

ON THE EQUATION

$$f(n^2 + nm + m^2) = f(n)^2 + f(n)f(m) + f(m)^2$$

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Abstract. We give all solutions of the equation

$$f(n^2 + nm + m^2) = f(n)^2 + f(n)f(m) + f(m)^2,$$

where f is an arbitrary complex valued function defined on $\mathbb{N}$.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively.

A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative if

$$f(nm) = f(n)f(m) \quad \text{for every } n, m \in \mathbb{N}, (n, m) = 1.$$

Let $\mathcal{M}$ be the set of complex-valued multiplicative functions.

Let $Q(x, y) \in \mathbb{Z}[x, y]$ be a polynomial with two variable and let A, B be subsets of $\mathbb{N}$. We are interested in finding solutions $f : \mathbb{N} \rightarrow \mathbb{C}$ to an equation of the form

$$(1.1) \quad f(Q(a, b)) = Q(f(a), f(b)) \quad \text{for every } a \in A, b \in B.$$

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If $Q(x, y) = x + y$, $A = B = \mathbb{N}$, then it can be shown that there is a single family of solutions, namely $f(n) = cn$, where $c = f(1) \in \mathbb{C}$ is an arbitrary number.

In 1992, C. Spiro [16] considered the equation (1.1) in the case when $Q(x, y) = x + y$ and $A = B = \mathcal{P}$. She proved that if a real-valued multiplicative function f satisfies

$$f(p+q) = f(p) + f(q) \quad (\forall p, q \in \mathcal{P}) \quad \text{and} \quad f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P},$$

then $f(n)$ is the identity function.

In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong [10] proved that if a function $f \in \mathcal{M}$ satisfy the condition

$$f(p+m^2) = f(p) + f(m^2) \quad \text{for every} \quad p \in \mathcal{P}, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

For some generalizations of the above results, we refer to the other works of P. V. Chung and B. M. Phong [2], K. K. Chen and Y. G. Chen [3], A. Dubickas and P. Sarka [4], J.-H. Fang [5], B. M. Phong [11]–[14].

In 2014 B. Bojan [1] considered the case $Q(x, y) = x^2 + y^2$, $A = B = \mathbb{N}$. He determined all solutions of those $f : \mathbb{N} \rightarrow \mathbb{C}$ for which

$$f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

It is proved that all solutions $f(n)$ of this equation satisfy

$$f(n) \in \left\{ 0, \frac{\varepsilon(n)}{2}, \varepsilon(n)n \right\},$$

where $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n^2 + m^2) = 1$ for every $n, m \in \mathbb{N}$.

I. Kátai and B. M. Phong posed the following conjecture for the cases when $Q(x, y) = x^2 + Dy^2$ and $A = B = \mathbb{N}$:

Conjecture 1. (I. Kátai and B. M. Phong [6]) *Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation*

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

- $f(n) = 0$ for every $n \in \mathbb{N}$,
- $f(n) = \frac{\epsilon(n)}{D+1}$ for every $n \in \mathbb{N}$,
- $f(n) = \epsilon(n)n$ for every $n \in \mathbb{N}$,

where $\epsilon(n) \in \{-1, 1\}$ and $\epsilon(n^2 + Dm^2) = 1$ for every $n, m \in \mathbb{N}$.

B. M. M. Khanh in [7] and [8] proved this conjecture. In [9] she given all functions $f : \mathbb{N} \rightarrow \mathbb{C}$ which satisfy the equation

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \text{for every } n, m \in \mathbb{N},$$

where $k \in \mathbb{N}_0$, $K \in \mathbb{C}$.

Recently Poo Sung Park [15] proved the following results

Theorem A. *If $f \in \mathcal{M}$ and*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N},$$

then f is the identity function.

Our general purpose is to determine all those $f : \mathbb{N} \rightarrow \mathbb{C}$ for which

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N},$$

where $D \in \mathbb{Z}$ is a fixed number.

In this paper we will consider the case $D = 1$. It seems to us that the cases of other D need some new ideas.

Some of the lemmas used in the proof are formulated in more general form, containing D if it true in this form.

Theorem 1. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Then the following assertions holds:

- (a) *If $f(1) = 0$, then $f(n) = 0$ for every $n \in \mathbb{N}$.*
- (b) *If $f(1) = \frac{1}{3}$, then $f(n) = \frac{1}{3}$ for every $n \in \mathbb{N}$.*
- (c) *If $f(1) = 1$, then $f(n) = n$ for every $n \in \mathbb{N}$.*

We note that Theorem A is a direct consequence of Theorem 1.

2. Lemmas

In this section we prove some lemmas.

Lemma 1. *Let $D \in \mathbb{N}$ be fixed and $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies*

$$(2.1) \quad f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

If one of the following assertions holds:

- (a) $f(n) = 0$ for every $n \leq 8D + 20$,
- (b) $f(n) = \frac{1}{D+2}$ for every $n \leq 8D + 20$,
- (c) $f(n) = n$ for every $n \leq 8D + 20$,

then the corresponding one of (a), (b), (c) holds for all $n \in \mathbb{N}$.

Proof. We shall prove the case (c). The proof of (a) and (b) is similar way.

Assume that the conditions (2.1) and (c) are satisfied. Assume that $f(n) = n$ for all $n < N$, where $N \geq 8D + 21$. We will prove that $f(N) = N$.

If $N = 2k + 1$, then $N = 2k + 1 \geq 8D + 21$ and $k \geq 4D + 10$. Consequently

$$\begin{aligned} 1 \leq k - D - 2 < N, \quad 1 \leq 2k - 2D - 1 < N, \\ 1 \leq k - 3D - 7 < N \quad \text{and} \quad 1 \leq 2k - 6D - 5 < N, \end{aligned}$$

which with our assumptions imply

$$\begin{aligned} f(k - D - 2) &= k - D - 2, \quad f(2k - 2D - 1) = 2k - 2D - 1, \\ f(k - 3D - 7) &= k - 3D - 7, \quad f(2k - 6D - 5) = 2k - 6D - 5. \end{aligned}$$

By using (2.1), the above relations with the following relations

$$\begin{aligned} (2k + 1)^2 + D(2k + 1)(k - D - 2) + (k - D - 2)^2 &= \\ = (2k - 2D - 1)^2 + D(2k - 2D - 1)(k + 2) + (k + 2)^2 \end{aligned}$$

and

$$\begin{aligned} (2k + 1)^2 + D(2k + 1)(k - 3D - 7) + (k - 3D - 7)^2 &= \\ = (2k - 6D - 5)^2 + D(2k - 6D - 5)(k + 5) + (k + 5)^2 \end{aligned}$$

show that

$$\begin{aligned} f(2k + 1)^2 + Df(2k + 1)(k - D - 2) + (k - D - 2)^2 &= \\ = (2k - 2D - 1)^2 + D(2k - 2D - 1)(k + 2) + (k + 2)^2 \end{aligned}$$

and

$$\begin{aligned} f(2k + 1)^2 + Df(2k + 1)(k - 3D - 7) + (k - 3D - 7)^2 &= \\ = (2k - 6D - 5)^2 + D(2k - 6D - 5)(k + 5) + (k + 5)^2. \end{aligned}$$

It is easily shown from the last two relations that $f(2k + 1) = 2k + 1$, $f(N) = N$.

