

CHARACTERIZATION OF THE IDENTITY FUNCTION WITH AN EQUATION FUNCTION

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Abstract. Let $D \in \{2, 3\}$ and $f : \mathbb{N} \rightarrow \mathbb{C}$. We prove that if
$$f(n^2 + Dnm + m^2) = f(n)^2 + Df(n)f(m) + f(m)^2 \quad \text{for every } n, m \in \mathbb{N},$$

then f is the identity function.

1. Introduction

A characterization of the identity was studied by Spiro [5], De Koninck, Káta and Phong [2] and by others. Recently Poo Sung Park [4] proved the following result:

Theorem A. *If f is a multiplicative function and*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N},$$

then f is the identity function.

Our general purpose is to determine all those $f : \mathbb{N} \rightarrow \mathbb{C}$ for which

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N},$$

where $D \in \mathbb{Z}$ is a fixed number.

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B. M. Phong and R. B. Szeidl [3] consider the case $D = 1$, they proved the following theorem:

Theorem B. ([3]) *If $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arbitrary function with $f(1) = 1$ and*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m)$$

holds for every $n, m \in \mathbb{N}$, then f is the identity function.

It is obvious that if f is multiplicative, then $f(1) = 1$, consequently Theorem B improves Theorem A.

We prove the following results:

Theorem 1. *Assume that f is an arithmetical function $\mathbb{N} \rightarrow \mathbb{C}$ with $f(1) = 1$ and $D \in \{2, 3\}$. If*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Then $f(n) = n$ for every $n \in \mathbb{N}$.

We note that $f(1) = 1$ if a function f is multiplicative, therefore the following result follows directly from Theorem 1.

Corollary 1. *Assume that a multiplicative function f and $D \in \{2, 3\}$ satisfy*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Then $f(n) = n$ for every $n \in \mathbb{N}$.

We think that the following assertion is true:

Conjecture 1. *Assume that $f \in \mathcal{M}$ and $D \in \mathbb{N}$ satisfy the following equation*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

- $f(n) = 0$ for every $n \in \mathbb{N}$,
- $f(n) = \frac{1}{D+2}$ for every $n \in \mathbb{N}$,
- $f(n) = n$ for every $n \in \mathbb{N}$.

We cannot prove Conjecture 1 for other D . It seems that they need some new ideas.

2. Proof of Theorem 1 for the case $D = 2$

First we prove the following lemma.

Lemma 1. *Assume that a function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies $f(1) = 1$ and*

$$(2.1) \quad f(n^2 + 2nm + m^2) = f^2(n) + 2f(n)f(m) + f^2(m)$$

for every $n, m \in \mathbb{N}$. Then

$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 3.$$

Proof. Assume that a function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies $f(1) = 1$. We infer from (2.1) that

$$f((n+m)^2) = (f(n) + f(m))^2 \quad \text{for every } n, m \in \mathbb{N},$$

consequently

$$\begin{cases} f(4) &= f(2^2) = f((1+1)^2) = (f(1) + f(1))^2 = 4, \\ f(9) &= f(3^2) = f((1+2)^2) = (f(1) + f(2))^2 = (f(2) + 1)^2, \\ f(16) &= f(4^2) = f((1+3)^2) = (f(2) + f(2))^2 = 4f(2)^2. \end{cases}$$

Consequently

$$\begin{cases} f(25) &= f((4+1)^2) = (f(4) + f(1))^2 = 25, \\ f(36) &= f((4+2)^2) = (f(4) + f(2))^2 = (f(2) + 4)^2, \end{cases}$$

which imply

$$\begin{cases} 4f(2)^2 &= f(16) = f((3+1)^2) = (f(3) + 1)^2, \\ 25 &= f(25) = f((3+2)^2) = (f(3) + f(2))^2, \\ (f(2) + 4)^2 &= f(36) = f((3+3)^2) = (f(3) + f(3))^2 = 4f(3)^2. \end{cases}$$

It easily follows from these relations that

$$\begin{cases} 4f(3)^2 &= (f(2) + 4)^2, \\ 16f(2)^2 &= 4(f(3) + 1)^2 = (f(2) + 4)^2 + 8f(3) + 4, \\ 100 &= 4(f(3) + f(2))^2 = (f(2) + 4)^2 + 8f(2)f(3) + 4f(2)^2. \end{cases}$$

Consequently

$$(2.2) \quad 8f(3) = 15f(2)^2 - 8f(2) - 20$$

and

$$(2.3) \quad 8f(2)f(3) = -5f(2)^2 - 8f(2) + 84.$$

We infer from (2.2) that

$$f(3) = \frac{15}{8}f(2)^2 - f(2) - \frac{5}{2}$$

and so we get from (2.2) that

$$8f(2)\left(\frac{15}{8}f(2)^2 - f(2) - \frac{5}{2}\right) = -5f(2)^2 - 8f(2) + 84.$$

