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THE FACTORIZATION OF COMPRESSED CHEBYSHEV POLYNOMIALS AND OTHER POLYNOMIALS RELATED TO MULTIPLE-ANGLE FORMULAS

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Abstract. In this paper, the so-called compressed Chebyshev polynomials are studied, which are obtained from the original Chebyshev polynomials by suitable scaling transformations. By introducing the concept of cyclotomic pre-polynomials that are in one-to-one correspondence with the well known cyclotomic polynomials, explicit formulas for the irreducible factorization of all four kinds of compressed Chebyshev polynomials are developed. The correspondence between the cyclotomic pre-polynomials and cyclotomic polynomials is based on a polynomial transformation. The same transformation maps all four kinds of the compressed Chebyshev polynomials into polynomials having the property that all their roots are primitive roots of unity, and consequently they are easy to factorize. Generalizations of the applied methods, suitable for the factorization of similar classes of polynomials are also given in the paper.

1. Introduction

Chebyshev polynomials are very important in the fields of numerical analysis, approximation theory and differential equations, see, e.g. Natanson [7]. For

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readers, who are mainly interested in the application of Chebyshev-like polynomials to approximation theory and related areas, we suggest to consider and test for applicability some of the more general polynomials that are studied in the last two sections of this paper.

There are known results for the irreducible factorization of the Chebyshev polynomials of the first and the second kinds in Hsiao [1], Rivlin [9] and Rayes, Trevisan and Wang [8]. Their main relevant results are formulated within the frames of Theorems 1 and 2 in [8]. In this paper, we give direct constructive methods for the factorization of certain scaled equivalents of all four kinds of Chebyshev polynomials (which we call compressed Chebyshev polynomials) by the help of cyclotomic pre-polynomials and cyclotomic polynomials. However, not only the method but also the presentation is quite different, which can be seen if we compare our formula shown in (54) with the formula given in [8, Theorem 1], and similarly (44) with the formula given in [8, Theorem 2].

The structure of the paper is as follows. In Section 2 we introduce the four kinds of compressed Chebyshev polynomials, whose leading coefficients are 1's for any positive integer n. In Section 3 the basic trigonometric identities, followed by explicit formulas and recurrence relations for the compressed Chebyshev polynomials are shown. A polynomial transformation is introduced in Section 4 that proves to be useful for the irreducible factorization of the compressed Chebyshev polynomials. As an example, we show that $x^n U_n\left(x+\frac{1}{x}\right) = (x^{2n+2}-1)/(x^2-1) = \sum_{k=0}^{n} x^{2k}$, where $U_n(x)$ is the compressed Chebyshev polynomial of the second kind. Section 5 deals with some properties of the cyclotomic polynomials and cyclotomic pre-polynomials that are important for the factorization of the compressed Chebyshev polynomials. (We use the terminus "cyclotomic pre-polynomial" for the polynomials that can be mapped into cyclotomic polynomials by the transformation introduced in Section 4.) By the help of the first five preparatory sections, in Section 6 the factorization of the other compressed Chebyshev polynomials, each considered as a special case of the expression $U_n(x) \pm U_{n-k}(x)$, is accomplished. Finally, in Section 7 a factorization formula for an even more general class of polynomials is stated and proved.

A large amount of numerical results belonging to the paper can be found in [4] as well as in the unpublished Hungarian language manuscript
drive.google.com/file/d/14l6-o4FHNKOlLsm16wwE-OfGhkfdbxU4/view

2. Compression of the Chebyshev polynomials

The two primary sequences of Chebyshev polynomials are the so called Chebyshev polynomials of the first kind, usually noted by $T_n(x)$ and that of the second kind, usually noted by $U_n(x)$. By the help of these polynomials the sines and cosines of multiple angles can be expressed as

$$T_n(\cos\theta) = \cos n\theta, \tag{1}$$

and

$$\sin\theta\, U_n(\cos\theta) = \sin(n+1)\theta, \tag{2}$$

see, e.g., [6, 12, 15].

Two more kinds of Chebyshev polynomials are introduced in Mason and Handscomb [6], named as the third and fourth kind of Chebyshev polynomials and noted by $V_n(x)$ and $W_n(x)$.

