ON THE EQUATION $f(n^2 + Dm^2 + k) = f(n)^2 + Df(m)^2 + k$

Bui Minh Mai Khanh (Budapest, Hungary)

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Abstract. We give all solutions of the equation

$$f(n^2 + Dm^2 + k) = f(n)^2 + Df(m)^2 + k,$$

where $D, k \in \mathbb{N}$ are given and f is an arbitrary complex valued function defined on $\mathbb{N}$.

1. Introduction

Let $Q(x, y) \in \mathbb{Z}[x, y]$ be a polynomial with two variable and let A, B be subsets of $\mathbb{N}$. We are interested in finding solutions $f : \mathbb{N} \to \mathbb{C}$ to an equation of the form

$$f\big(Q(a, b)\big) = Q\big(f(a), f(b)\big) \quad \text{for every} \quad a \in A, b \in B. \tag{1.1}$$

If $Q(x, y) = x + y$, $A = B = \mathbb{N}$, then it can be shown that there is a single family of solutions, namely $f(n) = cn$, where $c = f(1) \in \mathbb{C}$ is an arbitrary number.

In 1992, C. Spiro [9] consider the equation (1.1) in the cases when $Q(x, y) = x+y$ and $A = B = \mathcal{P}$. She proved that if a real-valued multiplicative function f satisfies

$$f(p + q) = f(p) + f(q) \quad (\forall p, q \in \mathcal{P}) \quad \text{and} \quad f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P},$$

then $f(n)$ is the identity function.

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Let $\mathcal{M}$ $(\mathcal{M}^*)$ be the set of all multiplicative (completely multiplicative) functions, respectively. In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong [6] proved that if a function $f \in \mathcal{M}$ satisfies the condition

$$f\left(p+m^2\right) = f(p) + f\left(m^2\right) \quad \text{for every} \quad p \in \mathcal{P}, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

In 2014 B. Bojan considered the case when $Q(x,y) = x^2 + y^2$, $A = B = \mathbb{N}$. He determined all solutions of those $f : \mathbb{N} \to \mathbb{C}$ for which

$$f\left(n^2+m^2\right) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

It is proved in [1] that the solution $f(n)$ of the above equation is one of the followings:

- $\circ \quad f(n) = 0 \quad \text{for every} \quad n \in \mathbb{N},$
- $\circ \quad f(n) = \dfrac{\varepsilon_{1,0}(n)}{2} \quad \text{for every} \quad n \in \mathbb{N},$
- $\circ \quad f(n) = \varepsilon_{1,0}(n)n \quad \text{for every} \quad n \in \mathbb{N},$

where $\varepsilon_{D,k} : \mathbb{N} \to \{-1, 1\}$ is an arithmetical function such that

$$\varepsilon_{D,k}\left(n^2 + Dm^2 + k\right) = 1 \quad \text{for every} \quad n, m \in \mathbb{N}.$$

I. Kátai and B. M. Phong posed the following conjecture for the cases when $Q(x,y) = x^2 + Dy^2$ and $A = B = \mathbb{N}$:

Conjecture 1. (I. Kátai and B. M. Phong [2]) *Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation*

$$f\left(n^2+Dm^2\right) = f^2(n) + Df^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

- $\circ \quad f(n) = 0 \quad \textit{for every} \quad n \in \mathbb{N},$
- $\circ \quad f(n) = \dfrac{\varepsilon_{D,0}(n)}{D+1} \quad \textit{for every} \quad n \in \mathbb{N},$
- $\circ \quad f(n) = \varepsilon_{D,0}(n)n \quad \textit{for every} \quad n \in \mathbb{N}.$

In 2015 we proved in [3] that Conjecture 1 is true for $D = 2$ and $D = 3$. N. T. Nghia [7] proved that Conjecture 1 is true for $D \in \{4, 5\}$. In 2017 we given a complete solution for this question.

Theorem A. (B. M. M. Khanh [4]) *Conjecture 1 is true.*

Recently, in [5] we have been studied and given all solutions of the equation

$$f\big(n^2+m^2+k\big)=f^2(n)+f^2(m)+K \quad \text{for every} \quad n,m\in\mathbb{N},$$

where $k\in\mathbb{N}_0$ and $K\in\mathbb{C}$. We infer from the results of [5] that if $Q(x,y)= = x^2+y^2+k,\ \ k\in\mathbb{N}$ and $A=B=\mathbb{N}$, then the following theorem holds:

Theorem B. (B. M. M. Khanh [5]) *Assume that a non-negative integer k and an arithmetical function $f:\mathbb{N}\to\mathbb{C}$ satisfy the equation*

$$f\big(n^2+m^2+k\big)=f^2(n)+f^2(m)+k \quad \textit{for every} \quad n,m\in\mathbb{N}.$$

Then one of the following assertions holds:

- $f(n)=\dfrac{\varepsilon_{1,k}(n)}{4}\big(1-\sqrt{-8k+1}\big)$,
- $f(n)=\dfrac{\varepsilon_{1,k}(n)}{4}\big(1+\sqrt{-8k+1}\big)$,
- $f(n)=\varepsilon_{1,k}(n)n$.

In this paper we improve Theorem A and Theorem B as follows:

Theorem 1. *Assume that the numbers $k\in\mathbb{N}_0$, $D\in\mathbb{N}$ and the arithmetical function $f:\mathbb{N}\to\mathbb{C}$ satisfy the equation*

$$f\big(n^2+Dm^2+k\big)=f^2(n)+Df^2(m)+k \quad \textit{for every} \quad n,m\in\mathbb{N}.$$

Then one of the following assertions holds:

a) $f(n)=\varepsilon_{D,k}(n)\dfrac{1-\sqrt{1-4Dk-4k}}{2(D+1)}$ *for every* $n\in\mathbb{N}$,

b) $f(n)=\varepsilon_{D,k}(n)\dfrac{1+\sqrt{1-4Dk-4k}}{2(D+1)}$ *for every* $n\in\mathbb{N}$,

c) $f(n)=\varepsilon_{D,k}(n)n$ *for every* $n\in\mathbb{N}$.

We infer from Theorem 1 the following result.

Corollary 1. *Assume that the numbers $k\in\mathbb{N}_0$, $D\in\mathbb{N}$ and a multiplicative function $f:\mathbb{N}\to\mathbb{C}$ satisfy the equation*

$$f\big(n^2+Dm^2+k\big)=f^2(n)+Df^2(m)+k \quad \textit{for every} \quad n,m\in\mathbb{N}.$$

Then $f(n)=\varepsilon_{D,k}(n)n$ for every $n\in\mathbb{N}$, where $\varepsilon_{D,k}\in\mathcal{M}$, $\varepsilon_{D,k}(n)\in\{1,-1\}$ and $\varepsilon_{D,k}(n^2+Dm^2+k)=1$ for every $n,m\in\mathbb{N}$.

In the order to prove Theorem 1, we will need to the following result.

Theorem 2. *Assume that the numbers* $\Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma \in \mathbb{C}$, $D \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and the arithmetical functions* $F : \mathbb{N} \to \mathbb{C}$ *satisfy*

$$F(n) = \Gamma_2\chi_2(n) + \Gamma_3\chi_3(n) + \Gamma_4\chi_4(n-1) + \Gamma_5\chi_5(n) + \Gamma \tag{1.2}$$

and

$$F\big(n^2 + Dm^2 + k\big) = \big(F(n) + DF(m) + k\big)^2 \tag{1.3}$$

for every $n,\ m \in \mathbb{N}$. *Then*

$$\Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0$$

and one of the following assertions holds:

$$A)\qquad F(n) = \Gamma = \left(\frac{1 + \sqrt{1 - 4Dk - 4k}}{2(D+1)}\right)^2 \quad \text{for every} \quad n \in \mathbb{N},$$

$$B)\qquad F(n) = \Gamma = \left(\frac{1 - \sqrt{1 - 4Dk - 4k}}{2(D+1)}\right)^2 \quad \text{for every} \quad n \in \mathbb{N},$$

where $\chi_2(n) \pmod 2$, $\chi_3(n) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(n) \pmod 4$, $\chi_5(n) \pmod 5$ *are the real, non-principal Dirichlet characters, i.e.* $\chi_2(0) = 0$, $\chi_2(1) = 1$, $\chi_3(0) = 0$, $\chi_3(1) = \chi_3(2) = 1$, $\chi_4(0) = \chi_4(2) = 0$, $\chi_4(1) = 1$, $\chi_4(3) = -1$, $\chi_5(2) = \chi_5(3) = -1$, $\chi_5(1) = \chi_5(4) = 1$.

