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# EXTENDED CONVERGENCE ANALYSIS OF OPTIMAL EIGHTH ORDER METHOD FOR SOLVING NONLINEAR EQUATIONS

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**Abstract.** To estimate the locally unique solutions of nonlinear systems in Banach spaces, the local convergence analysis of an optimal eighth order iterative technique is accomplished in this paper. The convergence analysis was carried out, in earlier studies, using the Taylor series expansions involving derivatives up to the ninth order. But, due to the fact that such higher order derivatives may not exist always, or laborious to evaluate for larger systems, the applicability of this technique remains limited. In this regard, we provide the convergence analysis using different approach which utilizes derivatives and divided differences only up to first order. The applicability of method is extended by computing upper bounds on the convergence region and prescribing set of conditions on the existence of unique solution in that region. Numerical examples are provided for the verification of theoretical results.

## 1. Introduction

Most of the problems, in applied mathematics, occur naturally as systems of nonlinear equations. To represent this system mathematically, let us consider

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$X$ and $Y$ as Banach spaces, and $\Omega$ as an open convex subset of $X$. Further, suppose $F : \Omega \subset X \to Y$ is a Fréchet-differentiable mapping [12], then the aforesaid problem can be expressed as,

$$F(x) = 0. \tag{1.1}$$

Obtaining the closed form solution of nonlinear equations is rather complicated (see [5, 10]), but the iterative techniques provide the estimate of solution up to the desired accuracy, provided the initial estimate is sufficiently close to the solution. In the extensively available literature (see for example [5, 8, 13, 15, 16], and references therein), numerous iterative techniques have been presented in the given context.

The usual approach to estimate the convergence order of an iterative technique includes Taylor series expansions, which involve higher order derivatives ($F^{(i)},\quad i = 1, 2, ...$) and some set of assumption on $F^{(i)}$. But, such assumptions limit the applicability of techniques, since most of these involve only first order derivatives. As a matter of fact, we consider a real valued function $F : \Omega \subseteq \mathbb{R} \to \mathbb{R},\ \ \Omega = \left[-\frac{5}{2}\ , 2\right]$, defined by

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Consequently, we have

$$F'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2 \cos\left(\frac{1}{x}\right) + 2x(5 + 3\log(\pi^2 x^2)) + x(20x^2 - 1)\sin\left(\frac{1}{x}\right),$$

and

$$F'''(x) = \frac{1}{x}\left[(1 - 36x^2)\cos\left(\frac{1}{x}\right) + x\left[22 + 6\log(\pi^2 x^2) + (60x^2 - 9)\sin\left(\frac{1}{x}\right)\right]\right].$$

We note here that, $F'''(x)$ is unbounded in the given interval $\Omega$, and therefore, Taylor series expansion might not be appropriate approach for the convergence analysis of an iterative technique.

In addition, the choice of initial estimate significantly affects the rate of convergence as well as the error approximations, and in that sense, the convergence analysis of iterative techniques should include the measure of closeness of initial estimate to the solution. Many authors (see [1, 2, 3, 4, 6, 7, 9, 11, 14]) have adopted a suitable methodology to establish the local (or semilocal) convergence analysis by considering the hypothesis of Lipschitz-continuity on the first order derivatives only, and moreover, the upper bounds on errors and convergence radius be computed by defining some set of real parameters and functions. From this perspective, we shall study the local convergence analysis of an eighth order iterative technique, developed in [15], and which is described as follows,

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= \psi(x_n, y_n), \\ x_{n+1} &= z_n - T_n^{-1}[z_n, y_n; F][z_n, x_n; F]^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where '$\psi$' is a $4^{th}$ order iteration operator, and $T_n = 2[z_n, y_n; F] - [z_n, x_n; F]$. For any $x \in \Omega \subseteq \mathbb{R}^m$, $F'(x) : \Omega \to \mathbb{R}^m$ is the first Fréchet derivative [12], and $[\cdot, \cdot; F] : \Omega \times \Omega \to \mathcal{L}(X,\ Y)$ is a divided difference operator of order one, where $\mathcal{L}(X,\ Y)$ stands for the set of bounded linear mappings from $X = \mathbb{R}^m$ into $Y = \mathbb{R}^m$.