For higher degrees n, the coefficients of the Chebyshev polynomials soon become very large in absolute value, as their leading coefficients are powers of 2, namely 2^{n-1} in $T_n(x)$ and 2^n in the three other variants. By the help of a suitable scaling, the leading coefficients become 1's for any positive integer n, and all other coefficients become significantly smaller in comparison to the corresponding coefficients of the original Chebyshev polynomials. For this purpose, we introduce the compressed Chebyshev polynomials $T_n^*(x)$, $U_n^*(x)$, $V_n^*(x)$, $W_n^*(x)$ by the scaling formulas

$$\begin{aligned} T_n^*(x) &= 2T_n\left(\frac{x}{2}\right), \quad U_n^*(x) = U_n\left(\frac{x}{2}\right), \\ V_n^*(x) &= V_n\left(\frac{x}{2}\right), \quad W_n^*(x) = W_n\left(\frac{x}{2}\right). \end{aligned} \tag{3}$$

We have to notice that the first and the second kinds of these compressed Chebyshev polynomials were already introduced and studied, with using other names and other notation, by Wituła and Słota [11] and Koshy [5].

Beside the compression of the coefficients, another great advantage of using these compressed Chebyshev polynomials is that the factorization of these variants is much more comfortable and can be expressed by more concise and more aesthetic formulas as we shall see in Section 6. From these formulas it will be seen that the factorization of $T_n^*(x)$ and the factorization of the polynomial $x^{2n}+1$ are fully synchronous with each other, where the factors of the former are cyclotomic pre-polynomials, while the factors of the latter are the corresponding cyclotomic polynomials. Similar assertions are valid for the other three compressed Chebyshev polynomials $U_n^*(x)$, $V_n^*(x)$, and $W_n^*(x)$, respectively, if they are matched to the polynomials $\sum_{k=0}^{n} x^{2k}$, $\sum_{k=0}^{2n}(-1)^k x^k$, and $\sum_{k=0}^{2n} x^k$, respectively.

As in the remaining part of the paper we do not need the original Chebyshev polynomials, but only the compressed ones, we will omit the asterisk from their notation that we used only in this introductory section.

3. Independent definitions of the compressed Chebyshev polynomials

Similarly to the original Chebyshev polynomials, there are different ways to define the compressed Chebyshev polynomials. Trigonometric definitions for the compressed Chebyshev polynomials are as follows.

$$2\cos n\theta = T_n(2\cos\theta), \tag{4}$$

$$2\sin(n+1)\theta = (2\sin\theta)U_n(2\cos\theta), \tag{5}$$

$$2\cos\left(n+\frac{1}{2}\right)\theta = \left(2\cos\frac{1}{2}\theta\right)V_n(2\cos\theta), \tag{6}$$

$$2\sin\left(n+\frac{1}{2}\right)\theta = \left(2\sin\frac{1}{2}\theta\right)W_n(2\cos\theta). \tag{7}$$

As it is known, there are also useful hyperbolic identities with both the original and the compressed Chebyshev polynomials, see, e.g., [3, 4].

An independent definition for $U_n(x)$ by an explicit formula and another one by a recurrence relation can be derived easily from the similar formulas known for the original Chebyshev polynomials. Thus, for the compressed Chebyshev polynomial of the second kind we have the explicit formula

$$U_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}, \tag{8}$$

and the recurrence relation

$$U_0(x) = 1, \tag{9}$$

$$U_1(x) = x, \tag{10}$$

$$U_{n+1}(x) = xU_n(x) - U_{n-1}(x). \tag{11}$$

By arranging (11) to the form

$$U_{n-1}(x) = xU_n(x) - U_{n+1}(x), \tag{12}$$

we can extend the definition of $U_n(x)$ into the opposite direction, i.e. for negative integer values of n, giving

$$U_{-1}(x) = 0 \tag{13}$$

and

$$U_{-n}(x) = -U_{n-2}(x) \ \text{ if } \ n \geq 2, \tag{14}$$

thus, e.g., $U_{-2}(x) = -1$ and $U_{-3}(x) = -x$. The reason of this extension is that later on we consider several additive compositions of $U_n(x)$ with different orders, e.g. $U_n(x) - U_{n-1}(x) - U_{n-2}(x) + U_{n-3}(x)$ that may require the knowledge of $U_n(x)$ for a few negative integer values of n.