2. The proof of Theorem 2. Auxiliary lemmas

In this Section we assume that all assumptions of Theorem 2 are satisfied, i.e.

$$F(n) = \Gamma_2\chi_2(n) + \Gamma_3\chi_3(n) + \Gamma_4\chi_4(n-1) + \Gamma_5\chi_5(n) + \Gamma$$

and

$$F\big(n^2 + Dm^2 + k\big) = \big(F(n) + DF(m) + k\big)^2 \tag{2.1}$$

for every $n, m \in \mathbb{N}$.

In the order to prove Theorem 2, first we shall prove some lemmas.

Lemma 1. *We have* $\Gamma_4 = 0$.

Proof. It is obvious from our assumptions that $\{F(\ell)\}_1^\infty$ is periodic $\pmod{60}$, therefore

$$F\big(2^2+D2^2+k\big)=F\big(8^2+D2^2+k\big) \text{ and } F\big(2^2+D2^2+k\big)=F\big(8^2+D8^2+k\big),$$

which with (2.1) imply that

$$\big(F(2)+DF(2)+k\big)^2=\big(F(8)+DF(2)+k\big)^2$$

and

$$\big(F(2)+DF(2)+k\big)^2=\big(F(8)+DF(8)+k\big)^2.$$

Consequently

$$\begin{aligned}\big(F(2)+DF(2)+k\big)^2-\big(F(8)+DF(2)+k\big)^2=\\ =\big(F(2)-F(8)\big)\big((2D+1)F(2)+F(8)+2k\big)=0\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\big(F(2)+DF(2)+k\big)^2-\big(F(8)+DF(8)+k\big)^2=\\ &=(D+1)\big(F(2)-F(8)\big)\big((D+1)F(2)+(D+1)F(8)+2k\big)=0.\end{aligned} \tag{2.3}$$

Since

$$F(2)=\Gamma_3+\Gamma_4-\Gamma_5+\Gamma \quad \text{and} \quad F(8)=\Gamma_3-\Gamma_4-\Gamma_5+\Gamma,$$

we have $\Gamma_4=0$ if $F(2)=F(8)$. Assume now that $F(2)-F(8)\neq 0$. Then we infer from (2.2) and (2.3) that

$$(2D+1)F(2)+F(8)+2k=0 \quad \text{and} \quad (D+1)F(2)+(D+1)F(8)+2k=0,$$

consequently

$$DF(2)=DF(8).$$

This contradicts to the assumption $F(2)-F(8)\neq 0$. Thus, $\Gamma_4=0$ follows, which finishes the proof of Lemma 1. ∎

Lemma 2. *We have*

$$\begin{cases}\Gamma_2\Gamma_3 &= 0,\\ \Gamma_2\Gamma_5 &= 0,\\ \Gamma_3\Gamma_5 &= 0.\end{cases}$$

Proof. By using Lemma 1, we have $\Gamma_4=0$, and so the sequence

$$F(\ell)=\Gamma_2\chi_2(\ell)+\Gamma_3\chi_3(\ell)+\Gamma_5\chi_5(\ell)+\Gamma$$

is the periodic $\pmod{30}$ and (2.1) is true.

First we prove that

$$F(n)+F(n+4)-F(n+10)-F(n+24)=0, \tag{2.4}$$

$$F(n+1)+F(n+4)-F(n+16)-F(n+19)=0 \tag{2.5}$$

and

$$F(n)+F(n+25)-F(n+10)-F(n+15)=0 \tag{2.6}$$

hold for every $n\in\mathbb{N}$.

From the definition of $F(\ell)$, we have

$$\begin{aligned}
F(n)&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_5\chi_5(n)+\Gamma,\\
F(n+4)&=\Gamma_2\chi_2(n+4)+\Gamma_3\chi_3(n+4)+\Gamma_5\chi_5(n+4)+\Gamma=\\
&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+4)+\Gamma,\\
F(n+10)&=\Gamma_2\chi_2(n+10)+\Gamma_3\chi_3(n+10)+\Gamma_5\chi_5(n+10)+\Gamma=\\
&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n)+\Gamma,\\
F(n+24)&=\Gamma_2\chi_2(n+24)+\Gamma_3\chi_3(n+24)+\Gamma_5\chi_5(n+24)+\Gamma=\\
&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_5\chi_5(n+4)+\Gamma,
\end{aligned}$$

therefore

$$\begin{gathered}
F(n)+F(n+4)-F(n+10)-F(n+24)=\\
=\big(2\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n)+\Gamma_5\chi_5(n+4)+2\Gamma\big)-\\
-\big(2\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n)+\Gamma_5\chi_5(n+4)+2\Gamma\big)=0,
\end{gathered}$$

which proves (2.4).

In a similar way, we have

$$\begin{aligned}
F(n+1)&=\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+1)+\Gamma,\\
F(n+4)&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+4)+\Gamma,\\
F(n+16)&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+1)+\Gamma,\\
F(n+19)&=\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+4)+\Gamma,
\end{aligned}$$

therefore

$$\begin{gathered}
F(n+1)+F(n+4)-F(n+16)-F(n+19)=\\
=\big(\Gamma_2\chi_2(n)+\Gamma_2\chi_2(n+1)+2\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n+1)+\\
+\Gamma_5\chi_5(n+4)+2\Gamma\big)-\big(\Gamma_2\chi_2(n)+\Gamma_2\chi_2(n+1)+2\Gamma_3\chi_3(n+1)+\\
+\Gamma_5\chi_5(n+1)+\Gamma_5\chi_5(n+4)+2\Gamma\big)=0.
\end{gathered}$$

Thus, (2.5) is proved.

Finally, we prove (2.6). We have

$$\begin{aligned} F(n)&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_5\chi_5(n)+\Gamma,\\ F(n+25)&=\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n)+\Gamma,\\ F(n+10)&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n+1)+\Gamma_5\chi_5(n)+\Gamma,\\ F(n+15)&=\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n)+\Gamma_5\chi_5(n)+\Gamma, \end{aligned}$$

therefore

$$\begin{gathered} F(n)+F(n+25)-F(n+10)-F(n+15)=\\ =\big(\Gamma_2\chi_2(n)+\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n)+\Gamma_3\chi_3(n+1)+2\Gamma_5\chi_5(n)+2\Gamma\big)-\\ -\big(\Gamma_2\chi_2(n)+\Gamma_2\chi_2(n+1)+\Gamma_3\chi_3(n)+2\Gamma_3\chi_3(n+1)+2\Gamma_5\chi_5(n)+2\Gamma\big)=0. \end{gathered}$$

This finishes the proof (2.6).

