$ be the $\|.\|_2$-closure of $\mathcal{E}(\mathcal{A}_5)$. Then we define a complete orthonormal system for the space $L^{*2}(\mathcal{A}_5)$.

Theorem 7. *The set* $\{h_{a_0,\dots,a_r}\}$ *of* q*-multiplicative functions with*

$$h_{a_0,\dots,a_r}(n) := \prod_{j=0}^{r} \exp\left(\frac{2\pi i a_j}{q}\varepsilon_j(n)\right),$$

$a_j \in \mathbb{A}, j = 0,\dots,r,\ r \in \mathbb{N}_0$ *is a complete orthonormal system for* $L^{*2}(\mathcal{A}_5)$.

2.7. Almost q-multiplicative functions

Let f be a q-multiplicative function which assumes only the values $\{-1, 0, 1\}$ and define the sets

$$A_f^+ = \{n : f(n) = 1\}, \quad A_f^0 = \{n : f(n) = 0\} \text{ and } A_f^- = \{n : f(n) = -1\}$$

with characteristic functions f^+, f^0, and f^-, respectively. We define the algebra $\mathcal{A}_6$ to be algebra generated by the sets A_f^+, A_f^0, A_f^- for all q-multiplicative f with $f(\mathbb{N}) \subset \{-1, 0, 1\}$.

An arbitrary element A of $\mathcal{A}_6$ has a characteristic function which is a linear combination of q- multiplicative functions. From this and by the theorem of Delange (*Acta Arith.*, **21** (1972),285–298) the asymptotic density $\delta(A)$ exists. Let $\mathcal{E}(\mathcal{A}_6)$ be the space of simple functions on $\mathcal{A}_6$. Let $\mathcal{L}^*(\mathcal{A}_6)$ be the $\|.\|_1$ -closure of $\mathcal{E}(\mathcal{A}_6)$.

Definition 3. A function $f \in \mathcal{L}^*(\mathcal{A}_6)$ is called *almost q-ary multiplicative* functions.

Next, we define a complete ortonormal system for $L^{*2}(\mathcal{A}_6)$. Let

$$\mathcal{G} := \{f : \mathbb{N} \to \mathbb{R} : f\ q-multiplicative,\ f(n) \in \{-1, 0, 1\},$$
$$for\ all\ n \in \mathbb{N}\ with\ \|f\|_2 \neq 0\}.$$

Define the relation $\sim$ on $\mathcal{G}$ by

$$f \sim g \text{ if and only if } \sum_{l=0}^{\infty}\sum_{a=0}^{q-1}(1 - f(aq^l)g(aq^l)) < \infty.$$

Obviously, $\sim$ is an equivalence relation on $\mathcal{G}$.

Now, we choose a representative from each equivalence class which is $\neq 0$ for all $n \in \mathbb{N}$. We denote this set of representatives by $\mathcal{F}_3$. We consider

$$\mathcal{F}_4 := \{h_{a_0,\dots,a_r} f :\ f \in \mathcal{F}_3,\ a_j \in A,\ j = 0,\dots,r;\ r \in \mathbb{N}\}$$

and show the following

Theorem 8. *$\mathcal{F}_4$ is a complete orthonormal system for $L^{*2}(\mathcal{A}_6)$.*

3. On uniformly distributed functions ([156])

Let $g(n) > 0$ for every $n \in \mathbb{N}$, $S(x) = \sum_{ng(n)\le x} 1$. In [156] it is proved that

$$(*) \qquad \lim \frac{S(x)}{x} = C$$

holds with some positive constant C, if g has a continuous limit distribution F, for which

$$\int_0^\infty u^{-2} F(u) du < \infty,$$

and that

$$\sup_{u\in\mathbb{R}} |F_x(u) - F(u)| \ll \frac{1}{(\log\log x)^2},$$

where

$$F_y(u) = \tfrac{1}{x} \sharp\{n \le x \ : \ |g(n)| < u\}.$$

Let furthermore $g(n)(\log\log n) \ge C_1$ if $n \ge 10$, C_1 is a positive constant. Then

$$C = \int_0^\infty \frac{F(u)}{u^2} du.$$

In [157] the uniform distribution of function is extended for the set of shifted primes, and the following theorem is proved.

Let g be a multiplicative function, $g(n) \in \mathbb{R}_+$. Let $f(n) = \log n$. Assume that with some constant B

$$\frac{1}{(\log\log n)^B} \le g(n) \le (\log\log n)^B \quad \textit{if} \quad n \ge n_0$$

and that for every fixed C

$$f(q^r)(\log q^r)^C \to 0 \quad \textit{as} \quad q^r \to \infty.$$

Here q^r runs over the set of prime powers.

Under these conditions, we have

$$\lim \tfrac{1}{li\, x} \sharp\{p \le x | (p+1)g(p+1) \le x\} = A,$$

where A is explicitly countable constant.