In the same way, if $N = 2k$, then $N = 2k \geq 8D + 21$ and $k \geq 4D + 11$. Consequently

$$\begin{aligned} 1 \leq k - 2D - 5 < N, \quad 1 \leq 2k - 4D - 4 < N, \\ 1 \leq k - 4D - 10 < N \quad \text{and} \quad 1 \leq 2k - 8D - 8 < N, \end{aligned}$$

which with our assumptions imply

$$\begin{aligned} f(k - 2D - 5) &= k - 2D - 5, \quad f(2k - 4D - 4) = 2k - 4D - 4, \\ f(k - 4D - 10) &= k - 4D - 10, \quad f(2k - 8D - 8) = 2k - 8D - 8. \end{aligned}$$

By using (2.1), the above relations with the following relations

$$\begin{aligned} (2k)^2 + D(2k)(k - 2D - 5) + (k - 2D - 5)^2 &= \\ = (2k - 4D - 4)^2 + D(2k - 4D - 4)(k + 3) + (k + 3)^2 \end{aligned}$$

and

$$\begin{aligned} (2k)^2 + D(2k)(k - 4D - 10) + (k - 4D - 10)^2 &= \\ (2k - 8D - 8)^2 + D(2k - 8D - 8)(k + 6) + (k + 6)^2 \end{aligned}$$

show that

$$\begin{aligned} f(2k)^2 + Df(2k)(k - 2D - 5) + (k - 2D - 5)^2 &= \\ = (2k - 4D - 4)^2 + D(2k - 4D - 4)(k + 3) + (k + 3)^2 \end{aligned}$$

and

$$\begin{aligned} f(2k)^2 + Df(2k)(k - 4D - 10) + (k - 4D - 10)^2 &= \\ (2k - 8D - 8)^2 + D(2k - 8D - 8)(k + 6) + (k + 6)^2. \end{aligned}$$

It is easily shown from the last two relations that $f(2k) = 2k$, $f(N) = N$.

Lemma 1 is proved. ■

In the following we assume that $D = 1$ and $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies (2.1), i.e.

$$(2.2) \quad f(n^2 + nm + m^2) = f(n)^2 + f(n)f(m) + f(m)^2 \quad \text{for every } n, m \in \mathbb{N}.$$

Let

$$E := \{N \in \mathbb{N} \mid N = n^2 + nm + m^2 = u^2 + uv + v^2 \quad \text{with } n > u, m \leq n, v \leq u\}.$$

It is obvious that if $N \in E$, then we obtain from (2.2) that

$$f(N) = f^2(n) + f(n)f(m) + f^2(m) \quad \text{and} \quad f(N) = f^2(u) + f(u)f(v) + f^2(v)$$

which show that

$$(2.3) \quad E(N) = f^2(n) + f(n)f(m) + f^2(m) - f^2(u) - f(u)f(v) - f^2(v) = 0$$

for every $N \in E$. First we note from (2.2) that

$$(2.4) \quad \begin{cases} f(3) &= f^2(1) + f(1)f(1) + f^2(1), \\ f(7) &= f^2(2) + f(2)f(1) + f^2(1), \\ f(12) &= f^2(2) + f(2)f(2) + f^2(2), \\ f(13) &= f^2(3) + f(3)f(1) + f^2(1), \\ f(19) &= f^2(3) + f(3)f(2) + f^2(2), \\ f(21) &= f^2(4) + f(4)f(1) + f^2(1), \\ f(27) &= 3f^2(3), \\ f(28) &= f^2(4) + f(4)f(2) + f^2(2). \end{cases}$$

Lemma 2. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies (2.2). If*

$$f(1) = 0, \quad f(2) = 0 \quad \text{and} \quad f(4) = 0,$$

then $f(n) = 0$ for every $n \in \mathbb{N}$.

Proof. Let

$$\mathcal{S}_0 := \{n \in \mathbb{N} \mid f(n) = 0\}.$$

Since $D = 1$, by using Lemma 1, we need to prove that

$$\{1, 2, \dots, 28\} \subset \mathcal{S}_0.$$

We obtain from $f(1) = f(2) = f(4) = 0$ and (2.4) that

$$\{3, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{S}_0,$$

consequently

$$\{1, 2, 3, 4, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{S}_0$$

and

$$\begin{aligned} f^2(11) &= f^2(11) + f(11)f(2) + f^2(2) - 3f^2(7) = E(147) = 0, \\ -f^2(17) &= f^2(21) + f(21)f(7) + f^2(7) - f^2(17) - f(17)f(12) - f^2(12) = \\ &= E(637) = 0, \\ f^2(20) &= f^2(20) + f(20)f(7) - f^2(13) - f(13)f(12) - f^2(12) = E(469) = 0, \\ f^2(22) &= f^2(22) + f(22)f(1) + f^2(1) - 3f^2(13) = E(507) = 0, \\ f^2(23) &= f^2(23) + f(23)f(4) + f^2(4) - f^2(21) - f(21)f(7) - f^2(7) = \\ &= E(637) = 0, \end{aligned}$$

which gives

$$\{1, 2, 3, 4, 7, 11, 12, 13, 17, 19, 20, 21, 22, 23, 27, 28\} \subset \mathcal{S}_0.$$

Thus we have

$$\begin{aligned} f^2(5) &= f^2(28) + f(28)f(5) + f^2(5) - f^2(23) - f(23)f(12) - f^2(12) = \\ &= E(949) = 0, \\ -f^2(6) &= f^2(19) + f(19)f(3) + f^2(3) - f^2(17) - f(17)f(6) - f^2(6) = \\ &= E(427) = 0, \\ f^2(8) &= f^2(23) + f(23)f(8) + f^2(8) - f^2(19) - f(19)f(13) - f^2(13) = \\ &= E(777) = 0, \\ -f(9)^2 &= f(11)^2 + f(11)f(1) + f(1)^2 - f(9)^2 - f(9)f(4) - f(4)^2 = \\ &= E(133) = 0, \\ -f(10)^2 &= f(22)^2 + f(22)f(3) + f(3)^2 - f(17)^2 - f(17)f(10) - f(10)^2 = \\ &= E(559) = 0, \\ -3f(14)^2 &= f(22)^2 + f(22)f(4) + f(4)^2 - 3f(14)^2 = E(588) = 0, \\ -f(15)^2 &= f(20)^2 + f(20)f(7) + f(7)^2 - f(15)^2 - f(15)f(13) - f(13)^2 = \\ &= E(589) = 0, \\ -f(16)^2 &= f(23)^2 + f(23)f(1) + f(1)^2 - f(16)^2 - f(16)f(11) - f(11)^2 = \\ &= E(553) = 0, \\ f(24)^2 &= f(24)^2 + f(24)f(7) + f(7)^2 - f(21)^2 - f(21)f(11) - f(11)^2 = \\ &= E(793) = 0, \\ f(25)^2 &= f(25)^2 + f(25)f(2) + f(2)^2 - f(17)^2 - f(17)f(13) - f(13)^2 = \\ &= E(679) = 0, \\ f(26)^2 &= f(26)^2 + f(26)f(11) + f(11)^2 - 3f(19)^2 = E(1083) = 0, \end{aligned}$$

which implies

$$\begin{aligned} &\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \\ &15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\} \subset \mathcal{S}_0. \end{aligned}$$

Finally, we have

$$f(18)^2 = f(18)^2 + f(18)f(1) + f(1)^2 - f(14)^2 - f(14)f(7) - f(7)^2 = E(343) = 0,$$

and so

$$18 \in \mathcal{S}_0.$$

Thus, Lemma 2 is proved. ■

Lemma 3. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies (2.2). If*

$$f(1) = \frac{1}{3}, \quad f(2) = \frac{1}{3} \quad \text{and} \quad f(4) = \frac{1}{3},$$

then $f(n) = \frac{1}{3}$ for every $n \in \mathbb{N}$.