Clearly, technique (1.2) involves derivatives or divided differences of order not more than one, but the eighth order of convergence is proven in [15] using the derivatives up to order nine. Our objective is to weaken the conditions of [15] and to estimate the bounds on the convergence radius, which undoubtedly extend the applicability of the considered technique.

In what follows, analysis is developed in section 2, which includes the computation of bounds on convergence radius and defining the conditions for uniqueness of solution. Some numerical applications are given in Section 3. Section 4 contains the concluding remarks.

## 2. Analysis

To establish the local convergence analysis of technique (1.2), we first define some real parameters and functions. For Q $= [0, \infty)$, let the following suppositions (i–vi) hold.

(i) Equation:

$$v_0(t) - 1 = 0,$$

has a minimal root $\rho_0 \in Q - \{0\}$ for some continuous and non-decreasing function $v_0 : Q \to Q$. Set $Q_0 = [0,\ \rho_0)$.

(ii) Equation:

$$h_1(t) - 1 = 0,$$

has a minimal root $R_1 \in Q_0 - \{0\}$, where $h_1 : Q_0 \to Q$ is defined by

$$h_1(t) = \frac{\int_0^1 v((1-\theta)t)d\theta}{1 - v_0(t)},$$

for some continuous and non-decreasing function $v : Q_0 \to Q$.

(iii) Equation:

$$h_2(t, h_1(t)t) - 1 = 0,$$

has a minimal root $R_2 \in Q_0 - \{0\}$, for some continuous and non-decreasing function $h_2 : Q_0 \times Q_0 \to Q$.

(iv) Equation:

$$w_0(h_2(t, h_1(t)t)\ t,\ t) - 1 = 0,$$

has a minimal root $\rho_1 \in Q_0 - \{0\}$, for some continuous and non-decreasing function $w_0 : Q_0 \times Q_0 \to Q$.

(v) Equation:

$$p(t) - 1 = 0,$$

has a minimal root $\rho_2 \in Q_0 - \{0\}$, where $p : Q_0 \to Q$ is defined as,

$$p(t) = w_0(h_2(t, h_1(t)t)t, h_1(t)t) + w_1(t, h_1(t)t, h_2(t, h_1(t)t)t),$$

for some continuous and non-decreasing function $w_1 : Q_0 \times Q_0 \times Q_0 \to Q$. Set $Q_1 = [0, \rho)$, where $\rho = \min\{\rho_0, \rho_1, \rho_2\}$.

(vi) Equation:

$$h_3(t) - 1 = 0,$$

has a minimal root $R_3 \in Q_1 - \{0\}$, where $h_3 : Q_1 \to Q$ is defined as

$$h_3(t) = \left[\frac{w(h_2(t, h_1(t)t)t, t)}{1 - w_0(h_2(t, h_1(t)t)t, t)} + \frac{w_1(t, h_1(t)t, h_2(t, h_1(t)t)t)}{(1 - w_0(h_2(t, h_1(t)t)t, t))} \times \times \frac{w_2(h_2(t, h_1(t)t)t)}{(1 - p(t))}\right] h_2(t, h_1(t)t)$$

for some continuous and non-decreasing functions $w : Q_1 \times Q_1 \to Q$ and $w_2 : Q_1 \to Q$.

We shall aim to prove that:

$$R = \min\{R_1, R_2, R_3\}, \tag{2.1}$$

is the convergence radius for technique (1.2). Setting $G = [0, R)$, and by the definition of $R$, we have for all $t \in G$,

$$0 \le v_0(t) < 1, \tag{2.2}$$

$$0 \le w_0(h_2(t, h_1(t)t)t, t) < 1, \tag{2.3}$$

$$0 \le p(t) < 1, \tag{2.4}$$

and

$$(2.5) \qquad 0 \le h_1(t) < 1, \quad 0 \le h_2(t, h_1(t)t) < 1, \text{ and } 0 \le h_3(t) < 1.$$

By taking '$\alpha$' as center, we denote $\overline{B}(\alpha, R)$ as a closure of the open ball $B(\alpha, R)$ having radius equal to '$R$'. For the further analysis, '$\alpha$' is taken as the simple solution of (1.1). Before we proceed to the main result, it is required that the following conditions $(C_1 - C_5)$ hold.

