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A HYBRID INEQUALITY FOR THE NUMBER OF DIVISORS OF AN INTEGER

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Abstract. We establish an explicit inequality for the number of divisors of an integer n. It uses the size of n and its number of distinct prime divisors.

1. Introduction and notation

Let $\tau(n)$ and $\omega(n)$ be respectively the number of divisors and the number of distinct prime factors of n. In [2], the author and De Koninck have studied a variety of inequalities for the τ function. Among many helpful comments and suggestions, the anonymous referee of the said paper asked to justify and clarify some preliminary statements concerning the quality of our inequalities when compared to the well-known inequality of Wigert [7]

$$\tau(n) \le 2^{\frac{\log n}{\log\log n}(1+o(1))} \quad (n \to \infty).$$

The present author has therefore reworked some of his results to get to a nice statement which is also inspired by Théorème 2 of [1] and by [3]. Let us define $\vartheta := \vartheta(n)$ implicitly by $\omega(n) = \vartheta \frac{\log n}{\log\log n}$ for each $n \ge 16$. Then,

$$\tau(n) \le \exp\left(\vartheta \log\left(1 + \frac{1}{\vartheta}\right) \frac{\log n}{\log\log n}\left(1 + O\left(\frac{\log\log\log n}{\log\log n}\right)\right)\right).$$

In this paper, we make this result explicit and the best possible constant, here implicit, is found. A more precise result, not mentioned in [2], is also obtained in Theorem 1.2.

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Let us define the function

$$\rho(n) := \left(\frac{\log \tau(n)}{\omega(n)\log\left(1+\frac{1}{\vartheta(n)}\right)} - 1\right)\frac{\log\log n}{\log\log\log n}. \tag{1.1}$$

Theorem 1.1. *For each integer $n \geq 17$,*

$$\rho(n) \leq \rho(2^{26}3^{16}) = 2.00080128\ldots$$

The inequality is strict unless $n = 2^{26}3^{16}$.

Let $\mathcal{K}$ be the convex hull of the set $\left\{\left(\frac{1}{s}, \frac{\log(s+1)}{s}\right) : \ s \in \mathbb{N}\right\} \cup \{(0,0)\}$. We thus consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(\xi) := \sup\bigl(\{(\xi, z) : \ z \in \mathbb{R}\} \cap \mathcal{K}\bigr).$$

Theorem 1.2. *Let $\xi \in (0,1]$ be fixed. Let $\mathcal{J}$ be the ordered sequence of integers n satisfying*

$$\left|\omega(n) - \frac{\xi \log n}{\log\log n}\right| < \frac{\log n}{(\log\log n)^{3/2}}. \tag{1.2}$$

We have

$$\limsup_{\substack{n\to\infty \\ n\in\mathcal{J}}} \frac{\log \tau(n)\log\log n}{\log n} = f(\xi).$$

For each integer $k \geq 0$ we define n_k by $n_k = p_1 \cdots p_k$ (so that $n_0 = 1$) where p_k is the k-th prime number. We say that an integer is *primary* if it can be written as

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \qquad \alpha_1 \geq \cdots \geq \alpha_k$$

for some $k \geq 0$.

2. Preliminary lemmas

A well-known consequence of the prime number theorem is that

$$\max_{n\leq z} \omega(n) = \frac{\log z}{\log\log z} + O\left(\frac{\log z}{(\log\log z)^2}\right).$$

We need an explicit upper bound for our result.

Lemma 2.1. *For each integer $n \geq 3$ we have*

$$\omega(n) \leq \kappa \frac{\log n}{\log\log n}$$

where $\kappa = 1.3840127\ldots$ The inequality is strict unless $n = n_9$.