Proof. Let

$$\mathcal{S}_1 = \left\{ n \in \mathbb{N} \mid f(n) = \frac{1}{3} \right\}.$$

Since $D = 1$, by using Lemma 1, we need to prove that

$$\{1, 2, \dots, 28\} \subset \mathcal{S}_1.$$

It is obvious from $f(1) = \frac{1}{3}$, $f(2) = \frac{1}{3}$, $f(4) = \frac{1}{3}$ and (2.4) that

$$f(3) = f(7) = f(12) = f(13) = f(19) = f(21) = f(27) = f(28) = \frac{1}{3},$$

consequently

$$(2.5) \quad \{1, 2, 3, 4, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{S}_1.$$

We will prove in the following that

$$\{5, 6, 8, 9, 10, 11, 14, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26\} \in \mathcal{S}_1.$$

Let

$$I := \left\{ n \in \mathbb{N} \mid f(n) \in \left\{ -\frac{2}{3}, \frac{1}{3} \right\} \right\}.$$

Since

$$f(3n^2) = f(n^2 + n \cdot n + n^2) = 3f(n)^2,$$

we have

$$(2.6) \quad \text{If } n \in I \text{ and } 3n^2 \in I, \text{ then } n \in \mathcal{S}_1.$$

We write $N = [m, n]$ if $N = m^2 + mn + n^2$. It is enough to show

$$(2.7) \quad \text{If } m \in I \text{ and } n \in \mathcal{S}_1, \text{ then } N = [m, n] \in \mathcal{S}_1.$$

It is clear that if $m \in \mathcal{S}_1$, then (2.7) is true. Assume that $f(m) = -\frac{2}{3}$ and $f(n) = \frac{1}{3}$. Then

$$f(N) = f(m)^2 + f(m)f(n) + f(n)^2 = \frac{4}{9} - \frac{2}{9} + \frac{1}{9} = \frac{1}{3},$$

which completes the proof of (2.7).

We prove that $5 \in \mathcal{S}_1$. By using (2.6), we need to prove that $5 \in I$, $75 \in I$. By using (2.7), we have

$$(2.8) \quad \begin{cases} 37 = [4, 3] \in \mathcal{S}_1, & 48 = [4, 4] \in \mathcal{S}_1, & 79 = [7, 3] \in \mathcal{S}_1, \\ f(31) = f(5)^2 + f(5)f(1) + f(1)^2 = f(5)^2 + \frac{1}{3}f(5) + \frac{1}{9}, \\ f(61) = f(5)^2 + f(5)f(4) + f(4)^2 = f(5)^2 + \frac{1}{3}f(5) + \frac{1}{9}, \end{cases}$$

therefore we infer from (2.5) and (2.8) that

$$\begin{cases} E(1533) = f(37)^2 + f(37)f(4) + f(4)^2 - f(31)^2 - f(31)f(13) - f(13)^2 = \\ \quad = -\frac{1}{81}(3f(5) + 2)(3f(5) - 1)(9f(5)^2 + 3f(5) + 7) = 0, \\ E(6573) = f(79)^2 + f(79)f(4) + f(4)^2 - f(61)^2 - f(61)f(31) - f(31)^2 = \\ \quad = -\frac{1}{27}(3f(5) + 2)(3f(5) - 1)(9f(5)^2 + 3f(5) + 4) = 0. \end{cases}$$

These imply that $(3f(5) + 2)(3f(5) - 1) = 0$, and so $5 \in I$. We also have

$$\begin{cases} E(2587) = f(47)^2 + f(47)f(7) + f(7)^2 - f(37)^2 - f(37)f(21) - f(21)^2 = \\ \quad = \frac{1}{9}(3f(47) + 2)(3f(47) - 1) = 0, \\ E(6769) = f(75)^2 + f(75)f(13) + f(13)^2 - f(48)^2 - f(48)f(47) - f(47)^2 = \\ \quad = \frac{1}{3}(3f(75) + 1 + 3f(47))(f(75) - f(47)) = 0, \end{cases}$$

which show that $47 \in I$, and so we infer from the last relation that $75 \in I$. By using (2.6), we infer from facts $5 \in I$ and $3 \cdot 5^2 \in I$ that $5 \in \mathcal{S}_1$.

Thus, we can assume that

$$\{1, 2, 3, 4, 5, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{S}_1$$

By using (2.7), one can check that $31 = [5, 1] \in \mathcal{S}_1$, $49 = [5, 3] \in \mathcal{S}_1$, $147 = [7, 7]$, $172 = [12, 2] \in \mathcal{S}_1$, consequently

$$\begin{cases} E(31213) = f(172)^2 + f(172)f(9) + f(9)^2 - f(147)^2 - f(147)f(49) - \\ \quad - f(49)^2 = \frac{1}{9}(3f(9) + 2)(3f(9) - 1) = 0, \\ E(147) = f(11)^2 + f(11)f(2) + f(2)^2 - 3f(7)^2 = \\ \quad = \frac{1}{9}(3f(11) + 2)(3f(11) - 1) = 0, \\ E(259) = f(15)^2 + f(15)f(2) + f(2)^2 - f(13)^2 - f(13)f(5) - f(5)^2 = \\ \quad = \frac{1}{9}(3f(15) + 2)(3f(15) - 1) = 0, \end{cases}$$

$$\left\{ \begin{array}{l} E(637) = f(21)^2 + f(21)f(7) + f(7)^2 - f(17)^2 - f(17)f(12) - f(12)^2 = \\ \quad = -\frac{1}{9}(3f(17) + 2)(3f(17) - 1) = 0, \\ E(1027) = f(31)^2 + f(31)f(2) + f(2)^2 - f(19)^2 - f(19)f(18) - f(18)^2 = \\ \quad = -\frac{1}{9}(3f(18) + 2)(3f(18) - 1) = 0, \\ E(469) = f(20)^2 + f(20)f(3) + f(3)^2 - f(13)^2 - f(13)f(12) - f(12)^2 = \\ \quad = \frac{1}{9}(3f(20) + 2)(3f(20) - 1) = 0, \\ E(507) = f(22)^2 + f(22)f(1) + f(1)^2 - 3f(13)^2 = \\ \quad = \frac{1}{9}(3f(22) + 2)(3f(22) - 1) = 0, \\ E(637) = f(23)^2 + f(23)f(4) + f(4)^2 - f(21)^2 - f(21)f(7) - f(7)^2 = \\ \quad = \frac{1}{9}(3f(23) + 2)(3f(23) - 1) = 0, \end{array} \right.$$

we have $\{9, 11, 15, 17, 18, 20, 22, 23\} \subset I$, and so

$$\{1, 2, 3, 4, 5, 7, 9, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22, 23, 27, 28\} \subset I.$$

In the following we use the simple fact:

$$(2.9) \quad \text{If } a \in I \text{ and } (3f(a) + 1 + 3f(n))(f(a) - f(n)) = 0, \text{ then } n \in I.$$

It is obvious if $f(a) - f(n) = 0$. If $f(a) - f(n) \neq 0$, then

$$3f(a) + 1 + 3f(n) = 0 \quad \text{and} \quad f(n) = -\frac{3f(a) + 1}{3} = \begin{cases} -\frac{2}{3} & \text{if } f(a) = \frac{1}{3} \\ \frac{1}{3} & \text{if } f(a) = -\frac{2}{3}. \end{cases}$$

It is clear that $37 = [4, 3] \in \mathcal{S}_1$ and $39 = [5, 2] \in \mathcal{S}_1$, which with Maple, we obtain that

$$\begin{aligned} E(91) &= f(9)^2 + f(9)f(1) + f(1)^2 - f(6)^2 - f(6)f(5) - f(5)^2 = \\ &= \frac{1}{3}(3f(9) + 1 + 3f(6))(f(9) - f(6)) = 0, \\ E(1897) &= f(39)^2 + f(39)f(8) + f(8)^2 - f(37)^2 - f(37)f(11) - f(11)^2 = \\ &= -\frac{1}{3}(3f(8) + 1 + 3f(11))(f(11) - f(8)) = 0, \\ E(399) &= f(17)^2 + f(17)f(5) + f(5)^2 - f(13)^2 - f(13)f(10) - f(10)^2 = \\ &= \frac{1}{3}(3f(17) + 1 + 3f(10))(f(17) - f(10)) = 0, \end{aligned}$$

$$\begin{aligned} E(247) &= f(14)^2 + f(14)f(3) + f(3)^2 - f(11)^2 - f(11)f(7) - f(7)^2 = \\ &= -\frac{1}{3}(3f(11) + 1 + 3f(14))(f(11) - f(14)) = 0, \end{aligned}$$

$$\begin{aligned} E(793) &= f(24)^2 + f(24)f(7) + f(7)^2 - f(21)^2 - f(21)f(11) - f(11)^2 = \\ &= -\frac{1}{3}(3f(11) + 1 + 3f(24))(f(11) - f(24)) = 0, \end{aligned}$$