It is easy to prove regarding the three other kinds of compressed Chebyshev polynomials, that

$$T_n(x) = U_n(x) - U_{n-2}(x) \tag{15}$$

$$V_n(x) = U_n(x) - U_{n-1}(x) \tag{16}$$

$$W_n(x) = U_n(x) + U_{n-1}(x) \tag{17}$$

Now, from (8) and (15)–(17) we obtain the explicit formulas for $U_n(x)$, $V_n(x)$ and $W_n(x)$. Namely

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} - \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \binom{n-2-k}{k} x^{n-2-2k}, \tag{18}$$

$$V_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} x^{n-1-2k}, \tag{19}$$

$$W_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} x^{n-1-2k}, \tag{20}$$

where (18) is valid for $n \geq 1$, (19) and (20) are valid for $n \geq 0$. The initial values of these polynomials, and their recurrence relations are given by the help of (11) and (15)–(17) and are as follows.

$$T_0(x) = U_0(x) - U_{-2}(x) = 1 - (-1) = 2, \tag{21}$$

$$T_1(x) = U_1(x) - U_{-1}(x) = x - 0 = x, \tag{22}$$

$$T_{n+1}(x) = xT_n(x) - T_{n-1}(x), \tag{23}$$

$$V_0(x) = U_0(x) - U_{-1}(x) = 1 - 0 = 1, \tag{24}$$

$$V_1(x) = U_1(x) - U_0(x) = x - 1, \tag{25}$$

$$V_{n+1}(x) = xV_n(x) - V_{n-1}(x), \tag{26}$$

$$W_0(x) = U_0(x) + U_{-1}(x) = 1 + 0 = 1, \tag{27}$$

$$W_1(x) = U_1(x) + U_0(x) = x + 1, \tag{28}$$

$$W_{n+1}(x) = xW_n(x) - W_{n-1}(x). \tag{29}$$

4. A polynomial transformation and an example for its application

Let us consider the transformation of an arbitrary polynomial $P_n(x)$ of degree n into the polynomial $\overline{P_n(x)} = x^n P_n\left(x + \frac{1}{x}\right)$ of degree $2n$. For the product or for the sum of two polynomials $P_n(x)$ and $Q_m(x)$, where $n \geq m$, clearly

$$\overline{P_n(x) \cdot Q_m(x)} = \overline{P_n(x)} \cdot \overline{Q_m(x)} \tag{30}$$

and

$$\overline{P_n(x) \pm Q_m(x)} = \overline{P_n(x)} \pm x^{n-m}\overline{Q_m(x)}. \tag{31}$$

Consequently, if the polynomial $P_n(x)$ is divisible by the polynomial $Q_m(x)$, then

$$\overline{P_n(x)/Q_m(x)} = \overline{P_n(x)}/\overline{Q_m(x)}. \tag{32}$$

The special cases $\overline{aQ_m(x)} = a\overline{Q_m(x)}$ and $\overline{P_n(x) \pm b} = \overline{P_n(x)} \pm bx^n$ of (30) and (31) are also worth of a particular emphasis.

As an example, let us consider the compressed Chebyshev polynomial of the second kind. Obviously, the roots of our modified polynomial $U_n(x)$ are equal to the known roots (see [6, 12]) of the original Chebyshev polynomial of the second kind, multiplied by 2. Thus the roots of $U_n(x)$ are just

$$2\cos\frac{k\pi}{n+1}\ (k=1,\ldots,n), \tag{33}$$

and consequently

$$U_n(x)=\prod_{k=1}^{n}\left(x-2\cos\frac{k\pi}{n+1}\right). \tag{34}$$

From this, we obtain

$$\overline{U_n(x)}=\overline{\prod_{k=1}^{n}\left(x-2\cos\frac{k\pi}{n+1}\right)}=\prod_{k=1}^{n}\overline{\left(x-2\cos\frac{k\pi}{n+1}\right)}=$$

$$=\prod_{k=1}^{n}\left[x\left(x+\frac{1}{x}-2\cos\frac{k\pi}{n+1}\right)\right]=\prod_{k=1}^{n}\left(x^2-2x\cos\frac{k\pi}{n+1}+1\right)= \tag{35}$$

$$=\prod_{k=1}^{n}\left[\left(x-e^{\frac{k\pi i}{n+1}}\right)\left(x-e^{-\frac{k\pi i}{n+1}}\right)\right]=\prod_{k=1}^{n}\left(x-e^{\frac{k\pi i}{n+1}}\right)\prod_{k=1}^{n}\left(x-e^{-\frac{k\pi i}{n+1}}\right)$$

The last expression resembles the factored form of the polynomial $x^{2n+2}-1$ according to its roots, which are the $(2n+2)$th roots of unity, and so we have

$$x^{2n+2}-1=\prod_{k=0}^{n+1}\left(x-e^{\frac{k\pi i}{n+1}}\right)\prod_{k=1}^{n}\left(x-e^{-\frac{k\pi i}{n+1}}\right). \tag{36}$$