$C_1$ : For each $x \in \Omega$,

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \le v_0(\|x - \alpha\|).$$

Set $\Omega_0 = \Omega \cap B(\alpha, \rho_0)$ .

$C_2$ : For each $x, y \in \Omega_0$,

$$\|F'(\alpha)^{-1}(F'(y) - F'(x))\| \le v(\|y - x\|),$$

and

$$\|\psi(x, y) - \alpha\| \le h_2(\|x - \alpha\|, \|y - \alpha\|) \max\{\|x - \alpha\|, \|y - \alpha\|\}.$$

Set $\Omega_1 = \Omega \cap B(\alpha, \rho)$ .

$C_3$ : For each $x, y, z \in \Omega_1$,

$$\|F'(\alpha)^{-1}([z, x; F] - F'(\alpha))\| \le w_0(\|z - \alpha\|, \|x - \alpha\|),$$
$$\|F'(\alpha)^{-1}([z, x; F] - [z, \alpha; F])\| \le w(\|z - \alpha\|, \|x - \alpha\|),$$
$$\|F'(\alpha)^{-1}([z, y; F] - [z, x; F])\| \le w_1(\|x - \alpha\|, \|y - \alpha\|, \|z - \alpha\|),$$

and

$$\|F'(\alpha)^{-1}[x, \alpha; F]\| \le w_2(\|x - \alpha\|).$$

$C_4$ : $\overline{B}(\alpha,\ R) \subset \Omega$.

$C_5$ : There exists $R^* \ge R$, such that

$$\int_0^1 v_0(\tau R^*) d\tau < 1.$$

Set $\Omega_2 = \Omega \cap \overline{B}(\alpha, R^*)$.

Next, Conditions $(C_1 - C_5)$ are used in the main theorem presented below.

**Theorem 2.1.** *Under the conditions* $(C_1 - C_5)$ *and further choosing* $x_0 \in$ $\in B(\alpha, R) - \{\alpha\}$, *the iterations* $\{x_n\}$, *given by technique* (1.2), *exist and remain in* $B(\alpha, R)$ *for all* $n = 1, 2, \ldots$, *with* $\lim_{n\to\infty} x_n = \alpha$, *such that*

$$\|y_n - \alpha\| \le h_1(\|x_n - \alpha\|)\ \|x_n - \alpha\| < \|x_n - \alpha\| < R, \tag{2.6}$$

$$\|z_n - \alpha\| \le h_2(\|x_n - \alpha\|, \|y_n - \alpha\|)\ \|x_n - \alpha\| < \|x_n - \alpha\|, \tag{2.7}$$

*and*

$$\|x_{n+1} - \alpha\| \le h_3(\|x_n - \alpha\|)\ \|x_n - \alpha\| < \|x_n - \alpha\|, \tag{2.8}$$

*where the functions* $h_1$, $h_2$ *and* $h_3$ *are defined previously, and* $R$ *is defined in* (2.1). *Moreover, the limit point* $\alpha$ *is the only solution of equation* $F(x) = 0$ *in the domain* $\Omega_2$ *given in* $(C_5)$.