Proof. See [5]. It is also a consequence of (2.5) below along with some computations. ■

Lemma 2.2. *We have*

$$\sum_{j=1}^{k} \log\log p_j \geq k\log\log k \quad (k \geq 44), \tag{2.1}$$

$$\log n_k \leq 2k\log k \quad (k \geq 2), \tag{2.2}$$

$$\log\log n_k \geq \log k \quad (k \geq 3), \tag{2.3}$$

$$\log n_k \leq k\log\log n_k \quad (k \geq 3). \tag{2.4}$$

Proof. Inequalities (2.1) and (2.2) are simple consequences of Lemma 4.8 of [2] in (4.22) and (4.20) respectively. Inequality (2.3) can easily be obtained by induction. For (2.4), we first establish the inequality

$$\log n_k \geq k(\log k + \log\log k - 5/4) \quad (k \geq 2) \tag{2.5}$$

with induction by using the fact that $p_k \geq k\log k$ from [6]. So, we get a lower bound for $k\log\log n_k$ with (2.5) and an upper bound for $\log n_k$ with (4.20) from Lemma 4.8 of [2]. We leave the details to the reader. ■

Lemma 2.3. *For each fixed $k \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$, the function*

$$\log\left(1 + \frac{\exp(x)}{kx}\right)\left(1 + \frac{c\log x}{x}\right) \tag{2.6}$$

is strictly increasing for $x \geq 1$.

Proof. The derivative of (2.6) with respect to x is

$$\frac{-c(\log x - 1)(1+\theta)\log(1+\frac{1}{\theta}) + (x-1)(c\log x + x)}{x^2(1+\theta)},$$

where we write $\theta = \frac{kx}{\exp(x)}$ to simplify. We want to show that the numerator is positive for $x \geq 1$. We will show that the function

$$-(\log x - 1)(1+\theta)\log\left(1 + \frac{1}{\theta}\right) + (x-1)\log x$$

is positive for $x \geq 1$. It is the derivative of the numerator with respect to c. We observe that it implies the desired result for each fixed $c > 0$ and $x \geq 1$.

This result clearly holds when $x \in [1, \exp(1)]$. For $x > \exp(1)$, since the function $z \mapsto (1+z)\log\left(1+\frac{1}{z}\right)$ is strictly decreasing for $z > 0$, it is enough to prove the result in the particular case $k = 1$. We then use the inequality $\log\left(1 + \frac{\exp(x)}{x}\right) \le \log \frac{\exp(x+1)}{x}$ to deduce that it is enough to prove that

$$-(\log x - 1)\Big(1 + \frac{x}{\exp(x)}\Big)(x + 1 - \log x) + (x - 1)\log x \ge 0 \tag{2.7}$$

for $x > \exp(1)$. We observe that the left hand side of (2.7) can be written as

$$(x + 1 - 3\log x) + \Big(\log^2 x - \frac{(x+1)^2 \log x}{\exp(x)}\Big) + \frac{x^2 + x + x\log^2 x + \log x}{\exp(x)}$$

and the result follows from the fact that each of the above three terms is positive for $x > \exp(1)$. ■

Let $\gamma(n)$ stand for the product of the distinct prime factors of n.

Lemma 2.4. *For each integer $n \ge 2$,*

$$\tau(n) \le \prod_{p|n}\Big(\frac{\log n\gamma(n)}{\omega(n)\log p}\Big). \tag{2.8}$$

Proof. This is a famous inequality due to Ramanujan [4]. One can find a proof in Corollary 4.5 of [2] as well. ■

Lemma 2.5. *Let $k := \omega(n) \ge 74$. Then,*

$$\tau(n) < \Big(1 + \frac{\log n}{k\log k}\Big)^k.$$

Proof. This is Theorem 3.4 from [2]. In this paper we are using this result only for $k \ge 95$, which requires substantially fewer computations. ■

Lemma 2.6. *Let $\delta > 0$ and $[\alpha, \beta] \subseteq [a, b]$ be fixed. Let also $h \in C([a, b])$ satisfying $\max_{x\in[\alpha,\beta]} |h'(x)| \le M_1$. Assume that*

$$\max_{\substack{i\in\mathbb{Z}\\ \mu+i\delta\in[a,b]}} h(\mu + i\delta) \le M$$

for some $\mu \in [\alpha, \beta]$. Then,

$$\max_{x\in[\alpha,\beta]} h(x) < M + \delta M_1.$$

Proof. Let $z \in [\alpha, \beta]$ be fixed. There is a $j \in \mathbb{Z}$ with $|\mu + j\delta - z| < \delta$. The result follows from

$$h(z) = h(\mu + j\delta) + \int_{\mu+j\delta}^{z} h'(t)\mathrm{d}t.$$

■

3. Proof of Theorem 1.1

Throughout this proof, we often write $x = \log\log n$ to simplify the notation. Also, x is sometimes considered as a real variable when arguments from calculus have to be used. Furthermore, it is always assumed that $k = \omega(n)$ is fixed. Lemma 2.3 allows us to assume that n is primary when $\omega(n) = k \geq 3$. We used PARI/GP to verify that $\rho(n) < 2$ for each $17 \leq n \leq 10^9$. In particular, this verification along with Lemma 2.3 leave us with only primary integers to verify in the case where $k = 1, 2$ as well.