In the next step, we prove that

$$F(k)+F(k+4)-F(k+10)-F(k+24)=-2\Gamma_3\Gamma_5, \tag{2.7}$$

$$F(k+1)+F(k+4)-F(k+16)-F(k+19)=4\Gamma_2\Gamma_5 \tag{2.8}$$

and

$$F(k)+F(k+25)-F(k+10)-F(k+15)=2\Gamma_2\Gamma_3. \tag{2.9}$$

It follows from the definition of $F(\ell)=\Gamma_2\chi_2(\ell)+\Gamma_3\chi_3(\ell)+\Gamma_5\chi_5(\ell)+\Gamma$ that

$$\begin{aligned} F(1)&=\Gamma_2\chi_2(1)+\Gamma_3\chi_3(1)+\Gamma_5\chi_5(1)+\Gamma=\Gamma_2+\Gamma_3+\Gamma_5+\Gamma,\\ F(2)&=\Gamma_2\chi_2(2)+\Gamma_3\chi_3(2)+\Gamma_5\chi_5(2)+\Gamma=\Gamma_3-\Gamma_5+\Gamma,\\ F(4)&=\Gamma_2\chi_2(4)+\Gamma_3\chi_3(4)+\Gamma_5\chi_5(4)+\Gamma=\Gamma_3+\Gamma_5+\Gamma,\\ F(7)&=\Gamma_2\chi_2(7)+\Gamma_3\chi_3(7)+\Gamma_5\chi_5(7)+\Gamma=\Gamma_2+\Gamma_3-\Gamma_5+\Gamma,\\ F(10)&=\Gamma_2\chi_2(10)+\Gamma_3\chi_3(10)+\Gamma_5\chi_5(10)+\Gamma=\Gamma_3+\Gamma,\\ F(12)&=\Gamma_2\chi_2(12)+\Gamma_3\chi_3(12)+\Gamma_5\chi_5(12)+\Gamma=-\Gamma_5+\Gamma,\\ F(15)&=\Gamma_2\chi_2(15)+\Gamma_3\chi_3(15)+\Gamma_5\chi_5(15)+\Gamma=\Gamma_2+\Gamma,\\ F(25)&=\Gamma_2\chi_2(25)+\Gamma_3\chi_3(25)+\Gamma_5\chi_5(25)+\Gamma=\Gamma_2+\Gamma_3+\Gamma,\\ F(30)&=\Gamma_2\chi_2(30)+\Gamma_3\chi_3(30)+\Gamma_5\chi_5(30)+\Gamma=\Gamma. \end{aligned}$$

Since

$$1^2\equiv 1, 2^2\equiv 4, 10^2\equiv 10, 4^2\equiv 16, 7^2\equiv 19, 12^2\equiv 24, 30^2\equiv 0 \pmod{30},$$

and $F(\ell)$ is periodic $\pmod{30}$, we infer from (2.1) that

$$\begin{aligned}
F(k) &= \big(F(30) + DF(30) + k\big)^2 = \big((D+1)\Gamma + k\big)^2,\\
F(k+1) &= \big(F(1) + DF(30) + k\big)^2 = \big(\Gamma_2 + \Gamma_3 + \Gamma_5 + (D+1)\Gamma + k\big)^2,\\
F(k+4) &= \big(F(2) + DF(30) + k\big)^2 = \big(\Gamma_3 - \Gamma_5 + (D+1)\Gamma + k\big)^2,\\
F(k+10) &= \big(F(10) + DF(30) + k\big)^2 = \big(\Gamma_3 + (D+1)\Gamma + k\big)^2,\\
F(k+15) &= \big(F(15) + DF(30) + k\big)^2 = \big(\Gamma_2 + (D+1)\Gamma + k\big)^2,\\
F(k+16) &= \big(F(4) + DF(30) + k\big)^2 = \big(\Gamma_3 + \Gamma_5 + (D+1)\Gamma + k\big)^2,\\
F(k+19) &= \big(F(7) + DF(30) + k\big)^2 = \big(\Gamma_2 + \Gamma_3 - \Gamma_5 + (D+1)\Gamma + k\big)^2,\\
F(k+24) &= \big(F(12) + DF(30) + k\big)^2 = \big(-\Gamma_5 + (D+1)\Gamma + k\big)^2,\\
F(k+25) &= \big(F(25) + DF(30) + k\big)^2 = \big(\Gamma_2 + \Gamma_3 + (D+1)\Gamma + k\big)^2.
\end{aligned}$$

By using theses, we obtain that

$$\begin{aligned}
&F(k) + F(k+4) - F(k+10) - F(k+24) =\\
&= \big((D+1)\Gamma + k\big)^2 + \big(\Gamma_3 - \Gamma_5 + (D+1)\Gamma + k\big)^2 -\\
&\qquad - \big(\Gamma_3 + (D+1)\Gamma + k\big)^2 - \big(-\Gamma_5 + (D+1)\Gamma + k\big)^2 =\\
&= -2\Gamma_3\Gamma_5,
\end{aligned}$$

$$\begin{aligned}
&F(k+1) + F(k+4) - F(k+16) - F(k+19) =\\
&= \big(\Gamma_2 + \Gamma_3 + \Gamma_5 + (D+1)\Gamma + k\big)^2 + \big(\Gamma_3 - \Gamma_5 + (D+1)\Gamma + k\big)^2 -\\
&\qquad - \big(\Gamma_3 + \Gamma_5 + (D+1)\Gamma + k\big)^2 - \big(\Gamma_2 + \Gamma_3 - \Gamma_5 + (D+1)\Gamma + k\big)^2 =\\
&= 4\Gamma_2\Gamma_5
\end{aligned}$$

and

$$\begin{aligned}
&F(k) + F(k+25) - F(k+10) - F(k+15) =\\
&= \big((D+1)\Gamma + k\big)^2 + \big(\Gamma_2 + \Gamma_3 + (D+1)\Gamma + k\big)^2 -\\
&\qquad - \big(\Gamma_3 + (D+1)\Gamma + k\big)^2 - \big(\Gamma_2 + (D+1)\Gamma + k\big)^2 =\\
&= 2\Gamma_2\Gamma_3.
\end{aligned}$$

By applying (2.4), (2.5) and (2.6) for the case when $n = k$, we obtain from (2.7), (2.8) and (2.9) that

$$\Gamma_2\Gamma_5 = 0, \quad \Gamma_3\Gamma_5 = 0 \quad \text{and} \quad \Gamma_2\Gamma_3 = 0.$$

Lemma 2 is proved. ■

Lemma 3. *We have* $\Gamma_2=0$.

Proof. We will prove that $\Gamma_2=0$. Assume by contradiction that $\Gamma_2\neq 0$. Then we infer from Lemma 1 and Lemma 2 that $\Gamma_3=\Gamma_4=\Gamma_5=0$. Consequently

$$F(\ell)=\Gamma_2\chi_2(\ell)+\Gamma \tag{2.10}$$

is a periodic sequence $\pmod 2$ and (2.1) holds.

First we prove that

$$D\equiv 1 \pmod 2. \tag{2.11}$$

Assume by contradiction that $D\equiv 0 \pmod 2$. Then

$$n^2+Dm^2+k\equiv n^2+k \pmod 2,$$

therefore we obtain from (2.1) that

$$F\big(n^2+k\big)=F\big(n^2+Dm^2+k\big)=\big(F(n)+DF(m)+k\big)^2$$

holds for every $n,m\in\mathbb{N}$. This with $m=1$ and $m=2$ shows that

$$F\big(n^2+k\big)=\big(F(n)+DF(1)+k\big)^2=\big(F(n)+DF(2)+k\big)^2,$$

consequently

$$0=D\big(F(1)-F(2)\big)\big(2F(n)+DF(1)+DF(2)+2k\big).$$

Since $F(1)-F(2)=(\Gamma_2+\Gamma)-\Gamma=\Gamma_2\neq 0$, we have

$$F(n)=-\frac{DF(1)+DF(2)+2k}{2}\quad \text{for every}\quad n\in\mathbb{N},$$

and so

$$F(1)=F(2),\quad \Gamma_2=0.$$

This contradicts to the fact $\Gamma_2\neq 0$. Thus, (2.11) is proved.