Consequently

$$\begin{aligned} 0 &= f(2)(15f(2)^2 - 8f(2) - 20) + 5f(2)^2 + 8f(2) - 84 = \\ &= 3(5f(2)^3 - f(2)^2 - 4f(2) - 28) = 3(f(2) - 2)(5f(2)^2 + 9f(2) + 14), \end{aligned}$$

that is

$$(2.4) \quad (f(2) - 2)(5f(2)^2 + 9f(2) + 14) = 0.$$

On the other hand, we have

$$\begin{aligned} 4f(2)^2 &= f(4^2) = (f(3) + f(1))^2 = \left(\frac{15}{8}f(2)^2 - f(2) - \frac{5}{2} + 1\right)^2 = \\ &= \left(\frac{15}{8}f(2)^2 - f(2) - \frac{3}{2}\right)^2, \end{aligned}$$

which implies that

$$(2.5) \quad (5f(2) + 6)(f(2) - 2)(5f(2) + 2)(3f(2) - 2) = 0.$$

Finally, it is easy to check from (2.4) and (2.5) that

$$5f(2)^2 + 9f(2) + 14 \neq 0 \quad \text{if} \quad f(2) \in \left\{-\frac{6}{5}, -\frac{2}{5}, \frac{2}{3}\right\},$$

therefore

$$f(2) = 2 \quad \text{and} \quad f(3) = \frac{15}{8}f(2)^2 - f(2) - \frac{5}{2} = 3.$$

Lemma 1 is proved. ■

Now we complete the proof of Theorem 1 by showing the following lemma.

Lemma 2. *Assume that a function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies $f(1) = 1$ and*

$$(2.6) \quad f(n^2 + 2nm + m^2) = f^2(n) + 2f(n)f(m) + f^2(m)$$

for every $n, m \in \mathbb{N}$. Then

$$f(n) = n \quad \text{for every } n \in \mathbb{N}.$$

Proof. By using Lemma 1 we have $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$. We assume that

$$f(N - 1) = N - 1 \quad \text{for some } N \geq 4.$$

We will prove that

$$f(N) = N.$$

We infer from (2.1) that

$$f((N + 1)^2) = f(((N - 1) + 2)^2) = (f(N - 1) + f(2))^2 = (N + 1)^2$$

and

$$f((N + 2)^2) = f(f((N - 1) + 3)^2) = (f(N - 1) + f(3))^2 = (N + 2)^2.$$

But

$$f((N + 1)^2) = (f(N) + f(1))^2$$

and

$$f((N + 2)^2) = (f(N) + f(2))^2 = (f(N) + 2)^2,$$

consequently

$$\begin{aligned} 2f(N) + 3 &= (f(N) + 2)^2 - (f(N) + 1)^2 = \\ &= f((N + 2)^2) - f((N + 1)^2) = (N + 2)^2 - (N + 1)^2 = \\ &= 2N + 3. \end{aligned}$$

This implies that $f(N) = N$. Lemma 2 and so Theorem 1 are proved for the case $D = 2$. ■

3. Proof of Theorem 1 for the case $D = 3$. Lemmas

First we state the following result.

Lemma 3. (B. M. Phong and R. B. Szeidl [3]) *Let $D \in \mathbb{N}$ be fixed and the function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

If $f(n) = n$ for every $n \leq 8D + 20$, then $f(n) = n$ for every $n \in \mathbb{N}$.

Now assume that $D = 3$ and

$$(3.1) \quad f(n^2 + 3nm + m^2) = f^2(n) + 3f(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

In order to prove our theorem, by using Lemma 3, we need show that

$$f(n) = n \quad \text{for every } n \leq 8D + 20 = 8 \cdot 3 + 20 = 44.$$

First, we prove the following lemma.

Lemma 4. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ and*

$$(3.2) \quad f(n^2 + 3nm + m^2) = f(n)^2 + 3f(n)f(m) + f^2(m)$$

holds for every $n, m \in \mathbb{N}$. Then

$$f(k) = k \quad \text{for every } k \leq 4.$$

Proof. In the following we shall use the notation $[N, n, m, u, v] \in E$ if

$$N = n^2 + 3nm + m^2 = u^2 + 3uv + v^2 \quad \text{with } n > u, m \leq n, v \leq u.$$

It is obvious that if $[N, n, m, u, v] \in E$, then we obtain from (3.2) that

$$f(N) = f^2(n) + 3f(n)f(m) + f^2(m) \quad \text{and} \quad f(N) = f^2(u) + 3f(u)f(v) + f^2(v)$$

which show that

$$(3.3) \quad E(N) := f^2(n) + 3f(n)f(m) + f^2(m) - f^2(u) - 3f(u)f(v) - f^2(v) = 0.$$

We can find 6 numbers $[N, n, m, u, v] \in E$ such that $\max(n, m, u, v) \leq 79$ and $E(N)$ can written in terms of $f(2), f(3), f(4)$.