4. On uniformly summable functions and a problem of Halmos in ergodic theory ([155])

Let $(X, \mathcal{F}, \mu)$ be a measure space together with a measure-preserving transformation T, i.e. $T : X \to X$ and

$$\int\limits_X f(T^n)a)d\mu = \int\limits_X f(x)d\mu$$

for all $n = 0, 1, 2, \cdots$ and all $f \in L_1 := L_1(X, \mathcal{F}, \mu)$

Then P. Halmos (Lectures on Ergodic Theory, Chelsea Publ. Comp,, New York, (1956)) formulates Birkhoff's ergodic theorem as follows.

Individual Ergodic Theorem. If T is a measure preserving transformation and if $f \in L_1$, then $\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$ converges almost everywhere. The limit function f^* is integrable and invariant (i.e. $f^*(Tx) = f^*(x)$ almost everywhere). If $\mu(X) < \infty$, then $\int_X f^*(x)d\mu = \int_X f(x)d\mu$.

In his "Comments on the Ergodic Theorem" (see Lectures on Ergodic Theory, pp. 22–24) Halmos writes

I cannot resist the temptation of concluding these comments with an alternative "proof" of the ergodic theorem. If f is a complex-valued function on the non-negative integers, write $\int f(n)dn = \lim \frac{1}{n}\sum_{j=0}^{n-1} f(j)$ whenever the limit exists, and call such functions integrable. If T is a measure-preserving transformation on a space X and f is an integrable function on X, then

$$\int\int |f(T^n X)|dndx = \int\int |f(T^n x)|dxdn =$$

$$= \int\int |f(x)|dxdn = \int\int |f(x)|dx < \infty.$$

Hence, by "Fubini's theorem", $f(T^n x)$ is an integrable function of its two arguments and therefore, for almost every fixed x, it is an integrable function of n. Can any of this nonsense be made meaningful?

We assume $\mu(X) < \infty$ and define, for $f \in L_1$, the arithmetic function f_x by

$$f_x(n) = f(T^{n-1}x).$$

Then we show, that f_x lies in the space $\mathcal{L}^*$ of uniformly summable functions for almost all $x \in X$.

Next, the linear functional Λ on the vector space

$$\{s \in \mathcal{L}^\infty : s(n) = const \quad \text{for every} \quad n \in \mathbb{N}\},$$

defined by

$$\Lambda(s) = \lim_{N\to\infty} N^{-1} \sum_{n\leq N} s(n)$$

can be extended to a linear functional Λ^* on $\mathcal{L}^*$. We show, that, for all $f \in \mathcal{L}^*$, $\Lambda^*(f)$ can be written as an integral and solve the problem formulated by Halmos.

CONTINUATION OF LIST OF PUBLICATIONS

Karl-Heinz Indlekofer

The previous parts of his publications is published in this journal **22** (2003) 15–22, **39** (2013) 13–16 and **47** (2018) 45–46.

[153] Remarks on number theory over additive arithmetical semigroup, *Ukraïn. Mat. Zh.*, **72(3)** (2020), 371–390; reprinted in *Ukrainian Math. J.*, **72(3)** (2020), 422–445. (with E. Kaya)

[154] On the uniform distribution and uniform summability of positive valued multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 249–258.

[155] On uniformly summable functions and a problem of Halmos in ergodic theory, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 157–166.

[156] A remark on uniformly distributed functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 167–172. (with I. Kátai, O.I. Klesov and B.M. Phong)

[157] On some uniformly distributed functions on the set of shifted primes, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 173–181. (with I. Kátai and B.M. Phong)

[158] Orthonormal systems in spaces of number theoretical functions, *Lithuanian Math. J.*, **61** (2021), 373–381. (with E. Kaya and R. Wagner)

[159] A new proof of a result of Wirsing for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **55** (2023), (accepted).

Professor Karl-Heinz Indlekofer as governor of Tempus project in between the University of Paderborn (Germany) and some hungarian universities.

According to the decision of the European Economical Commission in 1990 the TEMPUS was established. It was a very fruitful program which supported the modernization of higher education in between countries in eastern Europa and countries in EU.

In the frame of TEMPUS I, II many projects were established between the university of Paderborn and hungarian universities. Professor Indlekofer was the leader of these projects. Appreciating his outstanding services and research activities the hungarian universities honored him with the following awards:

Order of Merit of Eötvös Loránd University (1992), Doctor Honoris Causa of Kossuth Lajos University, Debrecen (1992), Doctor Honoris Causa of Janus Pannonius University, Pécs (1996), Honorary Doctor and Professor of Eötvös Loránd University (2004).

As a byproduct, he initiated and created the twin-city relationship between Paderborn and Debrecen.