Assume now that $\Gamma_2\neq 0$ and $D\equiv 1 \pmod 2$. Thus, we have

$$F\big(n^2+m^2+k\big)=F\big(n^2+Dm^2+k\big)=\big(F(n)+DF(m)+k\big)^2 \tag{2.12}$$

for every $n,m\in\mathbb{N}$. This shows that

$$F(k)=F\big(2^2+2^2+k\big)=\big(F(2)+DF(2)+k\big)^2=\big((D+1)\Gamma+k\big)^2,$$
$$F(k)=F\big(1^2+1^2+k\big)=\big(F(1)+DF(1)+k\big)^2=\big((D+1)(\Gamma_2+\Gamma)+k\big)^2,$$

consequently

$$0 = \big((D+1)(\Gamma_2+\Gamma)+k\big)^2 - \big((D+1)\Gamma+k\big)^2 = \\ = (D+1)\Gamma_2\big((D+1)\Gamma_2 + 2(D+1)\Gamma + 2k\big)$$

and so

$$(D+1)\Gamma + k = -\frac{D+1}{2}\Gamma_2 \quad \text{and} \quad F(k) = \Big(\frac{D+1}{2}\Big)^2\Gamma_2^2.$$

By applying (2.12) with $n=1$ and $m=2$, we obtain

$$F(k+1) = F\big(1^2+2^2+k\big) = \big(F(1)+DF(2)+k\big)^2 = \\ = \big(\Gamma_2 + (D+1)\Gamma + k\big)^2 = \Big(\Gamma_2 - \frac{D+1}{2}\Gamma_2\Big)^2 = \Big(\frac{D-1}{2}\Big)^2\Gamma_2^2.$$

If k is even number, then $F(k)=F(2)$, $F(k+1)=F(1)$, and so

$$\begin{cases} \Big(\dfrac{D+1}{2}\Big)^2\Gamma_2^2 - \Gamma = 0, \\ \Big(\dfrac{D-1}{2}\Big)^2\Gamma_2^2 - (\Gamma_2+\Gamma) = 0. \end{cases}$$

Solving this system, using $\Gamma_2 \neq 0$, we have

$$\Gamma_2 = -\frac{1}{D}, \quad \Gamma = \Big(\frac{D+1}{2D}\Big)^2$$

and

$$k = -(D+1)\Gamma - \frac{D+1}{2}\Gamma_2 = -(D+1)\Big(\frac{D+1}{2D}\Big)^2 - \frac{D+1}{2}\Big(-\frac{1}{D}\Big) = \\ = -\frac{1+D^2+D+D^3}{4D^2} < 0,$$

which contradicts to the fact $k \in \mathbb{N}$.

Now we consider the case $2 \not| k$. Then $F(k)=F(1)$, $F(k+1)=F(2)$, and so

$$\begin{cases} \Big(\dfrac{D+1}{2}\Big)^2\Gamma_2^2 - (\Gamma_2+\Gamma) = 0, \\ \Big(\dfrac{D-1}{2}\Big)^2\Gamma_2^2 - \Gamma = 0. \end{cases}$$

Solving this system, using $\Gamma_2 \neq 0$, we have

$$\Gamma_2 = \frac{1}{D}, \quad \Gamma = \Big(\frac{D-1}{2D}\Big)^2$$

and
$$k=-\frac{1+D^2+D+D^3}{4D^2}<0.$$
This is impossible, because $k\in\mathbb{N}$. Thus we have proved that $\Gamma_2=0$.

Lemma 3 is proved. ■

Lemma 4. *We have* $\Gamma_3=0$.

Proof. Assume by contradiction that $\Gamma_3\neq 0$. Then from Lemmas 1–3 we have $\Gamma_2=\Gamma_4=\Gamma_5=0$, consequently

$$\begin{cases} F(\ell) & =\Gamma_3\chi_3(\ell)+\Gamma, \\ F(n^2+Dm^2+k) & =(F(n)+DF(m)+k)^2 \end{cases} \tag{2.13}$$

for every $n,m\in\mathbb{N}$ and

$$F(1)=\Gamma_3+\Gamma,\quad F(2)=\Gamma_3+\Gamma,\quad F(3)=\Gamma.$$

We infer from (2.13) that

$$\begin{cases} F(k)=\big(F(3)+DF(3)+k\big)^2=\big((D+1)\Gamma+k\big)^2, \\ F(k+1)=\big(F(1)+DF(3)+k\big)^2=\big((D+1)\Gamma+\Gamma_3+k\big)^2, \\ F(k+D)=\big(F(3)+DF(1)+k)\big)^2=\big((D+1)\Gamma+D\Gamma_3+k\big)^2, \\ F(k+D+1)=\big(F(1)+DF(1)+k\big)^2=\big((D+1)(\Gamma_3+\Gamma)+k\big)^2. \end{cases} \tag{2.14}$$

The case (A): $D\equiv 0 \pmod 3$.

Since a sequence $\{F(n)\}_{n=1}^{\infty}$ is periodic $\pmod 3$, we infer from (2.14) that

$$\begin{aligned} F(k)-F(k+D)&=\big((D+1)\Gamma+k\big)^2-\big((D+1)\Gamma+D\Gamma_3+k^2\big)= \\ &=-D\Gamma_3\big(2(D+1)\Gamma+D\Gamma_3+2k\big)=0 \end{aligned}$$

and

$$\begin{aligned} F(k+1)-F(k+D+1)&=((D+1)\Gamma+\Gamma_3+k^2-((D+1)(\Gamma_3+\Gamma)+k)^2= \\ &=-D\Gamma_3\big(2(D+1)\Gamma+(D+2)\Gamma_3+2k\big)=0. \end{aligned}$$

The last relations with $\Gamma_3\neq 0$ imply

$$\begin{cases} 2(D+1)\Gamma+D\Gamma_3+2k=0, \\ 2(D+1)\Gamma+(D+2)\Gamma_3+2k=0, \end{cases}$$

consequently
$$2\Gamma_3=0,$$
which is impossible.

The case (B): $D \equiv 1 \pmod 3$.

In this case, we distinguish three cases according to $k \pmod 3$.

The case (B1): $D \equiv 1 \pmod 3$, $k \equiv 1 \pmod 3$.

Then $F(k) - F(1) = 0$, $F(k+1) - F(2) = 0$ and $F(1+k+D) - F(3) = 0$, and so we have

$$\text{(B1)} \qquad \begin{cases} F(k) - F(1) = \big((D+1)\Gamma + k\big)^2 - \Gamma_3 - \Gamma = 0, \\ F(k+1) - F(2) = \big((D+1)\Gamma + \Gamma_3 + k\big)^2 - \Gamma_3 - \Gamma = 0, \\ F(k+D+1) - F(3) = \big((D+1)(\Gamma_3 + \Gamma) + k\big)^2 - \Gamma = 0. \end{cases}$$

Solving this system, we obtain that

$$\Gamma = \Big(\frac{2D+1}{2D(D+1)}\Big)^2, \quad \Gamma_3 = -\frac{1}{D(D+1)} \quad \text{and} \quad k = -\frac{4D^2+2D+1}{4D^2(D+1)}.$$

These are impossible, because $D \in \mathbb{N}$ and $k \in \mathbb{N}$. Thus, the case (B1) does not occur.

The case (B2): $D \equiv 1 \pmod 3$, $k \equiv 2 \pmod 3$.

In the case (B2), we infer from (2.14) that

$$F(k+D+1) - F(k) = F(1) - F(2) = 0,$$

and so

$$\begin{aligned} 0 = F(k+D+1) - F(k) &= \big((D+1)(\Gamma_3+\Gamma) + k\big)^2 - \big((D+1)\Gamma + k\big)^2 = \\ &= (D+1)\Gamma_3\big(2(D+1)\Gamma + (D+1)\Gamma_3 + 2k\big). \end{aligned}$$

Since $\Gamma_3 \neq 0$, we have

$$(D+1)\Gamma + k = -\frac{D+1}{2}\Gamma_3,$$

which with (2.14) show that

$$\text{(B2)} \quad \begin{cases} F(2) = F(k) = \big((D+1)\Gamma + k\big)^2 = \Big(\dfrac{D+1}{2}\Big)^2\Gamma_3^2 \\ F(3) = F(k+1) = \big((D+1)\Gamma + \Gamma_3 + k\big)^2 = \Big(\dfrac{D-1}{2}\Big)^2\Gamma_3^2 \\ F(1) = F(k+D+1) = \big((D+1)(\Gamma_3+\Gamma) + k\big)^2 = \Big(\dfrac{D+1}{2}\Big)^2\Gamma_3^2. \end{cases}$$

Since $F(1) = F(2) = \Gamma_3 + \Gamma$ and $F(3) = \Gamma$, we infer from (B2) that

$$\Gamma_3 = \frac{1}{D}, \quad \Gamma = \Big(\frac{D-1}{2D}\Big)^2 \quad \text{and} \quad k = -\frac{D^3+D^2+D+1}{4D^2},$$

which are impossible, because $k \in \mathbb{N}$. Thus, the case (B2) does not occur.