Our method as follows: Let

$$\mathcal{A} := \{n \in \mathbb{N} \mid f(n) = P(f(2), f(3), f(4)) \quad (n \leq 79) \quad \text{where} \\ P(x, y, z) \in \mathbb{Z}[x, y, z]\}.$$

Then we can show that there are 16 number in $\mathcal{A}$, namely

$$\mathcal{A} = \{1, 2, 3, 4, 5, 11, 19, 20, 29, 31, 41, 44, 45, 59, 61, 79\}.$$

By using the computer and Maple, we find those elements $[N, n, m, u, v] \in E$ such $n, m, u, v \in \mathcal{A}$, and so we get equation

$$E(N) = f^2(n) + 3f(n)f(m) + f^2(m) - f^2(u) - 3f(u)f(v) - f^2(v) = 0.$$

It is obvious that $E(N) = Q(P(f(2), f(3), f(4)))$, where $Q(x, y, z) \in \mathbb{Z}[x, y, z]$ and $\max(n, m, u, v) \leq 79$.

We can show that there is 6 such number $[N, n, m, u, v] \in E$, namely

$$\begin{aligned} &[605, 19, 4, 11, 11], [1805, 41, 1, 19, 19], [4805, 44, 19, 31, 31], \\ &[4205, 59, 4, 29, 29], [7205, 79, 4, 59, 19], [18605, 79, 44, 61, 61]. \end{aligned}$$

We will consider the following 4 equations $E(605)$, $E(4205)$, $E(7205)$ and $E(1805)$.

Since $f(1) = 1$, we obtain from (3.2) that

$$(3.4) \quad \begin{cases} f(5) &= f^2(1) + 3f(1)f(1) + f^2(1) = 5, \\ f(11) &= f^2(2) + 3f(2)f(1) + f^2(1) = f(2)^2 + 3f(2) + 1, \\ f(19) &= f^2(3) + 3f(3)f(1) + f^2(1) = f(3)^2 + 3f(3) + 1, \\ f(20) &= 5f(2)^2, \\ f(29) &= f^2(4) + 3f(4)f(1) + f^2(1) = f(4)^2 + 3f(4) + 1, \\ f(31) &= f^2(3) + 3f(3)f(2) + f^2(2), \\ f(41) &= f^2(5) + 3f(5)f(1) + f^2(1) = 41, \\ f(44) &= f^2(4) + 3f(4)f(2) + f^2(2), \\ f(45) &= 5f(3)^2, \\ f(59) &= f^2(5) + 3f(5)f(2) + f^2(2) = f(2)^2 + 15f(2) + 25, \\ f(61) &= f^2(4) + 3f(4)f(3) + f^2(3), \\ f(79) &= f^2(5) + 3f(5)f(3) + f^2(3) = f(3)^2 + 15f(3) + 25, \end{cases}$$

consequently, we have

$$\begin{aligned} E(605) &= f(19)^2 + 3f(19)f(4) + f(4)^2 - 5f(11)^2 = \\ &= (f(3)^2 + 3f(3) + 1)^2 + 3(f(3)^2 + 3f(3) + 1)f(4) + f(4)^2 - \\ &\quad - 5(f(2)^2 + 3f(2) + 1)^2 = \\ &= -5f(2)^4 + f(3)^4 - 30f(2)^3 + 6f(3)^3 + 3f(3)^2f(4) - 55f(2)^2 + \\ &\quad + 11f(3)^2 + 9f(3)f(4) + f(4)^2 - \\ &\quad - 30f(2) + 6f(3) + 3f(4) - 4 = 0, \end{aligned}$$

$$\begin{aligned}
E(4205) &= f(59)^2 + 3f(59)f(4) + f(4)^2 - 5f(29)^2 = \\
&= (f(2)^2 + 15f(2) + 25)^2 + 3(f(2)^2 + 15f(2) + 25)f(4) + f(4)^2 - \\
&\quad - 5(f(4)^2 + 3f(4) + 1)^2 = \\
&= f(2)^4 - 5f(4)^4 + 30f(2)^3 + 3f(2)^2f(4) - 30f(4)^3 + 275f(2)^2 + \\
&\quad + 45f(2)f(4) - 54f(4)^2 + 750f(2) + 45f(4) + 620 = 0, \\
E(7205) &= f(79)^2 + 3f(79)f(4) + f(4)^2 - f(59)^2 - 3f(59)f(19) - f(19)^2 = \\
&= (f(3)^2 + 15f(3) + 25)^2 + 3(f(3)^2 + 15f(3) + 25)f(4) + f(4)^2 - \\
&\quad - (f(2)^2 + 15f(2) + 25)^2 - \\
&\quad - 3(f(2)^2 + 15f(2) + 25)(f(3)^2 + 3f(3) + 1) - \\
&\quad - (f(3)^2 + 3f(3) + 1)^2 = \\
&= -f(2)^4 - 3f(3)^2f(2)^2 - 30f(2)^3 - 9f(2)^2f(3) - 45f(2)f(3)^2 + \\
&\quad + 24f(3)^3 + 3f(3)^2f(4) - 278f(2)^2 - 135f(2)f(3) + 189f(3)^2 + \\
&\quad + 45f(3)f(4) + f(4)^2 + 519f(3) - 795f(2) + 75f(4)) - 76 = 0
\end{aligned}$$