**Proof.** Inequalities (2.6)–(2.8) shall be shown using mathematical induction on '$n$'. Let $u \in B(\alpha, R) - \{\alpha\}$ be arbitrary. In view of (2.1), (2.2), and $(C_1)$, we have in turn that

$$\|F'(\alpha)^{-1}(F'(u) - F'(\alpha))\| \le v_0(\|u - \alpha\|) < v_0(R) < 1. \tag{2.9}$$

The existence of invertible mappings in Banach spaces (see [12]), together with (2.9), imply that $F'(u)^{-1} \in \mathcal{L}(Y, X)$, so that

$$||F'(u)^{-1}F'(\alpha)|| \le \frac{1}{1 - v_0(||u - \alpha||)}. \tag{2.10}$$

Consequently, $y_0$ exists by (2.10) for $u = x_0$, and is well defined by the first sub-step of technique (1.2) for $n = 0$. Then, we get

$$\begin{aligned} y_0 - \alpha &= x_0 - \alpha - F'(x_0)^{-1}F(x_0) = \\ &= F'(x_0)^{-1}F'(\alpha)\int_0^1 F'(\alpha)^{-1}\left[F'(x_0) - F'(\alpha + \tau(x_0 - \alpha))\right](x_0 - \alpha)d\tau. \end{aligned} \tag{2.11}$$

Using (2.1), (2.5), $(C_2)$, (2.10) (for $u = x_0$), and (2.11), we obtain the estimate as,

$$\begin{aligned} \|y_0 - \alpha\| &\le \frac{\int_0^1 v((1-\tau)\|x_0 - \alpha\|)\ d\tau}{1 - v_0(\|x_0 - \alpha\|)}\|x_0 - \alpha\| = \\ &= h_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\| < R, \end{aligned} \tag{2.12}$$

which proves $y_0 \in B(\alpha, R) \subseteq \Omega$ and (2.6) for $n = 0$. Further, using the second sub-step of (1.2) for $n = 0$, we have

$$z_0 - \alpha = \psi(x_0, y_0) - \alpha. \tag{2.13}$$

Then, by (2.1), (2.5), $(C_2)$, (2.12), and (2.13), we get

$$\begin{aligned}\|z_0-\alpha\| = \|\psi(x_0,y_0)-\alpha\| &\leq \\ &\leq h_2(\|x_0-\alpha\|,\|y_0-\alpha\|)\max\{\|x_0-\alpha\|,\|y_0-\alpha\|\} \leq \\ &\leq h_2(\|x_0-\alpha\|,h_1(\|x_0-\alpha\|)\|x_0-\alpha\|)\|x_0-\alpha\| < \\ &< \|x_0-\alpha\|, \end{aligned} \tag{2.14}$$

which proves $z_0 \in B(\alpha,\ R) \subseteq \Omega$ and (2.7) for $n=0$.

For the existence of iterate $x_1$, we shall first establish that $T_0^{-1},\ [z_0,x_0;F]^{-1} \in \in \mathcal{L}(Y,X)$. Using (2.1), (2.3), (2.4), and $(C_3)$, we have

$$\begin{aligned}\|F'(\alpha)^{-1}(T_0-F'(\alpha))\| &\leq \\ &\leq\ \|F'(\alpha)^{-1}([z_0,y_0;F]-F'(\alpha))\|+ \\ &\quad + \|F'(\alpha)^{-1}([z_0,y_0;F]-[z_0,x_0;F])\| \leq \\ &\leq\ w_0(\|z_0-\alpha\|,\|y_0-\alpha\|)+w_1(\|x_0-\alpha\|,\|y_0-\alpha\|,\|z_0-\alpha\|) \leq \\ &\leq\ p(\|x_0-\alpha\|) < p(R) < 1 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\|F'(\alpha)^{-1}([z_0,x_0;F]-F'(\alpha))\| &\leq \\ &\leq\ w_0(\|z_0-\alpha\|,\|x_0-\alpha\|) \leq \\ &\leq\ w_0(h_2(\|x_0-\alpha\|,h_1(\|x_0-\alpha\|)\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|) < \\ &<\ w_0(h_2(R,h_1(R)R)R,R) < 1. \end{aligned} \tag{2.16}$$