The case (B3): $D\equiv 1 \pmod 3$, $k\equiv 0 \pmod 3$.

In same way as above, in the case (B3) we have

$$\text{(B3)}\qquad \begin{cases} F(k)-F(3)=\big((D+1)\Gamma+k\big)^2-\Gamma=0,\\ F(k+1)-F(1)=\big((D+1)\Gamma+\Gamma_3+k\big)^2-\Gamma_3-\Gamma=0,\\ F(k+D+1)-F(2)=\big((D+1)(\Gamma_3+\Gamma)+k\big)^2-\Gamma_3-\Gamma=0.\end{cases}$$

Solving (B3), we obtain that

$$\Gamma_3=-\frac{1}{D+1},\quad \Gamma=-\frac{(D+2)^2}{4(D+1)^2}\quad\text{and}\quad k=-\frac{D(D+2)}{4(D+1)}.$$

These are impossible, because $k\in\mathbb{N}$. Thus (B3) and (B) do not occur.

The cases (C): $D\equiv 2 \pmod 3$.

The proof of (C) is similar as above, we infer from the following relation

$$\begin{aligned}0=&F(k+D+1)-F(k)=\big((D+1)(\Gamma_3+\Gamma)+k\big)^2-\big((D+1)\Gamma+k\big)^2=\\ =&(D+1)\Gamma_3\big(2(D+1)\Gamma+(D+1)\Gamma_3+2k\big).\end{aligned}$$

Since $\Gamma_3\neq 0$, we have

$$(D+1)\Gamma+k=-\frac{D+1}{2}\Gamma_3$$

and so

$$\text{(C)}\qquad \begin{cases} F(k)=\big((D+1)\Gamma+k\big)^2=\Big(\dfrac{D+1}{2}\Big)^2\Gamma_3^2,\\ F(k+1)=\big((D+1)\Gamma+\Gamma_3+k\big)^2=\Big(\dfrac{D-1}{2}\Big)^2\Gamma_3^2,\\ F(k+2)=F(k+D)=\big((D+1)\Gamma+D\Gamma_3+k\big)^2=\Big(\dfrac{D-1}{2}\Big)^2\Gamma_3^2.\end{cases}$$

Since $F(1)=F(2)=\Gamma_3+\Gamma$ and $F(3)=\Gamma$, we infer from (C) that $F(k)=F(3)$, $F(k+1)=F(k+2)=F(1)$, consequently

$$\begin{cases}\Gamma=\Big(\dfrac{D+1}{2}\Big)^2\Gamma_3^2,\\ \Gamma_3+\Gamma=\Big(\dfrac{D-1}{2}\Big)^2\Gamma_3^2.\end{cases}$$

Hence

$$\Gamma_3=-\frac{1}{D},\quad \Gamma=\Big(\frac{D+1}{2}\Big)^2\Gamma_3^2=\Big(\frac{D+1}{2D}\Big)^2$$

and

$$k = -\frac{D+1}{2}\Gamma_3 - (D+1)\Gamma = \frac{D+1}{2D} - (D+1)\Big(\frac{D+1}{2D}\Big)^2 = \\ = -\frac{D^3+D^2+D+1}{4D^2}.$$

The last relation contradicts to the fact $k \in \mathbb{N}$.

Thus, (C) and so Lemma 4 are proved. ∎

Lemma 5. *We have* $\Gamma_5 = 0$.

Proof. Assume by contradiction that $\Gamma_5 \neq 0$. Then from Lemmas 1–4 we have $\Gamma_2 = \Gamma_3 = \Gamma_4 = 0$, consequently

$$\begin{cases} F(\ell) & = \Gamma_5\chi_5(\ell) + \Gamma, \\ F(n^2+Dm^2+k) & = (F(n)+DF(m)+k)^2 \end{cases} \tag{2.15}$$

for every $n, m \in \mathbb{N}$. The elements of this sequence are:

$$F(1) = \Gamma_5 + \Gamma,\ F(2) = -\Gamma_5 + \Gamma,\ F(3) = -\Gamma_5 + \Gamma,\ F(4) = \Gamma_5 + \Gamma, F(5) = \Gamma.$$

We infer from (2.15) that

$$\begin{cases} F(k) = \big(F(5)+DF(5)+k\big)^2 = \big((D+1)\Gamma + k\big)^2, \\ F(k+1) = \big(F(1)+DF(5)+k\big)^2 = \big((D+1)\Gamma + \Gamma_5 + k\big)^2, \\ F(k+4) = \big(F(2)+DF(5)+k\big)^2 = \big((D+1)\Gamma - \Gamma_5 + k\big)^2. \end{cases} \tag{2.16}$$

In the proof of Lemma 5 we will distinguish five cases according to $k \pmod 5$.

∘ If $k \equiv 1 \pmod 5$, then $F(k) = F(1)$, $F(k+1) = F(2)$, $F(k+4) = F(5)$, and so

$$\begin{cases} \big((D+1)\Gamma + k\big)^2 = \Gamma_5 + \Gamma, \\ \big((D+1)\Gamma + \Gamma_5 + k\big)^2 = -\Gamma_5 + \Gamma, \\ \big((D+1)\Gamma - \Gamma_5 + k\big)^2 = \Gamma. \end{cases}$$

Solving this system, using $\Gamma_5 \neq 0$, we have

$$\Gamma = \frac{25}{16}, \quad \Gamma_5 = -\frac{3}{2} \quad \text{and} \quad k = -\frac{29}{16} - \frac{25}{16}D,$$

which contradicts to $k \in \mathbb{N}$.

∘ If $k \equiv 2 \pmod 5$, then $F(k) = F(2)$, $F(k+1) = F(3)$, $F(k+4) = F(6) = F(1)$, and so

$$\begin{cases} \big((D+1)\Gamma + k\big)^2 = -\Gamma_5 + \Gamma, \\ \big((D+1)\Gamma + \Gamma_5 + k\big)^2 = -\Gamma_5 + \Gamma, \\ \big((D+1)\Gamma - \Gamma_5 + k\big)^2 = \Gamma_5 + \Gamma. \end{cases}$$

Solving this system, using $\Gamma_5 \neq 0$, we have

$$\Gamma = \frac{5}{4}, \quad \Gamma_5 = 1 \quad \text{and} \quad k = -\frac{7}{4} - \frac{5}{4}D,$$

which contradicts to $k \in \mathbb{N}$.

∘ If $k \equiv 3 \pmod 5$, then $F(k) = F(3)$, $F(k+1) = F(4)$, $F(k+4) = F(7) = F(2)$, and so

$$\begin{cases} \big((D+1)\Gamma + k\big)^2 = -\Gamma_5 + \Gamma, \\ \big((D+1)\Gamma + \Gamma_5 + k\big)^2 = \Gamma_5 + \Gamma, \\ \big((D+1)\Gamma - \Gamma_5 + k\big)^2 = -\Gamma_5 + \Gamma. \end{cases}$$

Solving this system, using $\Gamma_5 \neq 0$, we have

$$\Gamma = \frac{5}{4}, \quad \Gamma_5 = 1 \quad \text{and} \quad k = -\frac{3}{4} - \frac{5}{4}D,$$

which contradicts to $k \in \mathbb{N}$.

∘ If $k \equiv 4 \pmod 5$, then $F(k) = F(4)$, $F(k+1) = F(5)$, $F(k+4) = F(8) = F(3)$, and so

$$\begin{cases} \big((D+1)\Gamma + k\big)^2 = \Gamma_5 + \Gamma, \\ \big((D+1)\Gamma + \Gamma_5 + k\big)^2 = \Gamma, \\ \big((D+1)\Gamma - \Gamma_5 + k\big)^2 = -\Gamma_5 + \Gamma. \end{cases}$$

Solving this system, using $\Gamma_5 \neq 0$, we have

$$\Gamma = \frac{25}{16}, \quad \Gamma_5 = -\frac{3}{2} \quad \text{and} \quad k = -\frac{21}{16} - \frac{25}{16}D,$$

which contradicts to $k \in \mathbb{N}$.