and

$$\begin{aligned}
E(1805) &= f(41)^2 + 3f(41)f(1) + f(1)^2 - 5f(19)^2 = 41^2 + 3 \cdot 41 + 1^2 - \\
&\quad - 5(f(3)^2 + 3f(3) + 1)^2 = \\
&= -5(f(3) - 3)(f(3) + 6)(f(3)^2 + 3f(3) + 20) = 0.
\end{aligned}$$

Let

$$\begin{aligned}
P_1(x, y, z) &= -5x^4 + y^4 - 30x^3 + 6y^3 + 3y^2z - 55x^2 + \\
&\quad + 11y^2 + 9yz + z^2 - 30x + 6y + 3z - 4, \\
P_2(x, y, z) &= x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + \\
&\quad + 45xz - 54z^2 + 750x + 45z + 620
\end{aligned}$$

and

$$\begin{aligned}
P_3(x, y, z) &= -x^4 - 3x^2y^2 - 30x^3 - 9x^2y - 45xy^2 + 24y^3 + 3y^2z - 278x^2 - \\
&\quad - 135xy + 189y^2 + 45yz + z^2 - 795x + 519y + 75z - 76.
\end{aligned}$$

It is clear to see that

$$(3.5) \quad \begin{cases} P_1(f(2), f(3), f(4)) = E(605) = 0, \\ P_2(f(2), f(3), f(4)) = E(4205) = 0, \\ P_3(f(2), f(3), f(4)) = E(7205) = 0. \end{cases}$$

Since

$$E(1805) = -5(f(3) - 3)(f(3) + 6)(f(3)^2 + 3f(3) + 20) = 0,$$

in the proof of Lemma 4 we shall distinguish three cases, namely

- (a) $f(3) = -6$,
- (b) $f(3)^2 + 3f(3) + 20 = 0$,
- (c) $f(3) = 3$.

We shall prove that the cases (a) and (b) are not occur.

Case (a). Assume that $f(3) = -6$. Then

$$0 = E(7205) - E(605) = -1575f(2) - 1926 - 277f(2)^2 - 144f(4) + 4f(2)^4,$$

which implies

$$f(4) = \frac{-1575f(2) - 1926 - 277f(2)^2 + 4f(2)^4}{144}.$$

Therefore, with the help of Maple and using a computer, we obtain the following values:

$$p := 20736E(605) = (f(2) + 6)(16f(2)^7 - 96f(2)^6 - 1640f(2)^5 - 2760f(2)^4 + 7033f(2)^3 + 208272f(2)^2 - 1116099f(2) - 786186) = 0$$

and

$$\begin{aligned} q := 429981696E(4205) = & -1280f(2)^{16} + 354560f(2)^{14} + 2016000f(2)^{13} - \\ & -34641120f(2)^{12} - 418824000f(2)^{11} + 54891440f(2)^{10} + 26418042000f(2)^9 + \\ & + 165565745371f(2)^8 + 1154380500f(2)^7 - 5225102615718f(2)^6 - \\ & -32941396108980f(2)^5 - 107581783487877f(2)^4 - 209955740732280f(2)^3 - \\ & -244519617134808f(2)^2 - 156273690782880f(2) - 42082903696464 = 0. \end{aligned}$$

Let

$$\begin{aligned} P(x) = & (x + 6)(16x^7 - 96x^6 - 1640x^5 - 2760x^4 + 7033x^3 + \\ & + 208272x^2 - 1116099x - 786186) \end{aligned}$$

and

$$\begin{aligned} Q(x) = & -1280x^{16} + 354560x^{14} + 2016000x^{13} - 34641120x^{12} - 418824000x^{11} + \\ & + 54891440x^{10} + 26418042000x^9 + 165565745371x^8 + 1154380500x^7 - \\ & - 5225102615718x^6 - 32941396108980x^5 - 107581783487877x^4 - \\ & - 209955740732280x^3 - 244519617134808x^2 - 156273690782880x - \\ & - 42082903696464. \end{aligned}$$

Then

$$P(f(2)) = p = 20736 \cdot E(605) = 0$$

and

$$Q(f(2)) = q = 429981696 \cdot E(4805) = 0.$$

But with help of computer, we have

$$(P(x), Q(x)) = 1,$$

which is impossible. The case (a) is not occur.