Consequently, both $T_0^{-1}$ and $[z_0,x_0;F]^{-1}$ exist by (2.15) and (2.16), respectively, and moreover

$$\|T_0^{-1}F'(\alpha)\| \leq \frac{1}{1-p(\|x_0-\alpha\|)}, \tag{2.17}$$

and

$$\begin{aligned}\|[z_0,x_0;F]^{-1}F'(\alpha)\| &\leq \\ &\leq \frac{1}{1-w_0(h_2(\|x_0-\alpha\|,h_1(\|x_0-\alpha\|)\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|)}. \end{aligned} \tag{2.18}$$

Thus, $x_1$ exists, and is well defined by the third sub-step of (1.2) for $n=0$. Further,

$$\begin{aligned}x_1-\alpha = z_0-\alpha-[z_0,x_0;F]^{-1}F(z_0)&+ \\ + T_0^{-1}(T_0-[z_0,y_0;F])[z_0,x_0;F]^{-1}F(z_0) &= \\ = [z_0,x_0;F]^{-1}\left([z_0,x_0;F]-[z_0,\alpha;F]\right)(z_0-\alpha)&+ \\ + T_0^{-1}\left([z_0,y_0;F]-[z_0,x_0;F]\right)[z_0,x_0;F]^{-1}F(z_0)&. \end{aligned} \tag{2.19}$$

Then, using (2.1), (2.5), ($C_2$), (2.14), and (2.17)–(2.19), we obtain that

$$\|x_1-\alpha\| \le \Bigg[\frac{w(\|z_0-\alpha\|,\|x_0-\alpha\|)}{1-w_0(\|z_0-\alpha\|,\|x_0-\alpha\|)}+ \\ +\frac{w_1(\|x_0-\alpha\|,\|y_0-\alpha\|,\|z_0-\alpha\|)w_2(\|z_0-\alpha\|)}{(1-w_0(\|z_0-\alpha\|,\|x_0-\alpha\|))(1-p(\|x_0-\alpha\|))}\Bigg]\|z_0-\alpha\| \le$$

$$\le h_3(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\| < R, \tag{2.20}$$

which proves $x_1 \in B(\alpha,\ R)$ and (2.8) for $n=0$. The induction process on '$n$', for the inequalities (2.6)–(2.8), is terminated if $x_0, y_0,\ z_0$ and $x_1$ are replaced by $x_n, y_n,\ z_n$ and $x_{n+1}$ respectively. In view of this, the estimate

$$\|x_{n+1}-\alpha\| < \beta\|x_n-\alpha\| < R, \tag{2.21}$$

where $\beta = h_3(||x_0-\alpha||) \in [0,1)$, eventually proves that $x_{n+1} \in B(\alpha,\ R)$ and $\lim_{n\to\infty} x_n = \alpha$.

Further, we claim that the solution '$\alpha$' is unique in the domain $\Omega_2 = \Omega \cap \cap \overline{B}(\alpha, R^*)$, as defined in ($C_5$). For this, consider $u \in \Omega_2$ such that $F(u)=0$. Then, by ($C_1$) and ($C_5$), and for $M = \int_0^1 F'(\alpha-\tau(u-\alpha))d\tau$, we get

$$\|F'(\alpha)^{-1}(M-F'(\alpha))\| \le \int_0^1 v_0(\tau\|u-\alpha\|)d\tau \le \\ \le \int_0^1 v_0(\tau R^*)d\tau < 1.$$

This proves the existence of $M^{-1}$, and ultimately $u=\alpha$ is obtained as,

$$0 = F(u) - F(\alpha) = M(u-\alpha). \qquad \blacksquare$$

**Special case:** To extend the above analysis, and further, for the purpose of numerical testing, we study a particular case by considering the fourth order iteration, developed in [8], at the second step of technique (1.2) as

$$\psi(x_n, y_n) = y_n - \left(2[y_n, x_n; F]^{-1} - F'(x_n)^{-1}\right) F(y_n).$$