The latter expression, given at (36) contains also the factors that correspond to the roots of unity $+1$ and -1 and that are not contained in the final expression, given at (35). By this, it has been established that

$$(x-1)(x+1)\overline{U_n(x)}=x^{2n+2}-1, \tag{37}$$

i.e.

$$\overline{U_n(x)}=\frac{x^{2n+2}-1}{x^2-1}. \tag{38}$$

5. Cyclotomic polynomials and cyclotomic pre-polynomials

According to the definition given in [13], the nth cyclotomic polynomial, that is usually denoted by $\Phi_n(x)$, for any positive integer n, is the unique irreducible polynomial with integer coefficients that is a divisor of $x^n - 1$ and is not a divisor of $x^k - 1$ for any $k < n$. Its roots are all nth primitive roots of unity $e^{\frac{2k\pi i}{n}}$, where k runs over the positive integers not greater than n and coprime to n. In other words, the nth cyclotomic polynomial is of degree $\varphi(n)$ (the Euler's totient function, see Wikipedia [14]), and is equal to

$$\Phi_n(x) = \prod_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} (x - e^{\frac{2k\pi i}{n}}). \tag{39}$$

The cyclotomic polynomials have several interesting properties, e.g, they have only integer coefficients; they are irreducible over the field of the rational numbers.

According to a well known existence theorem, for arbitrary n greater than 2, there exists a polynomial $\Psi_n(x)$ of degree $\varphi(n)/2$, for which $\overline{\Psi_n(x)} = \Phi_n(x)$ and

$$\Psi_n(x) = \prod_{\substack{0<k<n/2 \\ \gcd(k,n)=1}} \left(x - 2\cos\frac{2k\pi}{n}\right). \tag{40}$$

Similarly to the cyclotomic polynomials, also the cyclotomic pre-polynomials $\Psi_n(x)$ have only integer coefficients and are irreducible over the field of the rational numbers, see [2, Theorem 4.1].

Some interesting special cases of the cyclotomic pre-polynomials are as follows.

a) If p is an odd prime, then $\Psi_p(x) = W_{(p-1)/2}(x)$.

b) Under the same condition, $\Psi_{2p}(x) = V_{(p-1)/2}(x)$.

c) If $n = 2^q$ where $q \ge 2$, then $\Psi_n(x) = T_{n/4}(x)$.

d) If $n = 3^r$ where $r \ge 1$, then $\Psi_n(x) = T_{n/3}(x) + 1$.

e) If $n = 2^q 3^r$ where $q, r \ge 1$, then $\Psi_n(x) = T_{n/6}(x) - 1$.

For the proof, see Kéri [4]. A complete list of the cyclotomic pre-polynomials $\Psi_n(x)$ for $3 \le n \le 120$ can also be found at the same places.

6. Irreducible factorization of the compressed Chebyshev polynomials

As it is well known, each nth root of unity is a primitive dth root of unity for a unique d dividing n, and consequently

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{41}$$

From this it obviously follows that

$$x^n + 1 = \frac{x^{2n}-1}{x^n-1} = \frac{\prod_{d|2n} \Phi_d(x)}{\prod_{d|n} \Phi_d(x)} = \prod_{\substack{d|2n \\ d\nmid n}} \Phi_d(x). \tag{42}$$

These identities show that the number of factors in the irreducible factorization of $x^n - 1$ is $\tau(n)$, the number of positive divisors of n, while the number of factors in the irreducible factorization of $x^n + 1$ is $\tau(2n) - \tau(n)$, which is equal to the number of odd positive divisors of n. Both of these integer sequences are contained in the On-line Encyclopedia of Integer Sequences (OEIS), [10], the first one with identifier A000005, while the second one with identifier A001227.

For the irreducible factorization of $U_n(x)$, according to (38) and (41), we obtain for any positive integer n that

$$\overline{U_n(x)} = \frac{\prod_{d|2n+2} \Phi_d(x)}{\prod_{d|2} \Phi_d(x)} = \prod_{\substack{d\geq 3 \\ d|2n+2}} \Phi_d(x) = \prod_{\substack{d\geq 3 \\ d|2n+2}} \overline{\Psi_d(x)}, \tag{43}$$

which implies that, also for any positive integer n, the irreducible factorization of the compressed Chebyshev polynomial of the second kind can be expressed by the formula

$$U_n(x) = \prod_{\substack{d\geq 3 \\ d|2n+2}} \Psi_d(x). \tag{44}$$

According to (44), the number of irreducible factors in $U_n(x)$ is $\tau(2n+2) - 2$, which is A086327 in OEIS, [10].