Case (b). If $a(x, y, z)$ and $b(x, y, z)$ are polynomials in $(\mathbb{Q}[y, z])[x]$ with $b(x, y, z) \neq 0$, then there exist unique polynomials $q(x, y, z)$ and $r(x, y, z)$ in $(\mathbb{Q}[y, z])[x]$ with

$$a(x, y, z) = q(x, y, z)b(x, y, z) + r(x, y, z)$$

and such that the degree of $r(x, y, z)$ is smaller than the degree of $b(x, y, z)$ in x . The polynomials $q(x, y, z)$ and $r(x, y, z)$ are uniquely determined by $a(x, y, z)$ and $b(x, y, z)$. We denote by $\text{rem}(a(x, y, z), b(x, y, z), x)$ the $r(x, y, z)$. Similar way, we can define $\text{rem}(a(x, y, z), b(x, y, z), y)$ and $\text{rem}(a(x, y, z), b(x, y, z), z)$.

Now assume that $f(3)^2 + 3f(3) + 20 = 0$. Let

$$Q(x, y, z) := y^2 + 3y + 20.$$

Then

$$(3.6) \quad Q(f(2), f(3), f(4)) = 0.$$

Let

$$Q_1(x, y, z) = \text{rem}(P_1(x, y, z), Q(x, y, z), y), \quad Q_2(x, y, z) = P_2(x, y, z)$$

and

$$Q_3(x, y, z) = \text{rem}(-P_3(x, y, z), Q(x, y, z), y).$$

Then we have

$$Q_1(x, y, z) = -(5x^4 + 30x^3 + 55x^2 - z^2 + 30x + 57z - 356),$$

$$Q_2(x, y, z) = x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + 45xz - 54z^2 + 750x + 45z + 620$$

and

$$Q_3(x, y, z) = -x^4 - 30x^3 - 218x^2 + 36yz + z^2 + 105x - 312y + 15z - 2416.$$

It can check that

$$Q_1(x, y, z) = P_1(x, y, z) - Q(x, y, z)(y^2 + 3y + 3z - 18)$$

and

$$Q_3(x, y, z) = P_3(x, y, z) - 3Q(x, y, z)(x^2 + 15x - 8y - z - 39).$$

Thus, we infer from (3.5) and (3.6) that

$$(3.7) \quad Q_i(f(2), f(3), f(4)) = 0 \quad (i = 1, 2, 3).$$

Next, we define the following polynomials

$$\begin{aligned} Q_4(x, y, z) &= \text{rem}(-Q_2(x, y, z), Q_1(x, y, z), z) = \\ &= 125x^8 + 1500x^7 + 7250x^6 + 18000x^5 + 3000x^4z + 96369x^4 + 18000x^3z + \\ &\quad + 449940x^3 + 32997x^2z + 798920x^2 + 17955xz + \\ &\quad + 432720x + 812868z - 5778144 \\ Q_5(x, y, z) &= \text{rem}(Q_1(x, y, z), Q_4(x, y, z), y) = \\ &= 9(1000x^4 + 6000x^3 + 10999x^2 + 5985x + 270956)^2 Q_1(x, y, z) + \\ &\quad + Q_4(x, y, z)(125x^8 + 1500x^7 + 7250x^6 + 18000x^5 - 3000x^4z + 267369x^4 - \\ &\quad - 18000x^3z + 1475940x^3 - 32997x^2z + 2679749x^2 - 17955xz + 1456155x - \\ &\quad - 812868z + 40555332) = \\ &= 15625x^{16} + 375000x^{15} + 4062500x^{14} + 26250000x^{13} + 107029750x^{12} + \\ &\quad + 237342000x^{11} - 60155875x^{10} - 2383543125x^9 - 7842885634x^8 - \\ &\quad - 8072184150x^7 + 22066388781x^6 + 90163881705x^5 + 91053921601x^4 - \\ &\quad - 165337109910x^3 - 458418024684x^2 - 295729578720x + 894012947136. \end{aligned}$$

On the other hand, if we define $Q_6(x, y, z)$ and $Q_7(x, y, z)$ as follows

$$Q_6(x, y, z) = Q_1(x, y, z) - Q_3(x, y, z)$$

and

$$Q_7(x, y, z) = \text{rem}(144(3z - 26)^2 Q(x, y, z), Q_6(x, y, z), y)$$

then

$$Q_6(x, y, z) = -4x^4 + 163x^2 - 36yz - 135x + 312y - 72z + 2772,$$

and

$$\begin{aligned} Q_7(x, y, z) &= 16x^8 - 1304x^6 + 1080x^5 + 144x^4z + 8137x^4 - 44010x^3 - \\ &\quad - 5868x^2z + 769329x^2 + 4860xz + 23328z^2 - 622080x - 481680z + 7036272. \end{aligned}$$

Finally, we define $Q_8(x, y, z)$ by equation:

$$Q_8(x, y, z) = \text{rem}((1000x^4 + 6000x^3 + 10999x^2 + 5985x + 270956)^2 Q_7, Q_4, z).$$

