Annales Univ. Sci. Budapest., Sect. Comp. 53 (2022) 71–83

ALMOST BALCOBALANCING NUMBERS

Ahmet Tekcan and Meryem Yıldız

(Bursa, Türkiye)

Communicated by Imre Kátai

(Received December 30, 2021; accepted July 21, 2022)

Abstract. In this work, we determined the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and second type in terms of balancing and Lucas-balancing numbers.

1. Introduction

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1.1)

$$r = \frac{-2n - 1 + \sqrt{8n^2+1}}{2}. \tag{1.2}$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), $8(0)^2+1 = 1$ and $8(1)^2+1 = 3^2$ are perfect squares. So Behera and Panda accepted 0 and 1 to be balancing numbers.

Key words and phrases: Balancing numbers, cobalancing numbers, balcobalancing numbers, Pell equations, set of representations.

2010 Mathematics Subject Classification: 11B37, 11B39, 11D09, 11D79.

https://doi.org/10.71352/ac.53.071

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (1.3)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \tag{1.4}$$

From (1.4), $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then from (1.2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (1.4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \text{ and } c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively (see also [10, 11, 15, 16]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas-balancing number. In [18], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

for some positive integers n and r, then $an + b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For positive integers k, x, let $\Pi_k(x) = x(x+1)\ldots(x+k-1)$. Then it was proved in [5] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2,3,4\}$ all solutions were determined. In [25] Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers and in [17], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [3], Dash, Ota, Dash considered the t-balancing numbers for an integer $t \geq 1$. They called that n is a t-balancing number if the Diophantine equation

$$1 + 2 + \cdots + n - 1 = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t-balancer. Similarly a positive integer n is called a t-cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t-cobalancer. In [24], Tekcan and Aydın determined the general terms of t-balancing and Lucas t-balancing numbers, and in [21], Tekcan and Erdem determined the general terms of t-cobalancing and Lucas t-cobalancing numbers in terms of balancing and Lucas-balancing numbers. In [13], G.K. Panda and A.K. Panda defined that a positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)]| = 1 \tag{1.5}$$

holds for some positive integer r, which is called the almost balancer. By (1.5), they have two cases: If $[(n+1)+(n+2)+...+(n+r)]-[1+2+...+(n-1)] = 1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 9}}{2}, \tag{1.6}$$

and if $[(n + 1) + (n + 2) + ... + (n + r)] - [1 + 2 + ... + (n - 1)] = -1$, then n is called an almost balancing number of second type and r is called an almost balancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 7}}{2}. \tag{1.7}$$

(From (1.7), they noted that $8(1)^2 - 7 = 1^2$ and $8(2)^2 - 7 = 5^2$ are perfect squares. So they accepted 1 and 2 to be almost balancing numbers of second type). Let B^*_n denote the almost balancing number of first type and let B^{**}_n denote the almost balancing number of second type. Then from (1.6), B^*_n is an almost balancing number of first type if and only if $8(B^*_n)^2 + 9$ is a perfect square and from (1.7), B_n^{**} is an almost balancing number of second type if and only if $8(B_n^{**})^2 - 7$ is a perfect square. So

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} \quad \text{and} \quad C_n^{**} = \sqrt{8(B_n^{**})^2 - 7}$$

are integers which are called almost Lucas-balancing number of first type and of second type, respectively. In [19], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers and in [20], he considered the sums and spectral norms of all almost balancing numbers.

2. Almost balcobalancing numbers

In [22], Tekcan and Yıldız defined balcobalancing numbers, balcobalancers and Lucas-balcobalanciing numbers and determined the general terms of them. They sum both sides of (1.1) and (1.3), and they get the Diophantine equation

$$1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n = 2[(n+1)+(n+2)+\cdots+(n+r)] \tag{2.1}$$

and called a positive integer n is called a balcobalancing number if the Diophantine equation in (2.1) verified for some positive integer r which is called balcobalancer. From (2.1), they get

$$r = \frac{-2n-1+\sqrt{8n^2+4n+1}}{2}. \tag{2.2}$$

Let B_n^{bc} denote the balcobalancing number. Then from (2.2), B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2 + 4B_n^{bc} + 1$ is a perfect square. Thus

$$C_n^{bc} = \sqrt{8(B_n^{bc})^2 + 4B_n^{bc} + 1}$$

is an integer which are called Lucas-balcobancing number (see also [23]).