As regards the three other compressed Chebyshev polynomials, by applying the rule given at (31) to the right hand side expressions of identities (15)–(17), we obtain that for any positive integer n

$$\begin{aligned} x^n T_n\left(x + \frac{1}{x}\right) &= \overline{U_n(x)} - x^2 \cdot \overline{U_{n-2}(x)} = \\ &= \frac{x^{2n+2}-1}{x^2-1} - x^2 \cdot \frac{x^{2n-2}-1}{x^2-1} = x^{2n} + 1, \end{aligned} \tag{45}$$

$$x^n V_n\left(x+\frac{1}{x}\right) = \overline{U_n(x)} - x\overline{U_{n-1}(x)} = \tag{46}$$
$$= \frac{x^{2n+2}-1}{x^2-1} - x\cdot\frac{x^{2n}-1}{x^2-1} = \frac{x^{2n+1}+1}{x+1},$$

$$x^n W_n\left(x+\frac{1}{x}\right) = \overline{U_n(x)} + x\overline{U_{n-1}(x)} = \tag{47}$$
$$= \frac{x^{2n+2}-1}{x^2-1} + x\cdot\frac{x^{2n}-1}{x^2-1} = \frac{x^{2n+1}-1}{x-1}.$$

(Although (31) is not applicable for the case of $T_n(x)$ when $n = 1$, still the chain given in (45) is correct in this trivial case, too.)

As a generalization to the above three polynomial variants, consider $U_n(x)\pm \pm U_{n-k}(x)$. By using the same techniques as before, we obtain, that for $n \geq k$,

$$\overline{U_n(x)+U_{n-k}(x)} = \overline{U_n(x)} + x^k\cdot\overline{U_{n-k}(x)} =$$
$$= \frac{x^{2n+2}-1}{x^2-1} + x^k\cdot\frac{x^{2n-2k+2}-1}{x^2-1} = \frac{(x^k+1)(x^{2n-k+2}-1)}{\Phi_1(x)\Phi_2(x)} = \tag{48}$$
$$= \frac{1}{\Phi_1(x)\Phi_2(x)}\left[\prod_{\substack{d\mid 2k\\ d\nmid k}}\Phi_d(x)\right]\left[\prod_{d\mid 2n-k+2}\Phi_d(x)\right].$$

Here

$$\prod_{\substack{d\mid 2k\\ d\nmid k}}\Phi_d(x) = \begin{cases} \Phi_2(x)\prod_{\substack{d\geq 3\\ d\mid 2k\\ d\nmid k}}\Phi_d(x) & \text{if } k \text{ is odd},\\ \prod_{\substack{d\geq 3\\ d\mid 2k\\ d\nmid k}}\Phi_d(x) & \text{if } k \text{ is even}, \end{cases} \tag{49}$$

and

$$\prod_{d\mid 2n-k+2}\Phi_d(x) = \begin{cases} \Phi_1(x)\prod_{\substack{d\geq 3\\ d\mid 2n-k+2}}\Phi_d(x) & \text{if } k \text{ is odd},\\ \Phi_1(x)\Phi_2(x)\prod_{\substack{d\geq 3\\ d\mid 2n-k+2}}\Phi_d(x) & \text{if } k \text{ is even}. \end{cases} \tag{50}$$

Thus, independently from the parity of k, finally we arrive at

$$\overline{U_n(x)+U_{n-k}(x)} = \left[\prod_{\substack{d\geq 3\\ d\mid 2k\\ d\nmid k}}\Phi_d(x)\right]\left[\prod_{\substack{d\geq 3\\ d\mid 2n-k+2}}\Phi_d(x)\right], \tag{51}$$

and consequently

$$U_n(x) + U_{n-k}(x) = \left[\prod_{\substack{d\geq 3 \\ d\,|\,2k \\ d\,\nmid\, k}} \Psi_d(x) \right] \left[\prod_{\substack{d\geq 3 \\ d\,|\,2n-k+2}} \Psi_d(x) \right]. \tag{52}$$

For the factorization of $U_n(x) - U_{n-k}(x)$, by omitting the details, we obtain a similar formula, namely

$$U_n(x) - U_{n-k}(x) = \left[\prod_{\substack{d\geq 3 \\ d\,|\,k}} \Psi_d(x) \right] \left[\prod_{\substack{d\geq 3 \\ d\,|\,4n-2k+4 \\ d\,\nmid\, 2n-k+2}} \Psi_d(x) \right]. \tag{53}$$