One can check with computer that

$$\begin{aligned} Q_8(x, y, z) = & 50500000x^{16} + 1056000000x^{15} + 9552474000x^{14} + \\ & + 56193990000x^{13} + 331085315516x^{12} + 2106409113480x^{11} + \\ & + 11508205483916x^{10} + 50700142595880x^9 + 197646102014197x^8 + \\ & + 711483952342020x^7 + 2537656772345138x^6 + 7179892664435100x^5 + \\ & + 13657348154690953x^4 + 23102991892891800x^3 + 94982689780252416x^2 - \\ & - 20002804321668480x + 351745639528990464. \end{aligned}$$

By using of definitions of $Q_i(x, y, z)$ ($i = 1, 2, \dots, 8$), we obtain from (3.6) and (3.7) that

$$Q_i(f(2), f(3), f(4)) = 0 \quad (i = 1, 2, \dots, 8).$$

This is impossible, because we can show with computer that

$$(Q_5(x, y, z), Q_8(x, y, z)) = 1.$$

The case (b) is not occur.

Case (c). $f(3) = 3$.

Let

$$A_i(x, z) := P_i(x, 3, z) \quad (i = 1, 2, 3).$$

Then

$$(3.8) \quad A_i(f(2), f(4)) = P_i(f(2), f(3), f(4)) = 0 \quad (i = 1, 2, 3)$$

and

$$A_1(x, z) := -5x^4 - 30x^3 - 55x^2 + z^2 - 30x + 57z + 356,$$

$$A_2(x, z) := x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + 45xz - 54z^2 + 750x + 45z + 620,$$

$$A_3(x, z) := -x^4 - 30x^3 - 332x^2 + z^2 - 1605x + 237z + 3830,$$

Let $B(x, z) := A_1(x, z) - A_3(x, z)$. Then

$$(3.9) \quad \begin{cases} B(x, z) &= -4x^4 + 277x^2 + 1575x - 180z - 3474, \\ B(f(2), f(4)) &= A_1(f(2), f(4)) + A_3(f(2), f(4)) = 0. \end{cases}$$

$$\begin{aligned} B_1(x, z) &= \text{factor}(\text{rem}(32400A_1, B, z)) = \\ &= (x - 2)(x - 3)(16x^6 + 80x^5 - 1912x^4 - 22640x^3 - \\ &\quad - 200247x^2 - 964845x - 2006694), \end{aligned}$$

$$\begin{aligned}
B_2(x, z) &= \text{rem}(209952000A_2(x, z), B(x, z), z) = \\
&= -(x-2)(256x^{15} + 512x^{14} - 69888x^{13} - 542976x^{12} + 7100256x^{11} + \\
&\quad + 97965312x^{10} - 76393840x^9 - 6922389680x^8 - 28155424095x^7 + \\
&\quad + 149268389310x^6 + 1413378129186x^5 + 1734555943872x^4 - \\
&\quad - 11553800720103x^3 - 19538613395406x^2 + \\
&\quad + 77057131513356x - 52323738856488).
\end{aligned}$$

It follows from (3.8) and (3.9) that

$$(3.10) \quad B_1(f(2), f(4)) = 0 \quad \text{and} \quad B_1(f(2), f(4)) = 0.$$

One can prove that

$$(B_1(x, z), B_2(x, z)) = x - 2,$$

and so there are suitable polynomials $c_1(x, z) \in \mathbb{Q}[x, z], c_2(x, z) \in \mathbb{Q}[x, z]$ such that

$$x - 2 = c_1(x, z)B_1(x, z) + c_2(x, z)B_2(x, z).$$

This with (3.10) show that

$$f(2) - 2 = c_1(f(2), f(4))B_1(f(2), f(4)) + c_2(f(2), f(4))B_2(f(2), f(4)) = 0,$$

consequently

$$f(2) = 2.$$

If $f(2) = 2$, then

$$0 = B(f(2), f(4)) = B(2, f(4)) = -180f(4) + 720.$$

Therefore, we have $f(4) = 4$, and so the proof of Lemma 4 is completed. ■

Lemma 5. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ satisfying (3.2). Then*

$$f(6) = 6 \quad \text{and} \quad f(7) = 7.$$

Proof. Let

$$I := \{n \in \mathbb{N} \mid f(n) = n\}.$$

By using Lemma 4, we have $f(1) = 1, f(2) = 2, f(3) = 3$ and $f(4) = 4$. Consequently, we infer from (3.4) that

$$(3.11) \quad \{1, 2, 3, 4, 5, 11, 19, 20, 29, 31, 41, 44, 45, 59, 61, 79\} \subseteq I.$$

It is clear to check that

$$\begin{cases}
8405 &= 76^2 + 3 \cdot 76 \cdot 11 + 11^2 = 41^2 + 3 \cdot 41 \cdot 41 + 41^2, \\
51005 &= 124^2 + 3 \cdot 124 \cdot 79 + 79^2 = 101^2 + 3 \cdot 101 \cdot 101 + 101^2,
\end{cases}$$

consequently

$$\begin{cases} E(8405) &= f(76)^2 + 3f(76)f(11) + f(11)^2 - 5f(41)^2 = 0, \\ E(51005) &= f(124)^2 + 3f(124)f(79) + f(79)^2 - 5f(101)^2 = 0. \end{cases}$$