3.1. The case $k \geq 11000$

In this section, we will establish that $\rho(n) < 2$ for all the integers with at least 11000 distinct prime factors. Our main tool is Lemma 2.5. Recalling the definition of ϑ given in Section 1, we write

$$1 + \frac{\exp(x)}{k\log k} = 1 + \frac{1}{\vartheta} + \Big(\frac{\exp(x)}{k\log k} - \frac{1}{\vartheta}\Big)$$

where the term in parenthesis is positive if and only if $x \geq \log k$. If it is negative, then the result follows directly from Lemma 2.5. In the case where it is positive, we use the mean value theorem to get to

$$\begin{aligned}
\log\Big(1 + \frac{\exp(x)}{k\log k}\Big) &= \log\Big(1 + \frac{1}{\vartheta} + \Big(\frac{\exp(x)}{k\log k} - \frac{1}{\vartheta}\Big)\Big) \leq \\
&\leq \log\Big(1 + \frac{1}{\vartheta}\Big) + \frac{1}{1+\frac{1}{\vartheta}}\Big(\frac{\exp(x)}{k\log k} - \frac{1}{\vartheta}\Big) = \\
&= \log\Big(1 + \frac{1}{\vartheta}\Big)\Big(1 + \frac{1}{(\vartheta+1)\log(1+\frac{1}{\vartheta})}\frac{x - \log k}{\log k}\Big).
\end{aligned}$$

Now, since $(\vartheta+1)\log(1+\frac{1}{\vartheta}) \geq \log(\frac{1}{\vartheta}) =: t$ and $t = x - \log x - \log k$, we deduce that $\rho(n) < 2$ holds if

$$\frac{\log x + t}{t(x - \log x - t)} < \frac{2\log x}{x} \tag{3.1}$$

which is the case if $1 \leq t \leq \frac{x}{2}$ when $x \geq 11.66$. Indeed, by expanding, we find a parabola in t so that it is enough to verify (3.1) at $t = 1$ and at $t = \frac{x}{2}$. From there, it is an easy exercise that uses calculus. The details are left to the reader.

In the case where $t > \frac{x}{2}$, we use the fact that $\tau(n) < (\frac{2\log n}{k\log k})^k < (\frac{\log n}{k})^k$ (since $2^k \le \tau(n) < (1 + \frac{\log n}{k\log k})^k$) from Lemma 2.5. Thus, since we have

$$\begin{aligned} \log\Big(1+\frac{1}{\vartheta}\Big)\Big(1+\frac{2\log x}{x}\Big) &> \log\Big(\frac{1}{\vartheta}\Big)\Big(1+\frac{2\log x}{x}\Big) = \\ &= (x-\log x-\log k)\Big(1+\frac{2\log x}{x}\Big) = \\ &= (x-\log x-\log k)+t\frac{2\log x}{x} > \\ &> x-\log k, \end{aligned}$$

the result follows.

The remaining case is when $t < 1$, i.e. when $\frac{1}{\exp(1)} < \vartheta < 1.39$ (see Lemma 2.1). We then have

$$\frac{1}{(\vartheta+1)\log(1+\frac{1}{\vartheta})}\frac{x-\log k}{\log k} \le \frac{\log x+1}{1.25(x-\log x-1)} < \frac{2\log x}{x}$$

for $x \ge 11.66$. Since $\log\log n_{11000} > 11.66$, the result follows.