$$\begin{aligned} E(679) &= f(25)^2 + f(25)f(2) + f(2)^2 - f(17)^2 - f(17)f(13) - f(13)^2 = \\ &= -\frac{1}{3}(3f(17) + 1 + 3f(25))(f(17) - f(25)) = 0, \end{aligned}$$

$$\begin{aligned} E(1183) &= f(31)^2 + f(31)f(6) + f(6)^2 - f(26)^2 - f(26)f(13) - f(13)^2 = \\ &= \frac{1}{3}(3f(26) + 1 + 3f(6))(f(6) - f(26)) = 0, \end{aligned}$$

which with (2.9) imply that the numbers 6, 8, 10, 14, 24, 25, 26 belong to I . Finally, we prove that $16 \in I$. We infer from the relations

$$\begin{aligned} E(5833) &= f(56)^2 + f(56)f(31) + f(31)^2 - f(49)^2 - f(49)f(39) - f(39)^2 = \\ &= \frac{1}{9}(3f(56) + 2)(3f(56) - 1) = 0, \end{aligned}$$

$$\begin{aligned} E(3441) &= f(56)^2 + f(56)f(5) + f(5)^2 - f(49)^2 - f(49)f(16) - f(16)^2 = \\ &= \frac{1}{3}(3f(56) + 1 + 3f(16))(f(56) - f(16)) = 0, \end{aligned}$$

that $56 \in I$ and $16 \in I$.

Thus we have proved

$$A := \{1, 2, 3, 4, 5, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{S}_1$$

and

$$B := \{6, 8, 9, 10, 11, 14, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26\} \subset I.$$

Now we prove that

$$(2.10) \quad 3n^2 \in I \quad \text{for every } n \in B.$$

We note from (2.7) that the numbers $52 = [6, 2]$, $67 = [7, 2]$, $75 = [5, 5]$, $84 = [8, 2]$, $93 = [7, 4]$, $117 = [9, 3]$, $124 = [10, 2]$, $172 = [12, 2]$, $201 = [11, 5]$, $237 = [13, 4]$, $237 = [13, 4]$, $309 = [13, 7]$, $333 = [9, 12]$, $337 = [8, 13]$, $543 = [19, 7]$ are in $\mathcal{S}_1$. Thus, we have

$$\begin{aligned} E(12229) &= f(108)^2 + f(108)f(5) + f(5)^2 - f(75)^2 - f(75)f(52) - f(52)^2 = \\ &= \frac{1}{9}(3f(108) + 2)(3f(108) - 1) = 0, \end{aligned}$$

$$\begin{aligned}
E(56889) &= f(237)^2 + f(237)f(3) + f(3)^2 - f(192)^2 - f(192)f(75) - \\
&\quad - f(75)^2 = -\frac{1}{9}(3f(192) + 2)(3f(192) - 2) = 0, \\
E(104557) &= f(243)^2 + f(243)f(124) + f(124)^2 - f(201)^2 - f(201)f(172) - \\
&\quad - f(172)^2 = \frac{1}{9}(3f(243) - 1)(3f(243) + 2) = 0, \\
E(114589) &= f(337)^2 + f(337)f(3) + f(3)^2 - f(300)^2 - f(300)f(67) - \\
&\quad - f(67)^2 = -\frac{1}{9}(3f(300) + 2)(3f(300) - 1) = 0, \\
E(132867) &= f(363)^2 + f(363)f(3) + f(3)^2 - f(309)^2 - f(309)f(93) - \\
&\quad - f(93)^2 = \frac{1}{9}(3f(363) + 2)(3f(363) - 1) = 0, \\
E(347517) &= f(588)^2 + f(588)f(3) + f(3)^2 - f(543)^2 - f(543)f(84) - \\
&\quad - f(84)^2 = \frac{1}{9}(3f(588) + 2)(3f(588) - 1) = 0,
\end{aligned}$$

which imply that $108 = 3 \cdot 6^2 \in \mathcal{S}_1$, $192 = 3 \cdot 8^2 \in \mathcal{S}_1$, $243 = 3 \cdot 9^2 \in \mathcal{S}_1$, $300 = 3 \cdot 10^2 \in \mathcal{S}_1$, $363 = 3 \cdot 11^2 \in \mathcal{S}_1$, $588 = 3 \cdot 14^2 \in \mathcal{S}_1$. Thus we have proved

$$C := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 19, 21, 27, 28\} \subset \mathcal{S}_1.$$

Similar to the above proof, by using (2.7), the numbers $39 = [5, 2]$, $73 = [8, 1]$, $97 = [8, 3]$, $333 = [9, 12]$, $363 = [11, 11]$, $381 = [19, 1]$, $417 = [16, 7]$, $471 = [14, 11]$, $487 = [21, 2]$, $633 = [16, 13]$, $793 = [21, 11]$ are in $\mathcal{S}_1$, thus we have

$$\begin{aligned}
E(510229) &= f(675)^2 + f(675)f(73) + f(73)^2 - f(487)^2 - f(487)f(333) - \\
&\quad - f(333)^2 = \frac{1}{9}(3f(675) + 2)(3f(675) - 1) = 0, \\
E(592137) &= f(768)^2 + f(768)f(3) + f(3)^2 - f(471)^2 - f(471)f(417) - \\
&\quad - f(417)^2 = \frac{1}{9}(3f(768) + 2)(3f(768) - 1) = 0, \\
E(787023) &= f(867)^2 + f(867)f(39) + f(39)^2 - f(633)^2 - f(633)f(381) - \\
&\quad - f(381)^2 = \frac{1}{9}(3f(867) + 2)(3f(867) - 1) = 0, \\
E(1048477) &= f(972)^2 + f(972)f(97) + f(97)^2 - f(793)^2 - f(793)f(363) - \\
&\quad - f(363)^2 = \frac{1}{9}(3f(972) + 2)(3f(972) - 1) = 0,
\end{aligned}$$

consequently

$$3 \cdot 15^2 = 675 \in I, \quad 3 \cdot 16^2 = 768 \in I, \quad 3 \cdot 17^2 = 867 \in I, \quad 3 \cdot 18^2 = 972 \in I$$

and so

$$15 \in \mathcal{S}_1, \quad 16 \in \mathcal{S}_1, \quad 17 \in \mathcal{S}_1, \quad 18 \in \mathcal{S}_1.$$

For the proof of $3 \cdot 20^2 = 1200 \in I$ and $3 \cdot 22^2 = 1452 \in I$, we use (2.7) to get that the numbers $57 = [7, 1]$, $199 = [13, 2]$, $208 = [12, 4]$, $351 = [15, 6]$, $489 = [17, 8]$, $516 = [16, 10]$, $601 = [24, 1]$, $613 = [19, 9]$, $687 = [22, 7]$, $711 = [21, 9]$, $727 = [18, 13]$, $739 = [23, 7]$, $768 = [16, 16]$ belong to $\mathcal{S}_1$. Therefore, we have

$$\begin{aligned}
E(566257) &= f(752)^2 + f(752)f(1) + f(1)^2 - f(633)^2 - f(633)f(199) - \\
&\quad - f(199)^2 = \frac{1}{9}(3f(752) + 2)(3f(752) - 1) = 0, \\
E(1732864) &= f(1200)^2 + f(1200)f(208) + f(208)^2 - f(768)^2 - \\
&\quad - f(768)f(752) - f(752)^2 = -\frac{1}{3}(3f(752) + 1 + f(1200)) \cdot \\
&\quad \cdot (f(752) - f(1200)) = 0, \\
E(1169917) &= f(1052)^2 + f(1052)f(57) + f(57)^2 - f(727)^2 - f(727)f(516) - \\
&\quad - f(516)^2 = \frac{1}{9}(3f(1052) + 2)(3f(1052) - 1) = 0, \\
E(1732864) &= f(1452)^2 + f(1452)f(13) + f(13)^2 - f(1052)^2 - \\
&\quad - f(1052)f(613) - f(613)^2 = -\frac{1}{3}(3f(1052) + 1 + 3f(1452)) \cdot \\
&\quad \cdot (f(1052) - f(1452)) = 0.
\end{aligned}$$

Thus we have proved that $20 \in \mathcal{S}_1$ and $22 \in \mathcal{S}_1$.