But $11, 41, 79 \in I$ and

$$\begin{aligned} f(76) &= f(6)^2 + 3f(6)f(2) + f(2)^2 = f(6)^2 + 6f(6) + 4, \\ f(101) &= f(5)^2 + 3f(5)f(4) + f(4)^2 = 101, \\ f(124) &= f(6)^2 + 3f(6)f(4) + f(4)^2 = f(6)^2 + 12f(6) + 16, \end{aligned}$$

we have

$$\begin{cases} E(8405) &= (f(6)^2 + 6f(6) + 4)^2 + 3 \cdot 11 \cdot (f(6)^2 + 6f(6) + 4) + \\ &\quad + 11^2 - 5 \cdot 41^2 = \\ &= (f(6) - 6)(f(6) + 12)(f(6)^2 + 6f(6) + 113) = 0, \\ E(51005) &= (f(6)^2 + 12f(6) + 16)^2 + 3 \cdot 79 \cdot (f(6)^2 + 12f(6) + 16) + \\ &\quad + 79^2 - 5 \cdot 101^2 = \\ &= (f(6) - 6)(f(6) + 18)(f(6)^2 + 12f(6) + 377) = 0. \end{cases}$$

Let

$$a(x) := (x - 6)(x + 12)(x^2 + x + 113)$$

and

$$b(x) = (x - 6)(x + 18)(x^2 + 12x + 377).$$

Then

$$a(f(6)) = 0 \quad \text{and} \quad b(f(6)) = 0$$

and

$$\gcd(a(x), b(x)) = x - 6,$$

which imply that $f(6) = 6$. Thus, we have proved that $f(6) = 6$.

Now we prove $f(7) = 7$. Since

$$\begin{cases} 14801 &= 95^2 + 3 \cdot 95 \cdot 19 + 19^2 \quad \text{and} \quad 14801 = 80^2 + 3 \cdot 80 \cdot 31 + 31^2, \\ 31205 &= 121^2 + 3 \cdot 121 \cdot 41 + 41^2 \quad \text{and} \quad 31205 = 5 \cdot 79^2, \end{cases}$$

we have

$$\begin{cases} E(14801) &= f(95)^2 + 3f(95)f(19) + f(19)^2 - f(80)^2 - \\ &\quad - 3f(80)f(31) - f(31)^2 = 0, \\ E(31205) &= f(121)^2 + 3f(121)f(41) + f(41)^2 - 5f(79)^2 = 0. \end{cases}$$

One can check that

$$f(80) = 5f(4)^2 = 80,$$

$$f(95) = f(7)^2 + 3f(7)f(2) + f(2)^2 = f(7)^2 + 12f(7) + 4,$$

$$f(121) = f(7)^2 + 3f(7)f(3) + f(3)^2 = f(7)^2 + 9f(7) + 9.$$

Since $19 \in I$, $31 \in I$, $41 \in I$ and $79 \in I$, we obtain from above relations that

$$\left\{ \begin{array}{lcl} E(14801) & = & (f(7)^2 + 12f(7) + 4)^2 + 57(f(7)^2 + 12f(7) + 4) + \\ & & + 19^2 - 80^2 - 3 \cdot 80 \cdot 31 - 31^2 = \\ & = & (f(7) - 7)(f(7) + 13)(f(7)^2 + 6f(7) + 156) = 0, \\ E(31205) & = & (f(7)^2 + 9f(7) + 9)^2 + 123(f(7)^2 + 9f(7) + 9) + \\ & & + 41^2 - 5 \cdot 79^2 = \\ & = & (f(7) - 7)(f(7) + 16)(f(7)^2 + 9f(7) + 253) = 0. \end{array} \right.$$

Similar as above, if we put

$$c(x) := (x - 7)(x + 13)(x^2 + 6x + 156)$$

and

$$d(x) := (x - 7)(x + 16)(x^2 + 9x + 253),$$

then

$$c(f(7)) = b(f(7)) = 0 \quad \text{and} \quad \gcd(c(x), d(x)) = x - 7$$

imply that $f(7) = 7$.