As in (1.5), from (2.1), we say that a positive integer n is called an almost balcobalancing number if the Diophantine equation

$$\left| \begin{array}{c} 1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n \\ -2[(n+1)+(n+2)+\cdots+(n+r)] \end{array} \right| = 1 \tag{2.3}$$

holds for some positive integer r which is called almost balcobalancer.

From (2.3), we have two cases: If $[1+2+...+(n-1)+1+2+...+(n-$ $-1)+n] - 2[(n+1)+(n+2)+...+(n+r)] = 1$, then n is called an almost balcobalancing number of first type, r is called an almost balcobalancer of first type and in this case

$$r = \frac{-2n-1+\sqrt{8n^2+4n-3}}{2}. \tag{2.4}$$

For example $3, 39, 109$ are almost balcobalancing numbers of first type with almost balcobalancers of first type $1, 16, 45$. (From (2.4), we note that $8(1)^2 + +4(1) - 3 = 3^2$ is a perfect square, but in this case $r = 0$. Nevertheless we accept that 1 is an almost balcobalancing number of first type, just like Behera and Panda accepted 0 and 1 to be balancing numbers). If $[1 + 2 + ... + (n - -1) + 1 + 2 + ... + (n - 1) + n] - 2[(n + 1) + (n + 2) + ... + (n + r)] = -1$, then n is called an almost balcobalancing number of second type, r is called an almost balcobalancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 5}}{2}. \tag{2.5}$$

For example $5, 179, 6089$ are almost balcobalancing numbers of second type with almost balcobalancers of second type $2, 74, 2522$.

Let B_n^{bc*} denote the almost balcobalancing number of first type and let B_n^{bc**} denote the almost balcobalancing number of second type. Then by (2.4), B_n^{bc*} is an almost balcobalancing number of first type if and only if $8(B_n^{bc*})^2 + 4B_n^{bc*} - 3$ is a perfect square. So

$$C_n^{bc*} = \sqrt{8(B_n^{bc*})^2 + 4B_n^{bc*} - 3} \tag{2.6}$$

is an integer which is called almost Lucas-balcobalancing number of first type. Similarly from (2.5), B_n^{bc**} is an almost balcobalancing number of second type if and only if $8(B_n^{bc**})^2 + 4B_n^{bc**} + 5$ is a perfect square. So

$$C_n^{bc**} = \sqrt{8(B_n^{bc**})^2 + 4B_n^{bc**} + 5} \tag{2.7}$$

is an integer which is called almost Lucas-balcobalancing number of second type (We denote the almost balcobalancer of first type by R_n^{bc*} and denote almost balcobalancer of second type by R_n^{bc**}).

In this paper, we try to determine the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and second type in terms of balancing and Lucas-balancing numbers.

2.1. Almost balcobalancing numbers of first type

From (2.4), we notice that B_n^{bc*} is an almost balcobalancing number of first type if and only if $8(B_n^{bc*})^2 + 4B_n^{bc*} - 3$ is a perfect square. So we set

$$8(B_n^{bc*})^2 + 4B_n^{bc*} - 3 = y^2$$

for some positive integer y. If we multiply both sides of the last equation by 2, then we get

$$16(B_n^{bc*})^2 + 8B_n^{bc*} - 6 = 2y^2$$

and hence

$$(4B_n^{bc*}+1)^2-7=2y^2.$$

Taking $x=4B_n^{bc*}+1$, we get the Pell equation (see [1, 9])

$$x^2-2y^2=7. \tag{2.8}$$

Let Ω^{bc*} denotes the set of all (positive) integer solutions of (2.8), that is,

$$\Omega^{bc*}=\{(x,y):x^2-2y^2=7\}.$$

In order to determine the set of all integer solutions of (2.8), we need some notations. Let Δ be a non-square discriminant. Then the Δ-order O_Δ is defined to be the ring $O_\Delta=\{x+y\rho_\Delta : x,y\in\mathbb{Z}\}$, where $\rho_\Delta=\sqrt{\frac{\Delta}{4}}$ if $\Delta\equiv 0 \pmod 4$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta\equiv 1 (\mathrm{mod}\ 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta})=\{x+y\sqrt{\Delta}:x,y\in\mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ.