3.2. The case $44 \le k \le 10999$

We use inequality (2.8) into the definition of ρ (1.1) to get to

$$\rho(n) \le \Big(\frac{\log(\exp(x)+\log n_k)-\log k-\frac{1}{k}\sum_{j=1}^{k}\log\log p_j}{\log(1+\frac{1}{\vartheta})}-1\Big)\frac{x}{\log x} = \tag{3.2}$$

$$= \frac{x\log(\frac{\exp(x)+\log n_k}{\exp(x)+kx})+x\log x-\frac{x}{k}\sum_{j=1}^{k}\log\log p_j}{\log x\log(1+\frac{\exp(x)}{kx})}. \tag{3.3}$$

Since $x \ge \log\log n_k$ and from inequality (2.4), we deduce that the first term in the numerator in (3.3) is negative. Thus, from inequality (2.1) and $\log(1+\frac{\exp(x)}{kx}) > x-\log x-\log k$, we find that $\rho(n) < 2$ if

$$x\log x + x\log\log k - 2\log x(\log x+\log k) > 0 \tag{3.4}$$

which is the case when $x \ge 1.8\log k$. To prove this fact, we first write $x = z\log k$. We then show that the derivative with respect to z of the left hand side of (3.4) is positive and also that it is positive at $z = 1.8$ for each $k \ge 44$. Then, in light of (2.3), we have $x \ge \log\log n_k > \log k$. We can now assume that $x \in [\log k, 1.8\log k]$.

It remains to verify that $\rho(n) < 2$ for each $x \in [\log\log n_k, 1.8\log k]$ and each fixed value of k with a limited number of computations. To do so, we will work with the function

$$(3.5)\quad h_k(x) := \Big(\frac{\log(\exp(x)+\log n_k)-\log k-\frac{1}{k}\sum_{j=1}^{k}\log\log p_j}{\log(1+\frac{\exp(x)}{kx})}-1\Big)\frac{x}{\log x},$$

which is the right hand side of (3.2), and use Lemma 2.6.

Let us first establish that

$$(3.6)\qquad \max_{x\in[\log\log n_k,1.8\log k]}|h_k'(x)| \le \frac{400}{81}\frac{\log^2 k}{\log\log k}+\frac{130}{27}\frac{\log k}{\log\log k}.$$

We have

(3.7)

$$h_k'(x) = \Big(\frac{1}{\log x}-\frac{1}{\log^2 x}\Big)\Big(\frac{W}{\ell}-1\Big)+\frac{x}{\log x}\Big(\frac{\exp(x)}{\ell(\exp(x)+\log n_k)}-\frac{W}{\ell^2}\frac{\frac{1}{\vartheta}-\frac{1}{x\vartheta}}{1+\frac{1}{\vartheta}}\Big),$$

where we wrote $\ell := \log(1+\frac{1}{\vartheta})$ and $W := \log(\exp(x)+\log n_k)-\log k-$ $-\frac{1}{k}\sum_{j=1}^{k}\log\log p_j$ to simplify. Clearly,

$$(3.8)\qquad |h_k'(x)| \le \frac{1}{\log x}\Big|\frac{W}{\ell}-1\Big|+\frac{x}{\log x}\Big(\frac{1}{\ell}+\frac{W}{\ell^2}\Big)\quad (x\ge\exp(1)).$$

From there, we use the fact that $\vartheta \le 1.39$, i.e. $\ell > \frac{27}{50}$, and we establish that $W < 0.8\log k$ uniformly for $x\in[\log\log n_k, 1.8\log k]$. It allows us to conclude that (3.6) holds since $-1 < \frac{W}{\ell}-1 < \frac{W}{\ell}$ given that $\log 2 \le \frac{\log\tau(n)}{k} \le W$ so that $\Big|\frac{W}{\ell}-1\Big| < \frac{50}{27}W$. Now, from (2.1) and (2.2),

$$\begin{aligned} W &< \log(k^{1.8}+2k\log k)-\log k-\log\log k = \\ &= \log(\frac{k^{0.8}}{\log k}+2) \le 0.8\log k \end{aligned}$$

and the desired inequality follows.

We verify that the right hand side of (3.6) is an increasing function of k on the interval $[44, 10999]$. Thus it is less than $M_1 = 215$. For this reason, we set $\delta = 0.002$ and we thus have $\delta M_1 = 0.43$. Using Maple, we evaluate $h_k(x)$ at each increment of 0.002 in the interval $x\in[\log k, 1.8\log k]$ for each $k\in[44, 10999]$. This verification finishes the proof that $\rho(n) < 2$ for each such value of k.