∘ If $k \equiv 5 \pmod 5$, then $F(k) = F(5)$, $F(k+1) = F(6) = F(1)$, $F(k+4) = F(9) = F(4)$, and so

$$\begin{cases} \big((D+1)\Gamma + k\big)^2 = \Gamma, \\ \big((D+1)\Gamma + \Gamma_5 + k\big)^2 = \Gamma_5 + \Gamma, \\ \big((D+1)\Gamma - \Gamma_5 + k\big)^2 = \Gamma_5 + \Gamma. \end{cases}$$

Solving this system, using $\Gamma_5 \neq 0$, we have

$$\Gamma = 0, \quad \Gamma_5 = 1 \quad \text{and} \quad k = 0,$$

which contradicts to $k \in \mathbb{N}$.

Lemma 5 is proved. ■

3. The proof of Theorem 2

It follows from Lemmas 1–5 that

$$\Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0 \quad \text{and} \quad F(\ell) = \Gamma \quad \text{for every} \quad \ell \in \mathbb{N}.$$

Thus, the relation (1.3) gives

$$\Gamma = \big(\Gamma + D\Gamma + k\big)^2,$$

which implies

$$\Gamma = \frac{1 - 2Dk - 2k \pm \sqrt{1 - 4Dk - 4k}}{2(D+1)^2} = \left(\frac{1 \pm \sqrt{1 - 4Dk - 4k}}{2(D+1)}\right)^2.$$

Thus, we proved that

$$F(\ell) = \Gamma = \left(\frac{1 \pm \sqrt{1 - 4Dk - 4k}}{2(D+1)}\right)^2 \quad \text{for every} \quad \ell \in \mathbb{N}.$$

Theorem 2 is proved. ∎

4. The proof of Theorem 1. Lemmas

In this section we assume that the function $T, G, H : \mathbb{N} \to \mathbb{C}$ and the numbers $k \in \mathbb{N}_0$, $D \in \mathbb{N}$, $U, V \in \mathbb{C}$, $U \neq 0$ satisfy the relation

$$T(n^2 + Dm^2 + k) = G(n) + UH(m) + V \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{4.1}$$

Lemma 6. *Assume that the function T, G, $H : \mathbb{N} \to \mathbb{C}$ and the numbers $k \in \mathbb{N}_0$, $D \in \mathbb{N}$, $U, V \in \mathbb{C}$, $U \neq 0$ satisfy (4.1). Then*

$$\begin{aligned} H\big(\ell + 12m\big) = {} & H(\ell + 9m) + H(\ell + 8m) + H(\ell + 7m) - \\ & - H(\ell + 5m) - H(\ell + 4m) - H(\ell + 3m) + H(\ell) \end{aligned} \tag{4.2}$$

holds for every $\ell, m \in \mathbb{N}$ and

$$\begin{cases} H(7) & = & 2H(5) - H(1), \\ H(8) & = & 2H(5) + H(4) - 2H(1), \\ H(9) & = & H(6) + 2H(5) - H(2) - H(1), \\ H(10) & = & H(6) + 3H(5) - H(3) - 2H(1), \\ H(11) & = & H(6) + 4H(5) - H(3) - H(2) - 2H(1), \\ H(12) & = & H(6) + 4H(5) + H(4) - H(2) - 4H(1). \end{cases} \tag{4.3}$$

Proof. We note from (4.1) that

$$T\big(x^2 + Dy^2 + k\big) = G(|x|) + UH(|y|) + V \quad \text{for every} \quad x, y \in \mathbb{Z} \setminus \{0\}. \tag{4.4}$$

First we prove the following assertion:

$$H(n + 2m) - H(|n - 2m|) = H(2n + m) - H(|2n - m|) \tag{4.5}$$

for every $n, m \in \mathbb{N}, n \neq 2m, m/2$.

Assume that the numbers $n, m \in \mathbb{N}$ satisfy the conditions $n \neq 2m, n \neq m/2$. If $Dn - 2m \neq 0$, then we infer from (4.1) and the next relations

$$(Dn + 2m)^2 + D(n - 2m)^2 + k = (Dn - 2m)^2 + D(n + 2m)^2 + k$$

and

$$(Dn + 2m)^2 + D(2n - m)^2 + k = (Dn - 2m)^2 + D(2n + m)^2 + k$$

that

$$G(Dn + 2m) + UH(|n - 2m|) + V = G(|Dn - 2m|) + UH(n + 2m) + V$$

and

$$G(Dn + 2m) + UH(|2n - m|) + V = G(|Dn - 2m|) + UH(2n + m) + V.$$

These imply that

$$\begin{aligned} G(Dn + 2m) - G(|Dn - 2m|) &= UH(n + 2m) - UH(|n - 2m|) = \\ &= UH(2n + m) - UH(|2n - m|), \end{aligned}$$

which prove (4.5) in the case $Dn - 2m \neq 0$.

If $Dn - 2m = 0$, then $2Dn - m = 4m - m = 3m \neq 0$. In this case, we infer from (4.1) and the next relations

$$(2Dn + m)^2 + D(n - 2m)^2 + k = (2Dn - m)^2 + D(n + 2m)^2 + k$$

and

$$(2Dn + m)^2 + D(2n - m)^2 + k = (2Dn - m)^2 + D(2n + m)^2 + k$$

that

$$G(2Dn + m) + UH(|n - 2m|) + V = G(|2Dn - m|) + UH(n + 2m) + V$$

and

$$G(2Dn + m) + UH(|2n - m|) + V = G(|2Dn - m|) + UH(2n + m) + V.$$

Similar as above, we deduce from these relations

$$\begin{aligned} G(2Dn+m)-G(|2Dn-m|) &= UH(n+2m)-UH(|n-2m|) = \\ &= UH(2n+m)-UH(|2n-m|), \end{aligned}$$

which prove (4.5) in the case $Dn-2m=0$, and so (4.5) is proved.

Applications of (4.5) in the cases

$$(n,m)\in\{(1,3);\ (2,3);\ (1,4);\ (1,5);\ (3,4);\ (2,5)\}$$

prove that (4.3) holds for $H(7)$, $H(8)$, $H(9)$, $H(11)$, $H(10)$ and $H(12)$. Thus, (4.3) is proved.

Now we prove (4.2). By applying (4.5) with $n=\ell+2t$, we have

$$H(\ell+4t)-H(\ell)=H(2\ell+5t)-H(2\ell+3t) \quad \text{for every} \quad \ell,t\in\mathbb{N}.$$

This with $t=3m$ shows that

$$H(\ell+12m)-H(\ell)=H(2\ell+15m)-H(2\ell+9m).$$

Since $H(2x+5y)-H(2x+3y)=H(x+4y)-H(x)$ holds for every $x,y\in\mathbb{N}$, the last relation implies that

$$\begin{aligned} H(\ell+12m)-H(\ell) &= H(2\ell+15m)-H(2\ell+9m) = \\ &= \big[H\big(2(\ell+5m)+5m\big)-H\big(2(\ell+5m)+3m\big)\big]+ \\ &\quad +\big[H\big(2(\ell+4m)+5m\big)-H\big(2(\ell+4m)+3m\big)\big]+ \\ &\quad +\big[H\big(2(\ell+3m)+5m\big)-H\big(2(\ell+3m)+3m\big)\big] = \\ &= \big[H\big(\ell+9m\big)-H\big(\ell+5m\big)\big]+ \\ &\quad +\big[H\big(\ell+8m)\big)-H\big(\ell+4m\big)\big]+ \\ &\quad +\big[H\big(\ell+7m)\big)-H\big(\ell+3m\big)\big], \end{aligned}$$

which prove (4.2).

Lemma 6 is proved. ■

In the next lemma we shall follow a method in part similar to the one used in the proof of Lemma 2 of the paper [8].