Now we prove that

$$3 \cdot 23^2 \in I, \quad 3 \cdot 24^2 \in I, \quad 3 \cdot 25^2 \in I, \quad 3 \cdot 26^2 \in I \quad \text{and} \quad \{23, 24, 25, 26\} \subset \mathcal{S}_1.$$

Since $711 \in \mathcal{S}_1$, $601 \in \mathcal{S}_1$, $687 \in \mathcal{S}_1$, we have

$$\begin{aligned}
E(1294033) &= f(1131)^2 + f(1131)f(13) + f(13)^2 - f(711)^2 - f(711)f(601) - \\
&\quad - f(601)^2 = \frac{1}{9}(3f(1131) + 2)(3f(1131) - 1) = 0, \\
E(2528127) &= f(1587)^2 + f(1587)f(6) + f(6)^2 - f(1131)^2 - f(1131)f(687) - \\
&\quad - f(687)^2 = \frac{1}{3}(3f(1131) + 3f(1587) + 1) \cdot \\
&\quad \cdot (f(1131) - f(1587)) = 0,
\end{aligned}$$

that $1131 \in I$ and $1587 = 3 \cdot 23^2 \in I$, and so $23 \in \mathcal{S}_1$.

For the proof of $3 \cdot 24^2 \in I$, we use the facts that $351 \in \mathcal{S}_1$, $489 \in \mathcal{S}_1$, $727 \in \mathcal{S}_1$, $739 \in \mathcal{S}_1$ to get the following 3 relations

$$\begin{aligned}
E(533961) &= f(720)^2 + f(720)f(21) + f(21)^2 - f(489)^2 - f(489)f(351) - \\
&\quad - f(351)^2 = \frac{1}{9}(3f(720) + 2)(f(720) - 1) = 0,
\end{aligned}$$

$$\begin{aligned}
E(1596601) &= f(1256)^2 + f(1256)f(15) + f(15)^2 - f(739)^2 - f(739)f(720) - \\
&\quad - f(720)^2 = -\frac{1}{3}(3f(720) + 3f(1256) + 1) \cdot \\
&\quad \cdot (f(720) - f(1256)) = 0 \\
E(3019177) &= f(1728)^2 + f(1728)f(19) + f(19)^2 - f(1256)^2 - \\
&\quad - f(1256)f(727) - f(727)^2 = \\
&\quad -\frac{1}{3}(3f(1256) + 3f(1728) + 1) \cdot \\
&\quad \cdot (f(1256) - f(1728)) = 0,
\end{aligned}$$

which imply $720 \in I$, $1256 \in I$ and $1728 = 3 \cdot 24^2 \in I$. Therefore $24 \in \mathcal{S}_1$.

For the proof of $3 \cdot 25^2 \in I$, we infer from (2.7) that $37 = [4, 3]$, $43 = [6, 1]$, $52 = [6, 2]$, $403 = [14, 9]$, $652 = [22, 6]$, $723 = [17, 14]$, $777 = [19, 13]$ are in $\mathcal{S}_1$, consequently we have the following relations

$$\begin{aligned}
E(850269) &= f(895)^2 + f(895)f(52) + f(52)^2 - f(652)^2 - f(652)f(403) - \\
&\quad - f(403)^2 = \frac{1}{9}(3f(895) + 2)(3f(895) - 1) = 0, \\
E(1970839) &= f(1385)^2 + f(1385)f(37) + f(37)^2 - f(895)^2 - f(895)f(723) - \\
&\quad - f(723)^2 = -\frac{1}{3}(3f(895) + 3f(1385) + 1) \cdot \\
&\quad \cdot (f(895) - f(1385)) = 0, \\
E(3598099) &= f(1875)^2 + f(1875)f(43) + f(43)^2 - f(1385)^2 - \\
&\quad - f(1385)f(777) - f(777)^2 = -\frac{1}{3}(3f(1385) + 3f(1875) + 1) \cdot \\
&\quad \cdot (f(1385) - f(1875)) = 0,
\end{aligned}$$

that $895 \in I$, $1385 \in I$ and $1875 = 3 \cdot 25^2 \in I$, $25 \in \mathcal{S}_1$.

Finally, we prove that $3 \cdot 26^2 \in I$. Since $163 = [11, 3] \in \mathcal{S}_1$, $541 = [21, 4] \in \mathcal{S}_1$, $721 = [24, 5] \in \mathcal{S}_1$, $756 = [24, 6] \in \mathcal{S}_1$, $793 = [21, 11] \in \mathcal{S}_1$, we have

$$\begin{aligned}
E(1202583) &= f(1006)^2 + f(1006)f(163) + f(163)^2 - f(721)^2 - \\
&\quad - f(721)f(541) - f(541)^2 = \frac{1}{9}(3f(1006) + 2) \cdot \\
&\quad \cdot (3f(1006) - 1) = 0, \\
E(2344108) &= f(1524)^2 + f(1524)f(14) + f(14)^2 - f(1006)^2 - \\
&\quad - f(1006)f(756) - f(756)^2 = -\frac{1}{3}(3f(1006) + 3f(1524) + 1) \cdot \\
&\quad \cdot (f(1006) - f(1524)) = 0,
\end{aligned}$$

and

$$\begin{aligned} E(4159957) &= f(2028)^2 + f(2028)f(23) + f(23)^2 - f(1524)^2 - \\ &\quad - f(1524)f(793) - f(793)^2 = -\frac{1}{3}(3f(1524) + 3f(2028) + 1) \cdot \\ &\quad (f(1524) - f(2028)) = 0, \end{aligned}$$

that $1006 \in I$, $1524 \in I$ and $2028 = 3 \cdot 26^2 \in I$. Therefore we proved that $26 \in \mathcal{S}_1$.

The proof of Lemma 3 is completes. ■

Lemma 4. Assume that $D = 1$, $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies (2.3). If

$$f(1) = 1, \quad f(2) = 2 \quad \text{and} \quad f(4) = 4,$$

then $f(n) = n$ for every $n \in \mathbb{N}$.

Proof. Let

$$\mathcal{E} := \{n \in \mathbb{N} \mid f(n) = n\}.$$

In order to prove Lemma 4, by using Lemma 1, we need to show that

$$(2.11) \quad \{1, 2, \dots, 28\} \subset \mathcal{E}.$$

Since $1, 2, 4 \in \mathcal{E}$, we have

$$\{3, 7, 12, 13, 19, 21, 27, 28\} \subset \mathcal{E},$$

because $f(3) = f(1)^2 + f(1)f(1) + f(1)^2 = 3$, $f(7) = f(2)^2 + f(2)f(1) + f(1)^2 = 7$, $f(12) = f(2)^2 + f(2)f(2) + f(2)^2 = 12$, $f(13) = f(3)^2 + f(3)f(1) + f(1)^2 = 13$, $f(19) = f(3)^2 + f(3)f(2) + f(2)^2 = 19$, $f(21) = f(4)^2 + f(4)f(1) + f(1)^2 = 21$, $f(27) = 3f(3)^2 = 27$ and $f(28) = f(4)^2 + f(4)f(2) + f(2)^2 = 28$.

First we use the following relations

$$\begin{aligned} E(1533) &= f(37)^2 + f(37)f(4) + f(4)^2 - f(31)^2 - f(31)f(13) - f(13)^2 = \\ &= -(f(5) + 6)(f(5) - 5)(f(5)^2 + f(5) + 45) = 0, \end{aligned}$$

$$\begin{aligned} E(1603) &= f(39)^2 + f(39)f(2) + f(2)^2 - f(27)^2 - f(27)f(19) - f(19)^2 = \\ &= (f(5) - 5)(f(5) + 7)(f(5)^2 + 2f(5) + 45) = 0, \end{aligned}$$

$$\begin{aligned} E(637) &= f(23)^2 + f(23)f(4) + f(4)^2 - f(21)^2 - f(21)f(7) - f(7)^2 = \\ &= (f(23) + 27)(f(23) - 23) = 0, \end{aligned}$$

$$\begin{aligned} E(1957) &= f(37)^2 + f(37)f(12) + f(12)^2 - f(28)^2 - f(28)f(23) - f(23)^2 = \\ &= -(f(23) + 51)(f(23) - 23) = 0 \end{aligned}$$

to obtain that $5 \in \mathcal{E}$ and $23 \in \mathcal{E}$.