Now, by substituting $k = 2$ into (53), we obtain the irreducible factorization of $T_n(x)$, namely

$$T_n(x) = \prod_{\substack{d\,|\,4n \\ d\,\nmid\, 2n}} \Psi_d(x), \tag{54}$$

and then by substituting $k = 1$ into (53) and (52), we obtain the irreducible factorization of $V_n(x)$ and $W_n(x)$, namely

$$V_n(x) = \prod_{\substack{d\geq 3 \\ d\,|\,4n+2 \\ d\,\nmid\, 2n+1}} \Psi_d(x) \quad \text{and} \quad W_n(x) = \prod_{\substack{d\geq 3 \\ d\,|\,2n+1}} \Psi_d(x). \tag{55}$$

According to (54) and (55), the number of irreducible factors in $T_n(x)$ is $\tau(4n) - \tau(2n) = \tau(2n) - \tau(n)$, which is A001227 in OEIS, [10], while the number of irreducible factors in both of $V_n(x)$ and $W_n(x)$ is $\tau(2n+1) - 1$, which is A095374 in OEIS, [10].

Beside the three compressed Chebyshev polynomials, the general formula developed above for the factorization of $U_n(x) \pm U_{n-k}(x)$ is useful for other specific cases. We can substitute $k = n$ into (52) and (53) to obtain the irreducible factorization of $U_n(x) + 1$ and $U_n(x) - 1$, namely

$$U_n(x) + 1 = \left[\prod_{\substack{d\geq 3 \\ d\,|\,2n \\ d\,\nmid\, n}} \Psi_d(x) \right] \left[\prod_{\substack{d\geq 3 \\ d\,|\,n+2}} \Psi_d(x) \right] \tag{56}$$

and

$$U_n(x)-1=\left[\prod_{\substack{d\geq 3\\ d|n}}\Psi_d(x)\right]\left[\prod_{\substack{d\geq 3\\ d|2n+4\\ d\nmid n+2}}\Psi_d(x)\right]. \tag{57}$$

According to (56) and (57), the number of irreducible factors in $U_n(x)+1$ is $\tau(2n)-\tau(n)+\tau(n+2)-2$, which is A086375 in OEIS, [10], while the number of irreducible factors in $U_n(x)-1$ is $\tau(n)+\tau(2n+4)-\tau(n+2)-2$, which is A086389 in OEIS, [10].

7. Further generalization of the factorization formula to the sum of more $U_n(x)$ of different orders

As a wide class of polynomials that can be factorized by the help of cyclotomic polynomials, consider the polynomials having the form

$$Q_n(x)=U_n(x)+\sum_{\substack{R\subseteq S_j\\ R\neq\emptyset}}\left(\prod_{i\in R}\varepsilon_i\right)U_{n-\sum_{i\in R}k_i}(x), \tag{58}$$

where $S_j=\{1,2,\ldots,j\}$ for an arbitrary nonnegative integer j, k_i are different positive integers, and $\varepsilon_i=+1$ or $\varepsilon_i=-1$ for $i\in S_j$.

It can be seen that the right hand side of (58) is a sum of 2^j terms, and it reduces to $U_n(x)$ if $j=0$, to $U_n(x)\pm U_{n-k}(x)$ if $j=1$, etc.

By the help of (31), for $n\geq\sum_{i=1}^{j}k_i$ we obtain

$$\overline{Q_n(x)}=\overline{U_n(x)}+\sum_{\substack{R\subseteq S_j\\ R\neq\emptyset}}\left(\prod_{i\in R}\varepsilon_i\right)x^{\sum_{i\in R}k_i}\overline{U_{n-\sum_{i\in R}k_i}(x)}. \tag{59}$$

From this, by applying (38) to each term of (59), we arrive at

$$\begin{aligned}(x^2-1)\overline{Q_n(x)}=x^{2n+2}-1+\sum_{\substack{R\subseteq S_j\\ R\neq\emptyset}}\left(\prod_{i\in R}\varepsilon_i\right)x^{2n+2-\sum_{i\in R}k_i}-\\ -\sum_{\substack{R\subseteq S_j\\ R\neq\emptyset}}\left(\prod_{i\in R}\varepsilon_i\right)x^{\sum_{i\in R}k_i}\end{aligned} \tag{60}$$

which results in

$$(x^2-1)\overline{Q_n(x)}=\left(x^{2n+2-\sum_{i=1}^{j}k_i}-\prod_{i=1}^{j}\varepsilon_i\right)\prod_{i=1}^{j}\left(x^{k_i}+\varepsilon_i\right). \tag{61}$$

To prove the equivalence of the right hand sides of (60) and (61) it is enough to check and compare the coefficients in their terms.