Let $F(x,y)=ax^2+bxy+cy^2$ be an indefinite integral quadratic form ([4]) of discriminant $\Delta=b^2-4ac$. Then we can rewrite

$$F(x,y)=\frac{(xa+y\frac{b+\sqrt{\Delta}}{2})(xa+y\frac{b-\sqrt{\Delta}}{2})}{a}.$$

So the module M_F of F is

$$M_F=\{xa+y\frac{b+\sqrt{\Delta}}{2}:x,y\in\mathbb{Z}\}\subset\mathbb{Q}(\sqrt{\Delta}).$$

Therefore we get $(u+v\rho_\Delta)(xa+y\frac{b+\sqrt{\Delta}}{2})=x'a+y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \quad y']=\begin{cases} [x \quad y]\begin{bmatrix} u-\frac{b}{2}v & av \\ -cv & u+\frac{b}{2}v \end{bmatrix} & \text{if } \Delta\equiv 0 \,(\mathrm{mod}\ 4) \\ [x \quad y]\begin{bmatrix} u+\frac{1-b}{2}v & av \\ -cv & u+\frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta\equiv 1 (\mathrm{mod}\ 4). \end{cases} \tag{2.9}$$

Let m be any integer and let Ω denote the set of all integer solutions of $F(x,y)=$ $=ax^2+bxy+cy^2=m$. Then there is a bijection

$$\Psi:\Omega\to\{\gamma\in M_F:N(\gamma)=am\}.$$

The action of $O_{\Delta,1}^u=\{\alpha\in O_\Delta^u:N(\alpha)=1\}$ on Ω is most interesting when Δ is a positive non-square since $O_{\Delta,1}^u$ is infinite. Therefore the orbit of each solution will be infinite and so Ω is either empty or infinite. Since $O_{\Delta,1}^u$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ that is grater than 1 and let $\tau_\Delta=\varepsilon_\Delta$ if $N(\varepsilon_\Delta)=1$ or ε_Δ^2 if $N(\varepsilon_\Delta)=-1$.

Then every $O^u_{\Delta,1}$ orbit of integral solutions of $F(x,y)=m$ contains a solution $(x,y)\in\mathbb{Z}\times\mathbb{Z}$ such that $0\le y\le U$, where

$$U=\begin{cases} \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1-\frac{1}{\tau_\Delta}) & \text{if } am>0\\ \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1+\frac{1}{\tau_\Delta}) & \text{if } am<0.\end{cases}$$

So for finding a set of representatives of the $O^u_{\Delta,1}$ orbits of integral solutions of $F(x,y)=m$, we must find for each y_0 in the range $0\le y_0\le U$, whether Δy_0^2+4am is a perfect square or not since

$$ax_0^2+bx_0y_0+cy_0^2=m\Leftrightarrow \Delta y_0^2+4am=(2ax_0+by_0)^2. \tag{2.10}$$

If Δy_0^2+4am is a perfect square, then from (2.10)

$$x_0=\frac{-by_0\pm\sqrt{\Delta y_0^2+4am}}{2a}.$$

So there is a set of representatives $\mathrm{Rep}=\{[x_0\ \ y_0]\}$. For the matrix M derived from (2.9), the set of all integer solutions of $F(x,y)=m$ is

$$\{\pm(x,y):[x\ \ y]=[x_0\ \ y_0]M^n, n\in\mathbb{Z}\}.$$

If Δy_0^2+4am is not a perfect square, then there is no integer solution.

Now we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of* (2.8) *is*

$$\Omega^{bc*}=\{(x_{2n-1},y_{2n-1}),(x_{2n},y_{2n}):n\ge 1\},$$

where

$$x_{2n-1}=4B_{n-1}+3C_{n-1},\ \ y_{2n-1}=6B_{n-1}+C_{n-1},$$

$$x_{2n}=-4B_n+3C_n,\ \ y_{2n}=6B_n-C_n.$$

Proof. For the Pell equation in (2.8), the indefinite form is $F=(1,0,-2)$ of discriminant $\Delta=8$. So $\tau_\Delta=3+2\sqrt{2}$. Therefore the set of representatives is $\mathrm{Rep}=\{[\pm 3\quad 1]\}$ and $M=\begin{bmatrix}3&2\\4&3\end{bmatrix}$ by (2.9). Here we notice that

1. $[3\quad 1]M^{n-1}$ generates all integer solutions (x_{2n-1},y_{2n-1}) for $n\ge 1$
2. $[3\quad -1]M^n$ generates all integer solutions (x_{2n},y_{2n}) for $n\ge 1$.