Next, by using the facts

$$\{1, 2, 3, 4, 5, 7, 12, 13, 19, 21, 23, 27, 28\} \subset \mathcal{E},$$

we have

$$\begin{aligned}
E(4237) &= f(63)^2 + f(63)f(4) + f(4)^2 - f(52)^2 - f(52)f(21) - f(21)^2 = \\
&= 2(f(6) - 6)(f(6)^2 + 5f(6) + 34) = 0, \\
E(6357) &= f(76)^2 + f(76)f(7) + f(7)^2 - f(49)^2 - f(49)f(43) - f(43)^2 = \\
&= 3(f(6) - 6)(2f(6)^2 + 13f(6) + 113) = 0, \\
E(777) &= f(23)^2 + f(23)f(8) + f(8)^2 - f(19)^2 - f(19)f(13) - f(13)^2 = \\
&= (f(8) + 31)(f(8) - 8) = 0, \\
E(5089) &= f(67)^2 + f(67)f(8) + f(8)^2 - f(57)^2 - f(57)f(23) - f(23)^2 = \\
&= (f(8) + 75)(f(8) - 8) = 0, \\
E(259) &= f(15)^2 + f(15)f(2) + f(2)^2 - f(13)^2 - f(13)f(5) - f(5)^2 = \\
&= (f(15) + 17)(f(15) - 15) = 0, \\
E(4329) &= f(57)^2 + f(57)f(15) + f(15)^2 - f(48)^2 - f(48)f(27) - f(27)^2 = \\
&= (f(15) + 72)(f(15) - 15) = 0, \\
E(637) &= f(21)^2 + f(21)f(7) + f(7)^2 - f(17)^2 - f(17)f(12) - f(12)^2 = \\
&= -(f(17) + 29)(f(17) - 17) = 0, \\
E(1477) &= f(31)^2 + f(31)f(12) + f(12)^2 - f(27)^2 - f(27)f(17) - f(17)^2 = \\
&= -(f(17) + 44)(f(17) - 17) = 0, \\
E(1027) &= f(31)^2 + f(31)f(2) + f(2)^2 - f(19)^2 - f(19)f(18) - f(18)^2 = \\
&= -(f(18) + 37)(f(18) - 18) = 0, \\
E(1843) &= f(39)^2 + f(39)f(7) + f(7)^2 - f(31)^2 - f(31)f(18) - f(18)^2 = \\
&= -(f(18) + 49)(f(18) - 18) = 0, \\
E(469) &= f(20)^2 + f(20)f(3) + f(3)^2 - f(13)^2 - f(13)f(12) - f(12)^2 = \\
&= (f(20) + 23)(f(20) - 20) = 0, \\
E(1141) &= f(31)^2 + f(31)f(5) + f(5)^2 - f(20)^2 - f(20)f(19) - f(19)^2 = \\
&= -(f(20) + 39)(f(20) - 20) = 0.
\end{aligned}$$

These imply that $\{6, 8, 15, 17, 18, 20\} \subset \mathcal{E}$. In the following we will prove that

$$\{9, 10, 11, 14, 22, 25, 26\} \subset \mathcal{E}.$$

We have

$$\begin{aligned}
E(91) &= f(9)^2 + f(9)f(1) + f(1)^2 - f(6)^2 - f(6)f(5) - f(5)^2 = \\
&= (f(9) + 10)(f(9) - 9) = 0, \\
E(217) &= f(13)^2 + f(13)f(3) + f(3)^2 - f(9)^2 - f(9)f(8) - f(8)^2 = \\
&= -(f(9) + 17)(f(9) - 9) = 0, \\
E(399) &= f(17)^2 + f(17)f(5) + f(5)^2 - f(13)^2 - f(13)f(10) - f(10)^2 = \\
&= -(f(10) + 23)(f(10) - 10) = 0, \\
E(364) &= f(18)^2 + f(18)f(2) + f(2)^2 - f(12)^2 - f(12)f(10) - f(10)^2 = \\
&= -(f(10) + 22)(f(10) - 10) = 0, \\
E(147) &= f(11)^2 + f(11)f(2) + f(2)^2 - f(7)^2 - f(7)f(7) - f(7)^2 = \\
&= (f(11) + 13)(f(11) - 11) = 0, \\
E(511) &= f(19)^2 + f(19)f(6) + f(6)^2 - f(15)^2 - f(15)f(11) - f(11)^2 = \\
&= -(f(11) + 26)(f(11) - 11) = 0, \\
E(343) &= f(18)^2 + f(18)f(1) + f(1)^2 - f(14)^2 - f(14)f(7) - f(7)^2 = \\
&= -(f(14) + 21)(f(14) - 14) = 0, \\
E(5047) &= f(63)^2 + f(63)f(14) + f(14)^2 - f(43)^2 - f(43)f(39) - f(39)^2 = \\
&= (f(14) + 77)(f(14) - 14) = 0, \\
E(507) &= f(22)^2 + f(22)f(1) + f(1)^2 - 3f(13)^2 = \\
&= (f(22) + 23)(f(22) - 22) = 0, \\
E(532) &= f(22)^2 + f(22)f(2) + f(2)^2 - f(18)^2 - f(18)f(8) - f(8)^2 = \\
&= (f(22) + 24)(f(22) - 22) = 0, \\
E(679) &= f(25)^2 + f(25)f(2) + f(2)^2 - f(17)^2 - f(17)f(13) - f(13)^2 = \\
&= (f(25) + 27)(f(25) - 25) = 0, \\
E(889) &= f(27)^2 + f(27)f(5) + f(5)^2 - f(25)^2 - f(25)f(8) - f(8)^2 = \\
&= -(f(25) + 33)(f(25) - 25) = 0, \\
E(703) &= f(26)^2 + f(26)f(1) + f(1)^2 - f(23)^2 - f(23)f(6) - f(6)^2 = \\
&= (f(26) + 27)(f(26) - 26) = 0, \\
E(1183) &= f(31)^2 + f(31)f(6) + f(6)^2 - f(26)^2 - f(26)f(13) - f(13)^2 = \\
&= -(f(26) + 39)(f(26) - 26) = 0,
\end{aligned}$$

These imply that $\{9, 10, 11, 14, 22, 25, 26\} \subset \mathcal{E}$.

Finally, we prove that $16 \in \mathcal{E}$ and $24 \in \mathcal{E}$. Indeed, we infer from

$$\begin{aligned} E(273) &= f(16)^2 + f(16)f(1) + f(1)^2 - f(11)^2 - f(11)f(8) - f(8)^2 = \\ &= (f(16) + 17)(f(16) - 16) = 0, \end{aligned}$$

$$\begin{aligned} E(481) &= f(19)^2 + f(19)f(5) + f(5)^2 - f(16)^2 - f(16)f(9) - f(9)^2 = \\ &= -(f(16) + 25)(f(16) - 16) = 0, \end{aligned}$$

$$\begin{aligned} E(793) &= f(24)^2 + f(24)f(7) + f(7)^2 - f(21)^2 - f(21)f(11) - f(11)^2 = \\ &= (f(24) + 31)(f(24) - 24) = 0, \end{aligned}$$

$$\begin{aligned} E(793) &= f(31)^2 + f(31)f(8) + f(8)^2 - f(24)^2 - f(24)f(17) - f(17)^2 = \\ &= -(f(24) + 41)(f(24) - 24) = 0, \end{aligned}$$

that $16 \in \mathcal{E}$ and $24 \in \mathcal{E}$.