By noting $\varepsilon_0 = -\prod_{i=1}^{j} \varepsilon_i$ and $k_0 = 2n + 2 - \sum_{i=1}^{j} k_i$, (61) can be written as

$$\overline{Q_n(x)} = \frac{\prod_{i=0}^{j} \left(x^{k_i} + \varepsilon_i\right)}{x^2 - 1} =$$

$$= \frac{1}{x^2-1} \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i < 0}} \prod_{d | k_i} \Phi_d(x) \right) \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i > 0}} \prod_{\substack{d | 2k_i \\ d \nmid k_i}} \Phi_d(x) \right). \tag{62}$$

Now, if we denote by γ_1 the number of negative ε_i's in the set $S_j \cup \{0\}$, and by γ_2 the sum of the number of those negative ε_i's in the set $S_j \cup \{0\}$ for which k_i is even and the number of those positive ε_i's in the set $S_j \cup \{0\}$ for which k_i is odd, then we obtain

$$\overline{Q_n(x)} = [\Phi_1(x)]^{\gamma_1 - 1} [\Phi_2(x)]^{\gamma_2 - 1} \times$$

$$\times \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i < 0}} \prod_{\substack{d \geq 3 \\ d | k_i}} \Phi_d(x) \right) \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i > 0}} \prod_{\substack{d \geq 3 \\ d | 2k_i \\ d \nmid k_i}} \Phi_d(x) \right), \tag{63}$$

which leads to the formula that expresses the irreducible factorization of $Q_n(x)$ under general circumstances, namely

$$Q_n(x) = (x-2)^{(\gamma_1 - 1)/2} (x+2)^{(\gamma_2 - 1)/2} \times$$

$$\times \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i < 0}} \prod_{\substack{d \geq 3 \\ d | k_i}} \Psi_d(x) \right) \left(\prod_{\substack{i \in S_j \cup \{0\} \\ \varepsilon_i > 0}} \prod_{\substack{d \geq 3 \\ d | 2k_i \\ d \nmid k_i}} \Psi_d(x) \right). \tag{64}$$

(As obviously $\prod_{i=0}^{j} \varepsilon_i = -1$ and $\sum_{i=0}^{j} k_i$ is even, consequently both γ_1 and γ_2 are odd.)

For the relatively simple special case, when $j = 2$, some more details on the factorizations of $Q_n(x)$ can be given easily. Taking into account all the 16 possible variations according to the sign of ε_1 and ε_2, as well as the parity of k_1 and k_2, we find that $\gamma_1 = 3$ if and only if $\varepsilon_1 < 0$ and $\varepsilon_2 < 0$, independently on the parity of k_1 and k_2. In any other cases $\gamma_1 = 1$. For the value of γ_2, a more complicated condition has to be formulated, namely $\gamma_2 = 3$ if and only if $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, k_1 is odd and k_2 is odd, or $\varepsilon_1 > 0$, $\varepsilon_0 > 0$, k_1 is odd and

k_0 is odd (then $\varepsilon_2 < 0$, and k_2 is even), or $\varepsilon_2 > 0$, $\varepsilon_0 > 0$, k_2 is odd and k_0 is odd (then $\varepsilon_1 < 0$, and k_1 is even), or $\varepsilon_1 < 0$, $\varepsilon_2 < 0$, k_1 is even and k_2 is even. (Here $\varepsilon_0 = -\varepsilon_1\varepsilon_2$ and $k_0 = 2n+2-k_1-k_2$.) In any other cases $\gamma_2 = 1$. According to the above thoughts, we can state that

$$U_n(x) + \varepsilon_1 U_{n-k_1}(x) + \varepsilon_2 U_{n-k_2}(x) + \varepsilon_1\varepsilon_2 U_{n-k_1-k_2}(x) = E_\gamma \cdot E_0 \cdot E_1 \cdot E_2, \tag{65}$$