It can be easily seen that the n^{th} power of M is

$$M^n=\begin{bmatrix}C_n&2B_n\\4B_n&C_n\end{bmatrix}$$

for $n \geq 1$. So

$$\begin{aligned} [x_{2n-1} \quad y_{2n-1}] &= [3 \quad 1]M^{n-1} = [4B_{n-1} + 3C_{n-1} \quad 6B_{n-1} + C_{n-1}] \\ [x_{2n} \quad y_{2n}] &= [3 \quad -1]M^{n} = [-4B_{n} + 3C_{n} \quad 6B_{n} - C_{n}]. \end{aligned}$$

Thus the set of all integer solutions is $\Omega^{bc*} = \{(4B_{n-1}+3C_{n-1}, 6B_{n-1}+C_{n-1}), (-4B_n + 3C_n, 6B_n - C_n) : n \geq 1\}$. ∎

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned} B^{bc*}_{2n-1} &= \frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ B^{bc*}_{2n} &= \frac{4B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ C^{bc*}_{2n-1} &= 6B_{2n-1} - C_{2n-1}, \\ C^{bc*}_{2n} &= 6B_{2n-1} + C_{2n-1}, \\ R^{bc*}_{2n-1} &= \frac{16B_{2n-1} - 5C_{2n-1} - 1}{4}, \\ R^{bc*}_{2n} &= \frac{8B_{2n-1} - C_{2n-1} - 1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of (2.8) is $\Omega^{bc*} = \{(4B_{n-1}+3C_{n-1}, 6B_{n-1}+C_{n-1}), (-4B_n+3C_n, 6B_n-C_n) : n \geq 1\}$. Since $x = 4B^{bc*}_n + 1$, we get

$$B^{bc*}_{2n-1} = \frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}$$

for $n \geq 1$. Thus from (2.6), we obtain

$$\begin{aligned} C^{bc*}_{2n-1} &= \sqrt{8(B^{bc*}_{2n-1})^2 + 4B^{bc*}_{2n-1} - 3} = \\ &= \sqrt{8(\frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4})^2 + 4(\frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}) - 3} = \\ &= \sqrt{8B^2_{2n-1} - 12B_{2n-1}C_{2n-1} + \frac{9C^2_{2n-1}}{2} - \frac{7}{2}} = \\ &= \sqrt{8B^2_{2n-1} - 12B_{2n-1}C_{2n-1} + \frac{9}{2}(8B^2_{2n-1} + 1) - \frac{7}{2}} = \\ &= \sqrt{36B^2_{2n-1} - 12B_{2n-1}C_{2n-1} + 8B^2_{2n-1} + 1} = \\ &= \sqrt{36B^2_{2n-1} - 12B_{2n-1}C_{2n-1} + C^2_{2n-1}} = \end{aligned}$$

$$= \sqrt{(6B_{2n-1} - C_{2n-1})^2} =$$
$$= 6B_{2n-1} - C_{2n-1}.$$

Finally from (2.4), we deduce that

$$\begin{aligned} R^{bc*}_{2n-1} &= \frac{-2B^{bc*}_{2n-1} - 1 + C^{bc*}_{2n-1}}{2} = \\ &= \frac{-2(\frac{-4B_{2n-1}+3C_{2n-1}-1}{4}) - 1 + 6B_{2n-1} - C_{2n-1}}{2} = \\ &= \frac{16B_{2n-1} - 5C_{2n-1} - 1}{4}. \end{aligned}$$

Similarly it can be shown that $B^{bc*}_{2n} = \frac{4B_{2n-1}+3C_{2n-1}-1}{4}$, $C^{bc*}_{2n} = 6B_{2n-1}+C_{2n-1}$ and $R^{bc*}_{2n} = \frac{8B_{2n-1}-C_{2n-1}-1}{4}$. ■

In **Table 2.1**, the first ten terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type is given.

i	B_i^{bc*}	C_i^{bc*}	R_i^{bc*}
1	1	3	0
2	3	9	1
3	39	111	16
4	109	309	45
5	1333	3771	552
6	3711	10497	1537
7	45291	128103	18760
8	126073	356589	52221
9	1538569	4351731	637296
10	4282779	12113529	1773985