Consequently, (2.11) and Lemma 4 are proved. ■

3. Proof of Theorem 1

Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies

$$(3.1) \quad f(n^2 + nm + m^2) = f(n)^2 + f(n)f(m) + f(m)^2 \quad \text{for every } n, m \in \mathbb{N}.$$

In the following, let $f(1) = x$, $f(2) = y$ and $f(4) = z$. By using the computer and Maple program, we have

$$\begin{aligned} f(3) &= f(1)^2 + f(1)f(1) + f(1)^2 = 3f(1)^2 = 3x^2, \\ f(7) &= f(2)^2 + f(2)f(1) + f(1)^2 = x^2 + xy + y^2, \\ f(12) &= f(2)^2 + f(2)f(2) + f(2)^2 = 3f(2)^2 = 3y^2, \\ f(13) &= f(3)^2 + f(3)f(1) + f(1)^2 = 9x^4 + 3x^3 + x^2, \\ f(19) &= f(3)^2 + f(3)f(2) + f(2)^2 = 9x^4 + 3x^2y + y^2, \\ f(21) &= f(4)^2 + f(4)f(1) + f(1)^2 = x^2 + xz + z^2, \\ f(27) &= 3f(3)^2 = 27x^4, \\ f(28) &= f(4)^2 + f(4)f(2) + f(2)^2 = y^2 + yz + z^2, \\ f(37) &= f(4)^2 + f(4)f(3) + f(3)^2 = 9x^4 + 3x^2z + z^2, \\ f(48) &= 3f(4)^2 = 3z^2, \\ f(147) &= 3f(7)^2 = 3(x^2 + xy + y^2)^2, \\ f(157) &= f(12)^2 + f(12)f(1) + f(1)^2 = x^2 + 3xy^2 + 9y^4, \\ f(237) &= f(13)^2 + f(13)f(4) + f(4)^2 = \\ &= 81x^8 + 54x^7 + 27x^6 + 6x^5 + 9x^4z + x^4 + 3x^3z + x^2z + z^2, \end{aligned}$$

$$\begin{aligned}
f(309) &= f(13)^2 + f(13)f(7) + f(7)^2 = 81x^8 + 54x^7 + 36x^6 + 9x^5y + 9x^4y^2 + \\
&\quad + 9x^5 + 3x^4y + 3x^3y^2 + 3x^4 + 3x^3y + 4x^2y^2 + 2xy^3 + y^4, \\
f(463) &= f(21)^2 + f(21)f(1) + f(1)^2 = x^4 + 2x^3z + 3x^2z^2 + 2xz^3 + z^4 + \\
&\quad + x^3 + x^2z + xz^2 + x^2, \\
f(487) &= f(21)^2 + f(21)f(2) + f(2)^2 = x^4 + 2x^3z + 3x^2z^2 + 2xz^3 + z^4 + \\
&\quad + x^2y + xyz + yz^2 + y^2, \\
f(733) &= f(19)^2 + f(19)f(12) + f(12)^2 = \\
&= 81x^8 + 54x^6y + 54x^4y^2 + 15x^2y^3 + 13y^4.
\end{aligned}$$

We can find with the computer 6 numbers $n \in E$ such that $\max(x, y, u, v) \leq 800$ and $E(n)$ can be written in terms of x, y and z . In the following we shall use only 4 such number n , namely

$$\{1677, 3097, 677019, 112617\} \subset E.$$

For these numbers, we have

$$\begin{aligned}
E(1677) &= f(37)^2 + f(37)f(7) + f(7)^2 - f(28)^2 - f(28)f(19) - f(19)^2 = \\
&= -54x^6y + 54x^6z + 9x^6 + 9x^5y - 27x^4y^2 - 9x^4yz + 18x^4z^2 + \\
&\quad + 3x^4z + x^2z^2 + 3x^3yz - 9x^2y^3 - 3x^2yz^2 + 6x^2z^3 + x^4 + \\
&\quad + 2x^3y + 3x^2y^2 + 2xy^3 + xyz^2 - 2y^4 - 3y^3z - 3y^2z^2 - 2yz^3 = 0, \\
E(3097) &= f(48)^2 + f(48)f(13) + f(13)^2 - f(37)^2 - f(37)f(27) - f(27)^2 = \\
&= -972x^8 + 54x^7 - 135x^6z + 27x^6 - 27x^4z^2 + 6x^5 + 9x^3z^2 - \\
&\quad - 6x^2z^3 + x^4 + 3x^2z^2 + 8z^4 = 0, \\
E(677019) &= f(733)^2 + f(733)f(157) + f(157)^2 - f(487)^2 - f(487)f(463) - \\
&\quad - f(463)^2 = 6561x^{16} + 8748x^{14}y + 11664x^{12}y^2 + 8262x^{10}y^3 + \\
&\quad + 7371x^8y^4 + 243x^9y^2 + 3510x^6y^5 + 81x^{10} + 162x^7y^3 + \\
&\quad + 2115x^4y^6 + 54x^8y + 162x^5y^4 + 525x^2y^7 - 3x^8 - 12x^7z + \\
&\quad + 54x^6y^2 - 30x^6z^2 - 48x^5z^3 - 57x^4z^4 + 45x^3y^5 - 48x^3z^5 - \\
&\quad - 30x^2z^6 - 12xz^7 + 367y^8 - 3z^8 - 3x^7 - 3x^6y - 9x^6z - 9x^5yz - \\
&\quad - 18x^5z^2 + 15x^4y^3 - 18x^4yz^2 - 21x^4z^3 - 21x^3yz^3 - 18x^3z^4 - \\
&\quad - 18x^2yz^4 - 9x^2z^5 + 93xy^6 - 9xyz^5 - 3xz^6 - 3yz^6 - 4x^6 - \\
&\quad - x^5y - 8x^5z - 4x^4y^2 - 2x^4yz - 12x^4z^2 - 8x^3y^2z - 3x^3yz^2 - \\
&\quad - 8x^3z^3 + 40x^2y^4 - 12x^2y^2z^2 - 2x^2yz^3 - 4x^2z^4 - 8xy^2z^3 - \\
&\quad - xyz^4 - 4y^2z^4 - 2x^5 - x^4y - 2x^4z + 5x^3y^2 - x^3yz - 2x^3z^2 - \\
&\quad - 2x^2y^3 - x^2y^2z - x^2yz^2 - 2xy^3z - xy^2z^2 - 2y^3z^2 - x^2y^2 - y^4
\end{aligned}$$

and

$$\begin{aligned}
E(112617) &= f(309)^2 + f(309)f(48) + f(48)^2 - f(237)^2 - f(237)f(147) - \\
&\quad - f(147)^2 = (x^2 + xy + y^2 - z)(1458x^{12} + 1458x^{11} + 972x^{10} + \\
&\quad + 378x^9 + 126x^8 + 36x^7y + 54x^6y^2 + 36x^5y^3 + 18x^4y^4 + 24x^7 + \\
&\quad + 12x^6y - 9x^6z + 18x^5y^2 - 9x^5yz + 12x^4y^3 - 9x^4y^2z + 6x^3y^4 - \\
&\quad - 4x^6 - 20x^5y - 3x^5z - 42x^4y^2 - 3x^4yz + 18x^4z^2 - 52x^3y^3 - \\
&\quad - 3x^3y^2z - 46x^2y^4 - 24xy^5 - 8y^6 - 9x^4z - 17x^3yz + 6x^3z^2 - \\
&\quad - 25x^2y^2z - 16xy^3z - 8y^4z - 6x^2z^2 - 8xyz^2 - 8y^2z^2 - 8z^3) = \\
&= 0.
\end{aligned}$$

In the following we use a notation $\text{rem}(a(x, y, z), b(x, y, z), x)$ as follows: If $a(x, y, z)$, $b(x, y, z)$ are polynomials in $(\mathbb{Q}[y, z])[x]$ with $b(x, y, z) \neq 0$, then there exist unique polynomials $q(x, y, z)$ and $r(x, y, z)$ in $(\mathbb{Q}[y, z])[x]$ with

$$a(x, y, z) = q(x, y, z)b(x, y, z) + r(x, y, z)$$

and such that the degree of $r(x, y, z)$ is smaller than the degree of $b(x, y, z)$ in x . The polynomials $q(x, y, z)$ and $r(x, y, z)$ are uniquely determined by $a(x, y, z)$ and $b(x, y, z)$. We denote by $\text{rem}(a(x, y, z), b(x, y, z), x)$ the $r(x, y, z)$. Similar way, we can define $\text{rem}(a(x, y, z), b(x, y, z), y)$ and $\text{rem}(a(x, y, z), b(x, y, z), z)$.