where

$$E_\gamma = \begin{cases} (x-2)(x+2) & \text{if } \varepsilon_1 < 0,\ \varepsilon_2 < 0,\ k_1 \text{ is even and } k_2 \text{ is even}, \\ (x-2) & \text{if } \varepsilon_1 < 0,\ \varepsilon_2 < 0 \text{ and at least one of } k_1 \text{ and } k_2 \text{ is odd}, \\ (x+2) & \text{if } \varepsilon_1 > 0,\ \varepsilon_2 > 0,\ k_1 \text{ is odd and } k_2 \text{ is odd}, \\ & \text{or } \varepsilon_1 > 0,\ \varepsilon_2 < 0,\ k_1 \text{ is odd and } k_2 \text{ is even}, \\ & \text{or } \varepsilon_1 < 0,\ \varepsilon_2 > 0,\ k_1 \text{ is even and } k_2 \text{ is odd}, \\ 1 & \text{in any other case}, \end{cases} \tag{66}$$

$$E_0 = \begin{cases} \prod_{\substack{d \ge 3 \\ d \mid 2n+2-k_1-k_2}} \Psi_d(x) & \text{if } \varepsilon_1\varepsilon_2 > 0, \\ \prod_{\substack{d \ge 3 \\ d \mid 4n+4-2k_1-2k_2 \\ d \nmid 2n+2-k_1-k_2}} \Psi_d(x) & \text{if } \varepsilon_1\varepsilon_2 < 0, \end{cases} \tag{67}$$

and for $i = 1, 2$

$$E_i = \begin{cases} \prod_{\substack{d \ge 3 \\ d \mid k_i}} \Psi_d(x) & \text{if } \varepsilon_i < 0, \\ \prod_{\substack{d \ge 3 \\ d \mid 2k_i \\ d \nmid k_i}} \Psi_d(x) & \text{if } \varepsilon_i > 0. \end{cases} \tag{68}$$

Finally as a few examples let us apply (65)–(68) for $k_1 = 1$, $k_2 = 2$ (and then $k_0 = 2n-1$), where only ε_1 and ε_2 varies. For this case,

$$E_\gamma = \begin{cases} (x-2) & \text{if } \varepsilon_1 < 0 \text{ and } \varepsilon_2 < 0, \\ (x+2) & \text{if } \varepsilon_1 > 0 \text{ and } \varepsilon_2 < 0, \\ 1 & \text{if } \varepsilon_2 > 0, \end{cases} \tag{69}$$

$$E_0 = \begin{cases} \prod_{\substack{d\geq 3\\ d|2n-1}} \Psi_d(x) & \text{if } \varepsilon_1\varepsilon_2 > 0, \\ \prod_{\substack{d\geq 3\\ d|4n-2\\ d\nmid 2n-1}} \Psi_d(x) & \text{if } \varepsilon_1\varepsilon_2 < 0, \end{cases} \tag{70}$$

$$E_1 = 1 \ \text{ for all variations of } \varepsilon_1 \text{ and } \varepsilon_2, \tag{71}$$

$$E_2 = \begin{cases} 1 & \text{if } \varepsilon_2 < 0, \\ \Psi_4(x) = x & \text{if } \varepsilon_2 > 0. \end{cases} \tag{72}$$

Therefore we obtain, that for $n \geq 3$

$$\begin{gathered} U_n(x) + U_{n-1}(x) + U_{n-2}(x) + U_{n-3}(x) = \\ = x \cdot \prod_{\substack{d\geq 3\\ d|2n-1}} \Psi_d(x) \ \ / = x \cdot W_{n-1}(x)/, \end{gathered} \tag{73}$$

$$\begin{gathered} U_n(x) + U_{n-1}(x) - U_{n-2}(x) - U_{n-3}(x) = \\ = (x+2) \cdot \prod_{\substack{d\geq 3\\ d|4n-2\\ d\nmid 2n-1}} \Psi_d(x) \ \ / = (x+2) \cdot V_{n-1}(x)/, \end{gathered} \tag{74}$$

$$\begin{gathered} U_n(x) - U_{n-1}(x) + U_{n-2}(x) - U_{n-3}(x) = \\ = x \cdot \prod_{\substack{d\geq 3\\ d|4n-2\\ d\nmid 2n-1}} \Psi_d(x) \ \ / = x \cdot V_{n-1}(x)/, \end{gathered} \tag{75}$$

and

$$\begin{gathered} U_n(x) - U_{n-1}(x) - U_{n-2}(x) + U_{n-3}(x) = \\ = (x-2) \cdot \prod_{\substack{d\geq 3\\ d|2n-1}} \Psi_d(x) \ \ / = (x-2) \cdot W_{n-1}(x)/. \end{gathered} \tag{76}$$

It may be noticed, however, that (73)–(76) are valid also for $n = 1$ and $n = 2$ if we take $U_{-1}(x) = 0$ and $U_{-2}(x) = -1$ according to (13) and (14).

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