Lemma 7. *Assume that the function $T, G, H : \mathbb{N} \to \mathbb{C}$ and the numbers $D \in \mathbb{N}$, $U, V \in \mathbb{C}$, $U \neq 0$ satisfy (4.1). Let*

$$\begin{aligned}
A :=& \frac{1}{120}\big(H(6)+4H(5)-H(3)-H(2)-3H(1)\big),\\
\Gamma_2 :=& \frac{-1}{8}\big(H(6)-4H(5)+4H(4)-H(3)+3H(2)-3H(1)\big),\\
\Gamma_3 :=& \frac{-1}{3}\big(H(6)-2H(5)+2H(3)-H(2)\big),\\
\Gamma_4 :=& \frac{1}{4}\big(H(6)-2H(4)-H(3)+H(2)+H(1)\big),\\
\Gamma_5 :=& \frac{1}{5}\big(H(6)-H(5)-H(3)-H(2)+2H(1)\big),\\
\Gamma :=& \frac{1}{4}\big(H(6)-4H(5)+2H(4)+3H(3)+H(2)+H(1)\big),\\
F(\ell) :=& \Gamma_2\chi_2(\ell)+\Gamma_3\chi_3(\ell)+\Gamma_4\chi_4(\ell-1)+\Gamma_5\chi_5(\ell)+\Gamma,
\end{aligned}$$

where $\chi_2(\ell) \pmod 2$, $\chi_3(\ell) \pmod 3$ are the principal Dirichlet characters and $\chi_4(\ell) \pmod 4$, $\chi_5(\ell) \pmod 5$ are the real, non-principal Dirichlet characters, i.e.

$$\begin{aligned}
&\chi_2(0)=0,\ \chi_2(1)=1,\\
&\chi_3(0)=0,\ \chi_3(1)=\chi_3(2)=1,\\
&\chi_4(0)=\chi_4(2)=0,\ \chi_4(1)=1,\ \chi_4(3)=-1,\\
&\chi_5(0)=0,\ \chi_5(2)=\chi_5(3)=-1,\ \chi_5(1)=\chi_5(4)=1.
\end{aligned}$$

Then we have

$$H(\ell)=A\ell^2+F(\ell) \quad \text{for every} \quad \ell\in\mathbb{N}. \tag{4.6}$$

Proof. With the help of compute, we proved that (4.6) holds for $1 \le k \le 12$. Indeed, by using (4.3), we show with Maple that

$$H(\ell)-A\ell^2-F(\ell)=0 \quad \text{for every} \quad \ell=1,\dots,12,$$

where $F(\ell)=\Gamma_2\chi_2(\ell)+\Gamma_3\chi_3(\ell)+\Gamma_4\chi_4(\ell-1)+\Gamma_5\chi_5(\ell)+\Gamma$ for every $\ell\in\mathbb{N}$.

Assume that $H(k) = Ak^2 + F(k)$ holds for $\ell \le k \le \ell + 11$, where $\ell \ge 1$. By applying (4.2) with $m = 1$, we have

$$\begin{aligned}H(\ell+12) = {} & H(\ell+9) + H(\ell+8) + H(\ell+7) - H(\ell+5) - \\ & - H(\ell+4) - H(\ell+3) + H(\ell) = \\ = {} & A\big[(\ell+9)^2 + (\ell+8)^2 + (\ell+7)^2 - (\ell+5)^2 - \\ & - (\ell+4)^2 - (\ell+3)^2 + \ell^2\big] + \\ & + \big[F(\ell+9) + F(\ell+8) + F(\ell+7) - F(\ell+5) - \\ & - F(\ell+4) - F(\ell+3) + F(\ell)\big] = \\ = {} & A(\ell+12)^2 + F(\ell+12),\end{aligned}$$

which proves that (4.6) holds for $\ell + 12$ and so it is true for all ℓ. In the last relation we have used

$$\begin{aligned}& F(\ell+9) + F(\ell+8) + F(\ell+7) - F(\ell+5) - F(\ell+4) - F(\ell+3) + F(\ell) = \\ & = \Gamma_2\Big[\sum_{k=\ell+6}^{\ell+9} \chi_2(k) - \sum_{k=\ell+3}^{\ell+6} \chi_2(k) + \chi_2(\ell)\Big] + \\ & + \Gamma_3\Big[\sum_{k=\ell+7}^{\ell+9} \chi_3(k) - \sum_{k=\ell+3}^{\ell+5} \chi_3(k) + \chi_3(\ell)\Big] + \\ & + \Gamma_4\Big[\sum_{k=\ell+6}^{\ell+9} \chi_4(k-1) - \sum_{k=\ell+3}^{\ell+6} \chi_4(k-1) + \chi_4(\ell-1)\Big] + \\ & + \Gamma_5\Big[\sum_{k=\ell+6}^{\ell+10} \chi_5(k) - \sum_{k=\ell+2}^{\ell+6} \chi_5(k) - \chi_5(\ell+10) + \chi_5(\ell+2) + \chi_5(\ell)\Big] + \Gamma = \\ & = \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell+2) + \Gamma = \\ & = \Gamma_2\chi_2(\ell+12) + \Gamma_3\chi_3(\ell+12) + \Gamma_4\chi_4(\ell+11) + \Gamma_5\chi_5(\ell+12) + \Gamma = \\ & = F(\ell+12).\end{aligned}$$

Lemma 7 is proved. ■

5. Proof of Theorem 1

Assume that the numbers $k \in \mathbb{N}_0$, $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation

$$f\big(n^2 + Dm^2 + k\big) = f^2(n) + Df^2(m) + k \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Let $\mathcal{F}(n) := f^2(n)$ for every $n \in \mathbb{N}$ and $U = D$, $V = k$. Thus, we have

$$f\big(n^2 + Dm^2 + k\big) = \mathcal{F}(n) + D\mathcal{F}(m) + k \quad \text{for every} \quad n, m \in \mathbb{N}.$$

and

$$\mathcal{F}\big(n^2+Dm^2+k\big) = \big(\mathcal{F}(n) + D\mathcal{F}(m) + k\big)^2 \tag{5.1}$$

for every $n, m \in \mathbb{N}$. We shall use the notations of Lemmas 6–7 for the cases $G = \mathcal{F}$ and $H = \mathcal{F}$. Hence

$$\begin{aligned}
A :=& \frac{1}{120}\big(\mathcal{F}(6) + 4\mathcal{F}(5) - \mathcal{F}(3) - \mathcal{F}(2) - 3\mathcal{F}(1)\big),\\
\Gamma_2 :=& \frac{-1}{8}\big(\mathcal{F}(6) - 4\mathcal{F}(5) + 4\mathcal{F}(4) - \mathcal{F}(3) + 3\mathcal{F}(2) - 3\mathcal{F}(1)\big),\\
\Gamma_3 :=& \frac{-1}{3}\big(\mathcal{F}(6) - 2\mathcal{F}(5) + 2\mathcal{F}(3) - \mathcal{F}(2)\big),\\
\Gamma_4 :=& \frac{1}{4}\big(\mathcal{F}(6) - 2\mathcal{F}(4) - \mathcal{F}(3) + \mathcal{F}(2) + \mathcal{F}(1)\big),\\
\Gamma_5 :=& \frac{1}{5}\big(\mathcal{F}(6) - \mathcal{F}(5) - \mathcal{F}(3) - \mathcal{F}(2) + 2\mathcal{F}(1)\big),\\
\Gamma :=& \frac{1}{4}\big(\mathcal{F}(6) - 4\mathcal{F}(5) + 2\mathcal{F}(4) + 3\mathcal{F}(3) + \mathcal{F}(2) + \mathcal{F}(1)\big),\\
F(\ell) :=& \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell) + \Gamma,
\end{aligned}$$

From (4.6) we have

$$\mathcal{F}(\ell) = f^2(\ell) = A\ell^2 + F(\ell) \quad \text{for every} \quad \ell \in \mathbb{N}. \tag{5.2}$$

Lemma 8. *We have*

$$A \in \{0, 1\}.$$

Proof. We infer from (5.1) and (5.2) that

$$\begin{aligned}
&A(n^2+Dm^2+k)^2 + F(n^2+Dm^2+k) =\\
&= \Big(A(n^2+Dm^2) + F(n) + DF(m) + k\Big)^2
\end{aligned} \tag{5.3}$$

for every $n, m \in \mathbb{N}$. Since

$$|F(\ell)| \le |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every} \quad \ell \in \mathbb{N},$$

for each fix $n \in \mathbb{N}$, we infer from (5.3) that

$$\begin{aligned}
A &= \lim_{m\to\infty} \frac{A(n^2+Dm^2+k)^2 + F(n^2+Dm^2+k)}{(Dm^2)^2} =\\
&= \lim_{m\to\infty} \Big(\frac{A(n^2+Dm^2) + F(n) + DF(m) + k}{Dm^2}\Big)^2 = A^2.
\end{aligned}$$

Therefore, we have $A \in \{0, 1\}$, and so Lemma 8 is proved. ∎

Lemma 9. *Assume that $A = 1$. Then*

$$\mathcal{F}(n) = n^2, \quad f(n) = \pm n$$

and

$$f\big(n^2 + Dm^2 + k\big) = n^2 + Dm^2 + k$$

for every $n, m \in \mathbb{N}$.