Table 2.1 Almost balcobalancing numbers of first type

2.2. Almost balcobalancing numbers of second type

From (2.5), we notice that B_n^{bc**} is an almost balcobalancing number of second type if and only if $8(B_n^{bc**})^2 + 4B_n^{bc**} + 5$ is a perfect square. So we set

$$8(B_n^{bc**})^2 + 4B_n^{bc**} + 5 = y^2$$

for some positive integer y. If we multiply both sides of the last equation by 2, then we get

$$16(B_n^{bc**})^2 + 8B_n^{bc**} + 10 = 2y^2$$

and hence

$$(4B_n^{bc**} + 1)^2 + 9 = 2y^2.$$

Taking $x = 4B_n^{bc**} + 1$, we get the Pell equation

$$x^2 - 2y^2 = -9. \tag{2.11}$$

Let Ω^{bc**} denotes the set of all (positive) integer solutions of (2.11), that is,

$$\Omega^{bc**} = \{(x, y) : x^2 - 2y^2 = -9\}.$$

Then we can give the following theorem.

Theorem 2.3. *The set of all integer solutions of* (2.11) *is*

$$\Omega^{bc**} = \{(x_n, y_n) \ : \ n \geq 1\},$$

where

$$x_n = 12B_{n-1} + 3C_{n-1} \text{ and } y_n = 6B_{n-1} + 3C_{n-1}.$$

Proof. For the Pell equation in (2.11), the indefinite form is $F = (1, 0, -2)$ of discriminant $\Delta = 8$. So $\tau_\Delta = 3 + 2\sqrt{2}$. In this case, the set of representatives is $\text{Rep} = \{[\pm 3 \quad 3]\}$ and $M = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. Here $[3 \quad 3]M^{n-1}$ generates all integer solutions (x_n, y_n) for $n \geq 1$. Thus the set of all integer solutions is $\Omega^{bc**} = \{(12B_{n-1} + 3C_{n-1}, 6B_{n-1} + 3C_{n-1}) : n \geq 1\}$. ■

From Theorem 2.3, we can give the following result.

Theorem 2.4. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of second type are*

$$\begin{aligned} B_n^{bc**} &= \frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ C_n^{bc**} &= 6B_{2n-1} + 3C_{2n-1}, \\ R_n^{bc**} &= \frac{3C_{2n-1} - 1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. Note that $\Omega^{bc**} = \{(12B_{n-1} + 3C_{n-1}, 6B_{n-1} + 3C_{n-1}) : n \geq 1\}$ by Theorem 2.3. So we get

$$B_n^{bc**} = \frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}$$

for $n \geq 1$. Thus from (2.7), we obtain

$$\begin{aligned}
C_n^{bc**} &= \sqrt{8(B_n^{bc**})^2 + 4B_n^{bc**} + 5} = \\
&= \sqrt{8(\frac{12B_{2n-1} + 3C_{2n-1} - 1}{4})^2 + 4(\frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}) + 5} = \\
&= \sqrt{72B_{2n-1}^2 + 36B_{2n-1}C_{2n-1} + \frac{9C_{2n-1}^2}{2} + \frac{9}{2}} = \\
&= \sqrt{36B_{2n-1}^2 + 36B_{2n-1}C_{2n-1} + 9C_{2n-1}^2} = \\
&= \sqrt{(6B_{2n-1} + 3C_{2n-1})^2} = \\
&= 6B_{2n-1} + 3C_{2n-1}.
\end{aligned}$$

From (2.5), we deduce that $R_n^{bc**} = \frac{3C_{2n-1}-1}{4}$. ■

In **Table 2.2**, the first ten terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of second type is given.

i	B_i^{bc**}	C_i^{bc**}	R_i^{bc**}
1	5	15	2
2	179	507	74
3	6089	17223	2522
4	206855	585075	85682
5	7026989	19875327	2910674
6	238710779	675176043	98877242
7	8109139505	22936110135	3358915562
8	275472032399	779152568547	114104251874
9	9357939962069	26468251220463	3876185648162
10	317894486677955	899141388927195	131676207785642

Table 2.2 Almost balcobalancing numbers of second type

References

[1] **Barbeau, E.J.,** *Pell's Equation*, Springer-Verlag New York, 2003.