For the existence of the iterate $z_n$ for $n=0$, it is required to establish that the operator $[y_0, x_0; F]^{-1}$ exists. In this regard, using (2.1), (2.3), (2.5), and ($C_3$), we have

$$\|F'(\alpha)^{-1}([y_0,x_0;F]-F'(\alpha))\| \le w_0(\|y_0-\alpha\|,\|x_0-\alpha\|) \le \\ \le w_0(h_1(\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|) < \\ < w_0(h_1(R)R,R) < 1, \tag{2.22}$$

which implies that $[y_0, x_0; F]^{-1}$ exists, and

$$\|[y_0, x_0; F]^{-1} F'(\alpha)\| \leq \frac{1}{1 - w_0(\|y_0 - \alpha\|, \|x_0 - \alpha\|)}. \tag{2.23}$$

Further, we need to establish the definition of function $h_2 : Q_0 \times Q_0 \to Q$ which should satisfy the inequality (2.14), where $Q_0$ and $Q$ are defined already. Using the equations (2.10) and (2.23) along with the conditions $(C_1)$ and $(C_3)$, we obtain

$$\begin{aligned}
\|z_0 - \alpha\| = & \ \|\psi(x_0, y_0) - \alpha\| = \\
= & \ \|y_0 - \alpha - \left(2[y_0, x_0; F]^{-1} - F'(x_0)^{-1}\right) F(y_0)\| \leq \\
\leq & \ \big[1 + \|[y_0, x_0; F]^{-1} F'(\alpha)\| \|F'(\alpha)^{-1}(2F'(x_0) - [y_0, x_0; F])\| \times \\
& \times \|F'(x_0)^{-1} F'(\alpha)\| \|F'(\alpha)^{-1}[y_0, \alpha; F]\|\big] \|y_0 - \alpha\| \leq \\
\leq & \ \left[1 + \frac{(1 + 2v_0(\|x_0 - \alpha\|) + w_0(\|y_0 - \alpha\|, \|x_0 - \alpha\|))\, w_2(\|y_0 - \alpha\|)}{(1 - w_0(\|y_0 - \alpha\|, \|x_0 - \alpha\|))(1 - v_0(\|x_0 - \alpha\|))}\right] \times \\
& \times \|y_0 - \alpha\| \leq \\
\leq & \ h_2(\|x_0 - \alpha\|, \|y_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\| < R,
\end{aligned} \tag{2.24}$$

where,

$$h_2(t, s) = \left[1 + \frac{(1 + 2v_0(t) + w_0(s, t))\, w_2(s)}{(1 - w_0(s, t))(1 - v_0(t))}\right] h_1(t).$$

**Remark 2.1.** To claim the eighth order of convergence for the technique (1.2), instead of using Taylor series expansions, the term computational order of convergence (COC) [6] is defined as

$$\text{COC} = \ln \left\| \frac{x_{r+2} - \alpha}{x_{r+1} - \alpha} \right\| \Big/ \ln \left\| \frac{x_{r+1} - \alpha}{x_r - \alpha} \right\|, \quad \text{for each } r = 0, 1, 2, \ldots.$$

Notice that, to compute COC, the knowledge of exact solution $(\alpha)$ is required, but that may not always be known explicitly. In that case, the convergence order can be determined using the approximated computational order of convergence (ACOC) [6] as defined below,

$$\text{ACOC} = \ln \left\| \frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r} \right\| \Big/ \ln \left\| \frac{x_{r+1} - x_r}{x_r - x_{r-1}} \right\|, \quad \text{for each } r = 1, 2, \ldots.$$

Apparently, no calculation of derivative is involved to determine the order of convergence of an iterative technique, either by using COC, or ACOC.

**Remark 2.2.** To demonstrate the numerical testing in the next section, we shall use the approximation:

$$[y,x;F]=\frac{1}{2}(F'(x)+F'(y)),\ \text{or}\ \ [y,x;F]=\int_0^1 F'(x+\theta(y-x))d\theta.$$

## 3. Numerical results

In coherence with the theoretical deductions, we provide here the parameters and the estimated radius of convergence, as defined in Section 2, for each of the following numerical examples.