Proof. Assume that $A = 1$. In this case, we have $\mathcal{F}(n) = n^2 + F(n)$. Then we infer from (5.1) and (5.2) that

$$\begin{aligned} 0 &= \mathcal{F}\big(n^2 + Dm^2 + k\big) - \big(\mathcal{F}(n) + D\mathcal{F}(m) + k\big)^2 = \\ &= \big(n^2 + Dm^2 + k\big)^2 + F\big(n^2 + Dm^2 + k\big) - \\ &\quad - \big(n^2 + F(n) + Dm^2 + DF(m) + k\big)^2 = \\ &= 2D\big(n^2 + k - (n^2 + F(n) + DF(m) + k)\big)m^2 + W(n,m) = \\ &= -2D\big(F(n) + DF(m)\big)m^2 + W(n,m) \end{aligned} \tag{5.4}$$

holds for every $n, m \in \mathbb{N}$, where

$$\begin{aligned} W(n,m) :&= \big(n^2 + k\big)^2 + F\big(n^2 + Dm^2 + k\big) - \\ &\quad - \big(n^2 + F(n) + DF(m) + k\big)^2. \end{aligned} \tag{5.5}$$

Now let n, $a \in \mathbb{N}$ be fixed, $m \in \mathbb{N}$, $m \equiv a \pmod{60}$. Since $F(k)$ is a periodic function $\pmod{60}$, we have $n^2 + Dm^2 + k \equiv n^2 + Da^2 + k \pmod{60}$, and so

$$F\big(n^2 + Dm^2 + k\big) = F\big(n^2 + Da^2 + k\big)$$

and

$$n^2 + F(n) + DF(m) + k = n^2 + F(n) + DF(a) + k.$$

On the other hand

$$|W(n,m)| = \big|(n^2 + k)^2 + F(n^2 + Da^2 + k) - (n^2 + F(n) + DF(a) + k)^2\big| < \infty$$

and so we obtain from (5.4) that

$$F(n) + DF(a) = \lim_{\substack{m \to \infty \\ m \equiv a \pmod{60}}} \frac{W(n,m)}{2Dm^2} = 0.$$

Thus, we proved that $F(n) + DF(a) = 0$ holds for each fixed n, $a \in \mathbb{N}$, consequently

$$F(n) + DF(m) = 0 \quad \text{and} \quad W(n,m) = 0 \tag{5.6}$$

hold for every $n, m \in \mathbb{N}$. Applying (5.6) for the case $n = m$, we have

$$(D+1)F(n) = 0 \quad \text{for every} \quad n \in \mathbb{N}.$$

Consequently

$$F(n) = 0 \quad \text{for every} \quad m \in \mathbb{N}$$

and

$$\mathcal{F}(m) = f^2(m) = m^2, \quad f(m) = \pm m$$

and

$$f(n^2+Dm^2+k) = f^2(n) + Df^2(m) + k = n^2 + Dm^2 + k$$

for every $n, m \in \mathbb{N}$.

Lemma 9 is proved. ■

Now we complete the proof of Theorem 1.

Assume that $A = 0$. Then

$$F(n) = f(n)^2 = \Gamma_2\chi_2(n) + \Gamma_3\chi_3(n) + \Gamma_4\chi_4(n-1) + \Gamma_5\chi_5(n) + \Gamma$$

and

$$F(n^2+Dm^2+k) = (F(n) + DF(m) + k)^2.$$

We infer from Theorem 2 that

$$\Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0$$

and

$$F(n) = f^2(n) = \left(\frac{1 \pm \sqrt{1-4Dk-4k}}{2(D+1)}\right)^2, \quad f(n) = \pm\frac{1 \pm \sqrt{1-4Dk-4k}}{2(D+1)}.$$

We have

$$\begin{gathered} f\left(n^2+Dm^2+k\right) = f(n)^2 + Df(m)^2 + k = \\ = (D+1)\left(\frac{1 \pm \sqrt{1-4Dk-4k}}{2(D+1)}\right)^2 + k = \frac{1 \pm \sqrt{1-4Dk-4k}}{2(D+1)}. \end{gathered}$$

Thus, all assertions of Theorem 1 are proved.

The proof of Theorem 1 is finished. ■

6. Proof of Corollary 1

Assume that the numbers $k \in \mathbb{N}_0$, $D \in \mathbb{N}$ and a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation

$$f\big(n^2 + Dm^2 + k\big) = f^2(n) + Df^2(m) + k \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{6.1}$$

From Theorem 1, one of the following assertions holds:

$$\text{(a)} \qquad f(n) = \varepsilon_{D,k}(n)\frac{1 - \sqrt{1 - 4Dk - 4k}}{2(D+1)},$$

$$\text{(b)} \qquad f(n) = \varepsilon_{D,k}(n)\frac{1 + \sqrt{1 - 4Dk - 4k}}{2(D+1)},$$

$$\text{(c)} \qquad f(n) = \varepsilon_{D,k}(n)n,$$

where $\varepsilon_{D,k}(n) \in \{1, -1\}$ and $\varepsilon_{D,k}(n^2 + Dm^2 + k) = 1$ for every $n, m \in \mathbb{N}$.

Assume that (a) or (b) holds, then $f(n) = \varepsilon_{D,k}(n)C$, where $C \in \mathbb{C}$. Since $f \in \mathcal{M}$, by applying (6.1) with $n := n_1 \cdots n_t$, where $n_1, \ldots, n_t$ are pairwise relatively primes, we have

$$\begin{aligned} C &= \varepsilon_{D,k}\big((n_1 \cdot \ldots \cdot n_t)^2 + Dm^2 + k\big)C = \\ &= f^2(n_1 \cdot \ldots \cdot n_t) + Df^2(m) + k = f^2(n_1) \cdot \ldots \cdot f^2(n_t)^2 + DC^2 + k = \\ &= C^{2t} + DC^2 + k. \end{aligned}$$

Thus, $C^{2t} = C - DC^2 - k$ holds for every $t \in \mathbb{N}$, consequently

$$C^4 = C^2, \quad \text{and so} \quad C \in \{0, -1, 1\}.$$

Since f is multiplicative, $f(1) = 1$, therefore $C \neq 0$.

If $C = \pm 1$, then $C^{2t} = C - DC^2 - k$ implies $1 = \pm 1 - D - k$, and so $k = \pm 1 - 1 - D \notin \mathbb{N}$. This is a contradiction and so (a) and (b) do not occur.

Thus, the case (c) is true, i. e. $f(n) = \varepsilon_{D,k}(n)n$. It is obvious from $f \in \mathcal{M}$ that $\varepsilon_{D,k} \in \mathcal{M}$ and $\varepsilon_{D,k}(n^2 + Dm^2 + k) = 1$ for every $n, m \in \mathbb{N}$.

Corollary 1 is proved. ■

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B. M. M. Khanh
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
mbuiminh@yahoo.com