3.3. The case $1 \le k \le 43$

In this section, we complete the proof of Theorem 1.1. We will see that only the case $k = 2$ has some values of n for which $\rho(n) \ge 2$. The general idea is the same as in the previous section. We now set up what is needed to use Lemma 2.6 on the function $h_k(x)$.

Obviously we have $x \geq \log\log n_k$ and our main objective is to find an upper bound for an x that would realize $\rho(n) \geq 2$. To do so, we start from (3.2) and write

$$\begin{aligned} \rho(n) &\leq \Big(\frac{\log(\exp(x) + \log n_k) - \log k - \frac{1}{k}\sum_{j=1}^{k}\log\log p_j}{\log(1+\frac{1}{\vartheta})} - 1\Big)\frac{x}{\log x} \\ &\leq \Big(\frac{\log(2\exp(x)) - \log k - \frac{1}{k}\sum_{j=1}^{k}\log\log p_j}{x - \log x - \log k} - 1\Big)\frac{x}{\log x} \\ &= \frac{x\log 2x - \frac{x}{k}\sum_{j=1}^{k}\log\log p_j}{(x - \log x - \log k)\log x}. \end{aligned} \tag{3.9}$$

We then show that (3.9) is strictly less than 2 for $x \geq 9.36$ for each $k \in [1, 43]$. Now that we have the desired upper bound for x, we are ready for the final verification. From a previous verification, we know that $x \geq \log\log 10^9$. We need an upper bound for $|h_k'(x)|$ with $x \in [\log\log\max(10^9, n_k), 9.36]$ and for that we use (3.8). We still have $\ell \geq \frac{27}{50}$ and we get an upper bound for W directly from the fact that $x \leq 9.36$. We find that

$$\max_{x\in[\log\log\max(10^9,n_k),9.36]} |h_k'(x)| \leq 165 =: M_1.$$

For this reason, we choose $\delta = 0.00004$ so that $\delta M_1 = 0.0066$. We verify with a computer at each increment of 0.00004 in $[\log\log\max(10^9, n_k), 9.36]$ for $k = 1$ and $k \in [3, 43]$ and call the maximum M. We find that $M + \delta M_1 < 2$. Finally, for $k = 2$, we verify that $\max_{x\in[4,9.36]} h_2(x) < 2$ using the same method. For $x \leq 4$, it is enough to check the numbers of the shape $2^a \cdot 3^b$ with $a \leq 77$ and $b \leq 49$ that are larger than 10^9. We find that the maximum is only reached at $n = 2^{26} \cdot 3^{16}$. The proof is complete.

4. Proof of Theorem 1.2

Let us fix $\xi \in (0, 1]$ and choose an ϵ satisfying $\frac{\xi}{100} \geq \epsilon > 0$. We can assume that $\xi \in (1/(s + 1), 1/s]$ for some positive integer s.

We start with the lower bound. We choose a large z and we construct an integer $m = m_1^{s+1} m_2^s$ such that

$$m_1 = \prod_{j \leq (1-s\xi)\frac{\log z}{\log\log z}} p_j \quad \text{and} \quad m_2 = \prod_{\substack{j \leq \xi\frac{\log z}{\log\log z} \\ p \nmid m_1}} p_j.$$

We verify that $m = z\exp\bigl(O\bigl(\frac{\log z \log\log\log z}{\log\log z}\bigr)\bigr)$, so that inequality (1.2) holds and

$$\frac{\log\tau(m)\log\log m}{\log m} \geq f(\xi)(1+o(1)) \quad (m\to\infty). \tag{4.1}$$

We now turn to the upper bound. It is enough to prove the result for primary integers. We consider the ordered set $\mathcal{H}_\epsilon$ of primary integers n for which

$$\omega(n) < \frac{(\xi+\epsilon)\log n}{\log\log n}.$$

We define the constant

$$\mu := \limsup_{\substack{n\to\infty\\ n\in\mathcal{H}_\epsilon}} \frac{\log\tau(n)\log\log n}{\log n}.$$