**Example 3.1.** Initially, we take a look at the example given in the introduction section. Note that $\alpha=1/\pi$ is zero of this function. In this particular problem, we can choose

$$v_0(t)=v(t)=Lt,\ w_0(s,t)=w(s,t)=\frac{L}{2}(s+t),$$

$$w_1(r,s,t)=w_0(r,t)+w_0(s,t),\ \text{and}\ w_2(t)=2,$$

provided $L=\frac{2}{2\pi+1}[80+16\pi+(11+12\log 2)\pi^2]$.

Then, we obtain the estimates as

$$R_1=7.565\times 10^{-3},\ R_2=2.807\times 10^{-3},\ R_3=2.428\times 10^{-3},\ \text{and}\ R=2.428\times 10^{-3}.$$

**Example 3.2.** Consider the domain $\mathbb{R}^m$, for any integer $m\geq 2$, equipped with the max-norm $\|x\|=\max_{1\leq i\leq m}|x_i|$ for each $x=(x_1,\ldots,x_m)^T\in\mathbb{R}^m$. The corresponding matrix norm is given by $\|A\|=\max_{1\leq i\leq m}\sum_{j=1}^{j=m}|a_{ij}|$ for any $A=(a_{ij})_{1\leq i,j\leq m}\in\mathcal{L}(\mathbb{R}^m)$. Now consider the following two-point boundary value problem defined on the closed interval $[0,1]$ as,

$$\begin{aligned} u''+u^2&=0,\\ u(0)=u(1)&=0. \end{aligned}\tag{3.1}$$

To transform the Eq. (3.1) into a finite dimensional problem, the following divided difference approximations are used:

$$u_i''\approx\frac{u_{i+1}-2u_i+u_{i-1}}{h^2},\quad \text{for each}\ i=1,2,\ldots,k-1,$$

where $u_i=u(x_i)$, $x_i=0+ih$, and $h=1/k$. Note that the boundary conditions are considered as $u_0=u_k=0$. Therefore, Eq. (3.1) is transformed into the system of nonlinear equations, $F:\mathbb{R}^{k-1}\to\mathbb{R}^{k-1}$, which is given by

$$u_{i+1}-2u_i+h^2u_i^2+u_{i-1}=0,\quad i=1,2,\ldots,k-1.$$

The Fréchet derivative at any point $u = (u_1, \dots, u_{k-1})^T$ is given by

$$F'(u) = \begin{bmatrix} 2h^2u_1 - 2 & 1 & 0 & \cdots & 0 \\ 1 & 2h^2u_2 - 2 & 1 & \cdots & 0 \\ 0 & 1 & 2h^2u_3 - 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2h^2u_{k-1} - 2 \end{bmatrix}.$$

Taking $k = 11$ in particular, the above system reduces to 10 nonlinear equations for which the solution is given as, $\alpha = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$. Consequently, we can obtain that

$$\|F'(\alpha)^{-1}(F'(u) - F'(v))\| \leq \frac{30}{121}\|u - v\|,$$

for any $u, v$ in the given region.

So, we can choose

$$v_0(t) = v(t) = \frac{30}{121}t, \;\; w_0(s,t) = w(s,t) = \frac{15}{121}(s+t),$$
$$w_1(r,s,t) = w_0(r,t) + w_0(s,t), \text{ and } w_2(t) = 2,$$

and consequently, we obtain

$$R_1 = 2.6889, \;\; R_2 = 0.9975, \;\; R_3 = 0.8630, \text{ and } R = 0.8630.$$

**Example 3.3.** Consider the Hammerstein equation:

$$x(s) = \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt, \tag{3.2}$$

where

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t, \end{cases}$$

is the Green's function, defined as a kernel of (3.2), in the domain $[0,1] \times [0,1]$. In particular, we observe that

$$\left\|\int_0^1 G(s,\; t)dt\right\| \leq \frac{1}{8}.$$

Clearly, $\alpha(s) = 0$ is the solution of (3.2). By defining $F : D \subseteq C[0,1] \to$ $\to C[0,1]$, where $D = \overline{B}(0,1)$, as