[2] **Behera, A. and G.K. Panda,** On the square roots of triangular numbers, *The Fibonacci Quart.*, **37(2)** (1999), 98–105.

[3] **Dash, K.K., R.S. Ota and S. Dash,** t-Balancing numbers, *Int. J. Cont. Math. Sci.*, **7(41)** (2012), 1999–2012.

[4] **Flath, D.E.,** *Introduction to Number Theory*, Wiley, 1989.

[5] **Kovacs, T., K. Liptai and P. Olajos,** On (a, b)-balancing bumbers, *Publ. Math. Debrecen*, **77(3-4)**(2010), 485–498.

[6] **Liptai, K., F. Luca, A. Pinter and L. Szalay,** Generalized balancing numbers, *Indag. Mathem. N. S.*, **20(1)** (2009), 87–100.

[7] **Liptai, K.,** Fibonacci balancing numbers, *The Fibonacci Quarterly*, **42(4)** (2004), 330–340.

[8] **Liptai, K.,** Lucas balancing numbers, *Acta Math. Univ. Ostrav.*, **14** (2006), 43–47.

[9] **Jacobson, M. and H. Williams,** *Solving the Pell Equation*, CMS Books in Mathematics, Springer Science, Busines Media, LLC 2009.

[10] **Olajos, P.,** Properties of balancing, cobalancing and generalized balancing numbers, *Annales Mathematicae et Informaticae*, **37** (2010), 125–138.

[11] **Panda, G.K. and P.K. Ray,** Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bul. of Inst. of Math. Acad. Sinica*, **6(1)** (2011), 41–72.

[12] **Panda, G.K. and P.K. Ray,** Cobalancing numbers and cobalancers, *Int. J. Math. Math. Sci.*, **8** (2005), 1189–1200.

[13] **Panda, G.K. and A.K. Panda,** Almost balancing numbers, *Jour. of the Indian Math. Soc.*, **82(3-4)**(2015), 147–156.

[14] **Panda, G.K., T. Komatsu and R.K. Davala,** Reciprocal sums of sequences involving balancing and Lucas-balancing numbers, *Math. Reports*, **20(70)** (2018), 201–214.

[15] **Panda, A.K.,** *Some Variants of the Balancing Sequences*, Ph.D. dissertation, National Institute of Technology Rourkela, India, 2017.

[16] **Ray, P.K.,** *Balancing and Cobalancing Numbers*, Ph.D. dissertation, Department of Mathematics, National Institute of Technology, Rourkela, India, 2009.

[17] **Ray, P.K.,** Balancing and Lucas-balancing sums by matrix methods, *Math. Reports*, **17(67)** (2015), 225–233.

[18] **Szalay, L.,** On the resolution of simultaneous Pell equations, *Ann. Math. Inform.*, **34** (2007), 77–87.

[19] **Tekcan, A.,** Almost balancing, triangular and square triangular numbers, *Notes on Number Theory and Discrete Maths.*, **25(1)** (2019), 108–121.

[20] **Tekcan, A.,** Sums and spectral norms of all almost balancing numbers, *Creat. Math. Inform.*, **28(2)** (2019), 203–214.

[21] **Tekcan, A. and A. Erdem,** t-cobalancing numbers and t-cobalancers, *Notes on Number Theory and Discrete Maths.*, **26(1)** (2020), 45–58.

[22] **Tekcan, A. and M. Yıldız,** Balcobalancing numbers and balcobalancers, *Creative Mathematics and Informatics*, **30(2)** (2021), 203–222.

[23] **Tekcan, A. and M. Yıldız,** Balcobalancing numbers and balcobalancers II., *Creative Mathematics and Informatics*, **31(2)** (2022), 247–258.

[24] **Tekcan, A. and S. Aydın,** On t-balancers, t-balancing numbers and Lucas t-balancing numbers, *Libertas Mathematica*, accepted.

[25] **Tengely, S.,** Balancing numbers which are products of consecutive integers, *Publ. Math. Debrecen*, **83(1-2)** (2013), 197–205.

A. Tekcan and M. Yıldız
Bursa Uludag University
Faculty of Science
Department of Mathematics
Bursa, Türkiye
`tekcan@uludag.edu.tr, 502011002@ogr.uludag.edu.tr`