Let

$$\mathcal{H}_\epsilon^* := \{n\in\mathcal{H}_\epsilon\cap\mathbb{R}_{>x_0} :\ \Bigl|\frac{\log\tau(n)\log\log n}{\log n} - \mu\Bigr| \leq \epsilon^2\}$$

where $x_0(=x_0(\epsilon))$ is chosen large enough so that $\frac{\log\tau(n)\log\log n}{\log n} \geq \mu+\epsilon^2$ does not hold in $\mathcal{H}_\epsilon$. From Theorem 1.1, the integers u for which $\omega(u) \leq \frac{1/(s+1)\log u}{\log\log u}$ satisfy $\frac{\log\tau(u)\log\log u}{\log u} \leq f(1/(s+1))+\epsilon^2$. Thus, in view of (4.1), $\mathcal{H}_\epsilon^*$ contains only integers u such that $\omega(u) > \frac{1/(s+1)\log u}{\log\log u}$.

Each primary integer u can be written uniquely as $u := u^{**}\cdot u^*\cdot u^{(s+1)}\cdots u^{(1)}$ where

$$u^{(j)} := \prod_{\substack{p|u\\ p^j\|u}} p^j \quad (j=1,\dots,s+1)$$

and where u^{**} is the divisor of u formed of the (at most) $\lfloor\epsilon\omega(u)\rfloor$ first prime numbers. Let us assume, for a contradiction, that for some $n\in\mathcal{H}_\epsilon^*$ large enough we have $\omega(n^*)\geq\lfloor\epsilon\omega(n)\rfloor$. We will then find a primary integer n' satisfying

$$\omega(n') < \frac{(\xi+\epsilon)\log n'}{\log\log n'},$$

$$n\leq n'\leq n\exp\Bigl(\epsilon\log(c_1/\epsilon)\frac{\log n}{\log\log n}(1+o(1))\Bigr),$$

for some constant c_1, and for which $\tau(n')\geq(1+\frac{1}{8s^2})^{\lfloor\epsilon\omega(n)\rfloor}\cdot\tau(n)$. In this case, we will have

$$\exp\Bigl((\mu+c_2\epsilon)\frac{\log n}{\log\log n}\Bigr) \leq \tau(n') \leq \exp\Bigl((\mu+\epsilon^2)\frac{\log n'}{\log\log n'}\Bigr), \tag{4.2}$$

where c_2 is a constant depending only on ξ, from which we find a contradiction for ϵ small enough when n is large enough. This means that in fact $\omega(n^*) < < \epsilon\omega(n)$ and we have established that, for n large enough, $n := n''\cdot n^{(s+1)}\cdots n^{(1)}$ where n'' is made of at most the first $2\epsilon\omega(n)$ prime numbers.

We are thus ready to define this integer n'. We verify that the transformation of n which consists in replacing the largest prime factor q_1 of n^* by the smallest prime factor q_2 of $n^{(j)}$ (for the smallest $j \in \{1, \dots, s\}$ available), i.e. $n \mapsto \frac{q_2 n}{q_1}$, increases the value of τ by a factor $\geq 1 + \frac{1}{8s^2}$, increases the integer by a factor $\ll 1/\epsilon$ and transforms n into a new primary integer. Since $\xi > \frac{1}{s+1}$, it is possible to iterate this transformation $\lfloor \epsilon\omega(n) \rfloor$ times for ϵ small enough and n large enough. By doing so, starting with n, we end with an integer n' satisfying the 3 announced properties.

Now, from Theorem 1.1 we have

$$\tau(n'') \leq \exp\Big(c_3\epsilon \log(1/\epsilon) \frac{\log n}{\log\log n}\Big)$$

for some constant c_3. We deduce that $\tau(n'')$ is small when compared to $\tau(n)$. We can thus consider the integer $m := n/n'' = n^{(1)}\cdots n^{(s+1)}$ and optimize the value of $\tau(m)$ under the condition

$$\omega(m) = \frac{(\zeta + O(\epsilon)) \log n}{\log\log n}$$

for some $\zeta \leq \xi$.