Proof of (a). Assume that $x = f(1) = 0$. Then $E(3097) = 8z^4 = 0$ and so $z = f(4) = 0$. Therefore, we have $E(1677) = -2y^4 = 0$, which implies that $y = f(2) = 0$. Consequently, Lemma 2 implies that (a) of Theorem 1 holds.

Proof of (b). Assume that $x = f(1) = \frac{1}{3}$. Then

$$E(3097) = \frac{1}{81}(3z - 1)(6z + 1)(36z^2 + 3z + 4) = 0.$$

We have three cases:

- (i) $6z + 1 = 0$,
- (ii) $36z^2 + 3z + 4 = 0$,
- (iii) $3z - 1 = 0$.

Case (i). Assume that $x = f(1) = \frac{1}{3}$ and $z = -\frac{1}{6}$. Then

$$0 = 324E(1677) = -(3y - 1)(6y + 1)(36y^2 + 3y + 4)$$

and

$$\begin{aligned}
0 &= 26244E(112617) = \\
&= -(3y + 2)(3y - 1)(18y^2 + 6y + 5)(36y^2 + 30y + 13)(36y^2 - 6y + 7).
\end{aligned}$$

But

$$0 = gcd(324E(1677), 26244E(112617)) = 3y - 1,$$

which proves that $y = \frac{1}{3}$. On the other hand, if $x = \frac{1}{3}$, $z = -\frac{1}{6}$ and $y = \frac{1}{3}$, then $E(677019) = \frac{69}{256} \neq 0$, which is a contradiction. Thus, the case (i) does not occur.

Case (ii). Now assume that $x = f(1) = \frac{1}{3}$ and $Q(z) := 36z^2 + 3z + 4 = 0$. Let

$$Q_1 := rem(648E(1677), Q, z)$$

and

$$Q_2 := rem\left(\frac{9}{8}(12y + 1)^2 Q, Q_1, z\right).$$

Then

$$Q_2 = (36y^2 + 21y + 7)(72y^2 - 3y + 2).$$

Similar way, let

$$Q_3 := rem(11943936 \cdot E(677019), Q, z)$$

and

$$Q_4 := rem\left(\frac{6561}{16384}(221184y^3 + 27648y^2 + 9360y + 6541)^2 Q, Q_3, z\right).$$

Then with help of computer, we have

$$\begin{aligned} Q_4 = & 379972472325341184y^{16} + 120790522537574400y^{15} + \\ & + 127859283595689984y^{14} + 32218119998668800y^{13} + \\ & + 22173993529638912y^{12} + 4681113817227264y^{11} + \\ & + 2460889496613888y^{10} + 435555863548272y^9 + \\ & + 187636070089827y^8 + 27301842493713y^7 + \\ & + 10017117436986y^6 + 1170978598953y^5 + \\ & + 360330894360y^4 + 35176239549y^3 + 8145469377y^2 + \\ & + 550910895y + 136484728. \end{aligned}$$

But

$$gcd(Q_2, Q_4) = 1,$$

which is impossible. Thus, the case (ii) does not occur.

Case (iii). Now assume that $x = f(1) = \frac{1}{3}$ and $z = \frac{1}{3}$. Then

$$E(1677) = -\frac{1}{8}(3y + 2)(3y - 1)(18y^2 + 6y + 5) = 0$$

and

$$E(677019) = \frac{1}{6561}(3y-1)(802629y^7 + 395118y^6 + 256608y^5 + 99711y^4 + 44685y^3 + 14310y^2 + 4035y + 962) = 0.$$

But

$$0 = (8E(1677), 6561E(677019)) = 3y - 1,$$

which implies that $y = \frac{1}{3}$. Thus, we have proved that $x = y = z = \frac{1}{3}$, which with Lemma 2 implies that (b) of Theorem 1 holds.

Proof of (c). Assume that $f(1) = 1$. Then

$$E(3097) = (z-4)(8z^3 + 26z^2 + 89z + 221) = 0,$$

and so in the following we distinguish two cases:

$$(A) \quad 8z^3 + 26z^2 + 89z + 221 = 0,$$

$$(B) \quad z = 4.$$

First we consider the case (A). We will prove that this case does not occur. Let $x = 1$ and $P := 8z^3 + 26z^2 + 89z + 221 = 0$. Let

$$\begin{aligned} P_1 &= \text{rem}(4E(1677), P, z), \\ P_2 &= \text{rem}\left(\frac{1}{2}(6y^2 - 9y + 1)^2 P, P_1, z\right), \end{aligned}$$

and

$$P_3 = \text{rem}\left(\frac{1}{2}(48y^6 - 660y^5 + 80y^4 + 847y^3 - 5510y^2 + 8145y + 66)^2 P_1, P_2, z\right).$$

With the help of Maple and computer, we have

$$\begin{aligned} P_3 &= -(1024y^{12} + 5760y^{11} + 47232y^{10} + 132360y^9 + 778008y^8 + 951066y^7 + \\ &\quad + 3260222y^6 + 16475731y^5 + 16651871y^4 + 84676957y^3 + \\ &\quad + 161713315y^2 + 127018494y + 389604312) \cdot (6y^2 - 9y + 1)^2. \end{aligned}$$

On other hand, let

$$P_4 := \text{rem}(2E(112617), P, z), \quad P_4 = 0,$$

$$P_5 := \text{rem}((39y^2 + 39y + 119)^2 P, P_4, z), \quad P_5 = 0,$$

and

$$\begin{aligned} P_6 &= \text{rem}((2496y^{10} + 12480y^9 + 46616y^8 + 111584y^7 + 226986y^6 + \\ &\quad + 342830y^5 + 1788437y^4 + 3110712y^3 + 5747670y^2 + \\ &\quad + 4234599y + 141179756)^2 P_4, P_5, z), \quad P_6 = 0. \end{aligned}$$

With the help of Maple and computer, we have

$$\begin{aligned} P_6 = & -2 \left(8y^6 + 24y^5 + 74y^4 + 108y^3 + 215y^2 + 165y + 344 \right) \cdot \\ & \cdot \left(4096y^{18} + 36864y^{17} + 131072y^{16} + 212992y^{15} - 28800y^{14} - \right. \\ & - 1004416y^{13} - 8792640y^{12} - 40869888y^{11} - 78484864y^{10} - \\ & - 37231680y^9 + 201232250y^8 + 611161704y^7 + 4099068715y^6 + \\ & + 10417529621y^5 + 4618398473y^4 - 7546004973y^3 - 47172429173y^2 - \\ & \left. - 41744731095y - 672668750764 \right) (39y^2 + 39y + 119)^2. \end{aligned}$$

Consequently, P_3 and P_6 are impossible, because

$$P_3 = 0, \quad P_6 = 0 \quad \text{and} \quad \gcd(P_3, P_6) = 1.$$

Thus, we have proved that the case (A) does not occur.

Now we consider the case (B). Let $x = 1$ and $z = 4$. Then

$$E(1677) = -(y-2)(2y^3 + 23y^2 + 118y + 463) = 0$$

and

$$\begin{aligned} E(677019) = & (y-2)(367y^7 + 1259y^6 + 4726y^5 + 13007y^4 + 33586y^3 + \\ & + 75569y^2 + 161319y + 303195) = 0. \end{aligned}$$

Consequently

$$0 = \gcd(E(1677), E(677019)) = y - 2,$$

which gives $y = 2$. Thus we have proved that $x = 1$, $y = 2$, $z = 4$, and so Lemma 2 implies that (c) of Theorem 1 holds.

Theorem 1 is proved. ■

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