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt,$$

we simply have

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\left(\frac{3}{2}x(t)^{1/2} + x(t)\right) y(t)dt.$$

But with $F'(\alpha(s)) = I$, we get in turn that

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\right),$$

therefore we can choose

$$\begin{aligned} v_0(t) = v(t) &= \frac{1}{8}\left(\frac{3}{2}t^{1/2} + t\right), \\ w_0(s,t) = w(s,t) &= \frac{1}{16}\left(\frac{3}{2}t^{1/2} + \frac{3}{2}s^{1/2} + s + t\right), \\ w_1(r,s,t) = w_0(r,t) &+ w_0(s,t), \text{ and } w_2(t) = 2. \end{aligned}$$

Finally, we obtain

$$R_1 = 2.6303,\ R_2 = 0.6027,\ R_3 = 0.4797,\ \text{and } R = 0.4797.$$

## 4. Conclusions

An optimal eighth order iterative technique is comprehensively analyzed to establish its local convergence in Banach spaces. Contrary to the usual approach using Taylor series expansions, to study the convergence analysis, we developed here the generalized results using the assumptions only on first order derivatives or divided differences. Effectively, the applicability of given technique is extended to a wider section of the problems. Further, estimation of the bounds on convergence radius, both theoretically and numerically, satisfactorily favor our analysis.

## References

[1] **Amat, S., C. Bermúdez, M.A. Hernández and E. Martínez,** On an efficient k-th step iterative method for nonlinear equations, *J. Comput. Appl. Math.*, **302** (2016), 258–271.

[2] **Argyros, I.K., R. Behl and S.S. Motsa,** Local convergence of an optimal eighth-order method under weak conditions, *Algorithms*, **8** (2015), 645–655.

[3] **Argyros, I.K. and S. George,** Extending the applicability of a seventh order method without inverses of derivatives under weak conditions, *Int. J. Appl. Comput. Math.*, **6** (2020), 1–9.

[4] **Argyros, I.K. and S. Hilout,** Weaker conditions for the convergence of Newton's method, *J. Complexity*, **28** (2012), 364–387.

[5] **Argyros, I.K. and A.A. Magreñán,** *Iterative Methods and their Dynamics with Applications*, CRC Press, New York, 2017.

[6] **Argyros, I.K., J.R. Sharma and D. Kumar,** Ball convergence of the Newton-Gauss method in Banach spaces, *SeMA*, **74** (2017), 429–439.

[7] **Cordero, A., J.A. Ezquerro, M.A. Hernández and J.R. Torregrosa,** On the local convergence of fifth-order iterative method in Banach spaces, *Appl. Math. Comput.*, **251** (2015), 396–403.

[8] **Grau-Sánchez, M., A. Grau and M. Noguera,** Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, **218** (2011), 2377–2385.

[9] **Gutiérrez, J.M., A.A. Magreñán and N. Romero,** On the semilocal convergence of Newton-Kantorovich method under center-Lipschitz conditions. *Appl. Math. Comput.*, **221** (2013), 79–88.

[10] **Hoffman, J.D.,** *Numerical Methods for Engineers and Scientists*, McGraw-Hill Book Company, New York, 1992.

[11] **Magreñán, A.A. and I.K. Argyros,** New improved convergence analysis for the secant method, *Math. Comput. Simul.*, **119** (2016), 161–170.

[12] **Ortega, J.M. and W.C. Rheinboldt,** *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[13] **Petković, M.S. and L.D. Petković,** Families of optimal multipoint methods for solving nonlinear equations: A survey, *Appl. Anal. Discrete Math.*, **4** (2010), 1–22.

[14] **Sharma, J.R., I.K. Argyros and S. Kumar,** Ball convergence of an efficient eighth order iterative method under weak conditions, *Mathematics* **6** (2018), 260.

[15] **Sharma, J.R. and H. Arora,** A new family of optimal eighth order methods with dynamics for nonlinear equations, *Appl. Math. Comput.*, **273** (2016), 924–933.

[16] **Traub, J.F.,** *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1982.

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