By writing $k_1 + \cdots + k_{s+1} = \omega(m) =: k$, i.e. $k_j := \omega(n^{(j)})$, we find

$$\log(\tau(m)) = k_1 \log(2) + \cdots + k_{s+1} \log(s+2)$$

so that $\tau(m)$ is maximal when most of the k_j with j small are zero. For this reason, we will assume that $k_1, \dots, k_{s-1} = 0$ and that only k_s and k_{s+1} may be nonzero. We write $k_s = \alpha k$, so that $k_{s+1} = (1-\alpha)\cdot k$. From there, we can assume that $m = n^{(s)} \cdot n^{(s+1)}$ and now the problem is reduced to maximizing

$$\log(\tau(m)) = \big(\alpha \log(s+1) + (1-\alpha)\log(s+2)\big)\cdot k \tag{4.3}$$

for $m \leq n$ which can be written as

$$\begin{aligned}
\log n &\geq s \sum_{p|n^{(s)}} \log p + (s+1) \sum_{p|n^{(s+1)}} \log p = \\
&= \big(s\alpha k \log(\alpha k) + (s+1)(1-\alpha)k \log((1-\alpha)k)\big)(1+o(1)) = \\
&= \big(s\alpha + (s+1)(1-\alpha)\big)\cdot(k\log k)\cdot(1+o(1)) = \\
&= \big(s\alpha + (s+1)(1-\alpha)\big)\cdot(\zeta + O(\epsilon))\cdot(\log n)\cdot(1+o(1))
\end{aligned}$$

as $n \to \infty$. We deduce that $\alpha \geq s+1-1/\zeta+O(\epsilon)$ and by using this inequality in (4.3) we get

$$\begin{aligned}\log(\tau(m)) &\leq \big((\zeta(s+1)-1)\cdot\log(s+1)+(1-s\zeta)\cdot\log(s+2)+O(\epsilon)\big)\cdot\frac{\log n}{\log\log n} = \\ &= \big(f(\zeta)+O(\epsilon)\big)\cdot\frac{\log n}{\log\log n}\end{aligned}$$

so that

$$\begin{aligned}\log(\tau(n)) &\leq \big(f(\zeta)+O(\epsilon\log(1/\epsilon))\big)\cdot\frac{\log n}{\log\log n} \leq \\ &\leq \big(f(\xi)+O(\epsilon\log(1/\epsilon))\big)\cdot\frac{\log n}{\log\log n}.\end{aligned}$$

This is the desired upper bound. The proof is complete.

5. Concluding remarks

We have seen that $\rho(n) < 2$ given $\omega(n) \neq 2$. One can wonder if it is a good inequality. We can see directly from (3.3) that $\limsup_{\substack{n\to\infty \\ \omega(n)=k}} \rho(n) \leq 1$. Let us show that we have in fact equality. Indeed, let $(z_i)_{i\geq 1}$ be a strictly increasing sequence of sufficiently large integers. There are $\beta_1,\ldots,\beta_k$ satisfying $\max_j |\beta_j| < 1/2$ such that

$$\prod_{j=1}^{k}\Big(\frac{\log z_i n_k}{k\log p_j}+\beta_j\Big) =: \prod_{j=1}^{k}(\alpha_j+1) = \tau(m_i)$$

which defines the integer $m_i := p_1^{\alpha_1}\cdots p_k^{\alpha_k}$. We verify that

$$\log\tau(m_i) = k\log(\exp(x_i)+\log n_k) - k\log k - \sum_{j=1}^{k}\log\log p_j + O\Big(\frac{k\log n_k}{\exp(x_i)}\Big),$$

where $x_i := \log\log m_i$. Thus, by using this value of $\log\tau(m_i)$ in (1.1) (as we did to obtain (3.2)) we deduce as above that $\limsup_{\substack{n\to\infty \\ \omega(n)=k}} \rho(n) \geq 1$.

Let us now prove that

$$\limsup_{n\to\infty} \rho(n) = 1. \tag{5.1}$$

We already have the lower bound for each single value of $k \geq 1$. For the upper bound we use the main argument of Section 3.1. Precisely, for each fixed $\epsilon > 0$ we have

$$\frac{\log x + t}{t(x-\log x - t)} \leq \frac{\log x}{x}$$

for $t \leq (1-\epsilon)x$ when x is large enough. In the case where $t > (1-\epsilon)x$, we use inequality (3.3) to find $\limsup_{n\to\infty} \rho(n) \leq \frac{1}{1-\epsilon}$, from which (5.1) follows immediately.

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