Thus we have prove that $f(7) = 7$ and the proof of Lemma 5 is finished. ■

Lemma 6. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ satisfying (3.2). Then*

$$f(n) = n \quad \text{for every } n \in \{8, 15, 16, 22, 23, 28, 34, 35, 37\}.$$

Proof. By using Lemma 4 and Lemma 5, we have $f(n) = n$ for every $n \leq 7$, consequently

$$(3.12) \quad \{1, 2, 3, 4, 5, 6, 7, 11, 19, 20, 29, 31, 41, 44, 45, 59, 61, 79\} \subseteq I,$$

where

$$I = \{n \in \mathbb{N} \mid f(n) = n\}.$$

Since 1, 2, 5, 6, 7 $\in I$, we have

$$\begin{aligned}
 f(55) &= f(6)^2 + 3f(6)f(1) + f(1)^2 = 55, \\
 f(71) &= f(7)^2 + 3f(7)f(1) + f(1)^2 = 71, \\
 f(76) &= f(6)^2 + 3f(6)f(2) + f(2)^2 = 76, \\
 f(116) &= f(8)^2 + 3f(8)f(2) + f(2)^2 = f(8)^2 + 6f(8) + 4, \\
 f(121) &= f(7)^2 + 3f(7)f(3) + f(3)^2 = 121. \\
 f(125) &= 5f(5)^2 = 125.
 \end{aligned}$$

By using (3.12) and these relations, we shall prove

$$f(n) = n \quad \text{for every } n = 8, 15, 16, 22, 23, 28, 34, 35, 37.$$

In the proof, we need to find two elements $[N_1, x_1, y_1, u_1, v_1] \in E$ and $[N_2, x_2, y_2, u_2, v_2] \in E$ for which $E(N_1)$ and $E(N_1)$ can express as a polynomial of $f(e)$ for each $e \in \{8, 15, 16, 22, 23, 28, 34, 35, 37\}$. We find those elements with help of computer and Maple program.

◦ **Proof of $f(8) = 8$.**

Since $[589, 116, 1, 76, 31] \in E$ and $[4961, 59, 8, 55, 11] \in E$, we have

$$\left\{ \begin{aligned}
 E(589) &= f(116)^2 + 3f(116)f(1) + f(1)^2 - f(76)^2 - \\
 &\quad - 3f(76)f(31) - f(31)^2 = \\
 &= (f(8)^2 + 6f(8) + 4)^2 + 3 \cdot (f(8)^2 + 6f(8) + 4) + \\
 &\quad + 1 - 76^2 - 3 \cdot 76 \cdot 31 - 31^2 = \\
 &= (f(8) + 14)(f(8) - 8)(f(8)^2 + 6f(8) + 123), \\
 E(4961) &= f(59)^2 + 3f(59)f(8) + f(8)^2 - f(55)^2 - \\
 &\quad - 3f(55)f(11) - f(11)^2 = \\
 &= 59^2 + 3 \cdot 59 \cdot f(8) + f(8)^2 - 55^2 - 3 \cdot 55 \cdot 11 - 11^2 = \\
 &= (f(8) + 185)(f(8) - 8).
 \end{aligned} \right.$$

It is obvious that $E(589) \neq 0$ if $f(8) = -185$, consequently $f(8) = 8$.

◦ **Proof of $f(15) = 15$.**

Since $[589, 20, 3, 15, 7] \in E$ and $[2581, 45, 4, 31, 15] \in E$, we have

$$\left\{ \begin{array}{lcl} E(589) & = & f(20)^2 + 3f(20)f(3) + f(3)^2 - f(15)^2 - \\ & & - 3f(15)f(7) - f(7)^2 = \\ & = & 20^2 + 3 \cdot 20 \cdot 3 + 3^2 - f(15)^2 - 3 \cdot 7 \cdot f(15) - 7^2 = \\ & = & -(f(15) + 36)(f(15) - 15), \\ E(2581) & = & f(45)^2 + 3f(45)f(4) + f(4)^2 - f(31)^2 - \\ & & - 3f(31)f(15) - f(15)^2 = \\ & = & 45^2 + 3 \cdot 45 \cdot 4 + 4^2 - 31^2 - 3 \cdot 31f(15) - f(15)^2 = \\ & = & -(f(15) + 108)(f(15) - 15). \end{array} \right.$$

This implies that $f(15) = 15$.

◦ **Proof of $f(16) = 16$.**

Since $[3905, 61, 1, 41, 16] \in E$ and $[9680, 76, 16, 44, 44] \in E$, we have

$$\left\{ \begin{array}{lcl} E(3905) & = & f(61)^2 + 3f(61)f(1) + f(1)^2 - f(41)^2 - \\ & & - 3f(41)f(16) - f(16)^2 = \\ & = & 61^2 + 3 \cdot 61 + 1^2 - 41^2 - 3 \cdot 41f(16) - f(16)^2 = \\ & = & -(f(16) + 139)(f(16) - 16), \\ E(9680) & = & f(76)^2 + 3f(76)f(16) + f(16)^2 - f(44)^2 - \\ & & - 3f(44)f(44) - f(44)^2 = \\ & = & 76^2 + 3 \cdot 76f(16) + f(16)^2 - 5 \cdot 44^2 = \\ & = & (f(16) + 244)(f(16) - 16). \end{array} \right.$$

This implies that $f(16) = 16$.

◦ **Proof of $f(22) = 22$.**

Since $[2204, 44, 4, 22, 20] \in E$ and $[23111, 125, 19, 121, 22] \in E$, we have

$$\left\{ \begin{array}{lcl} E(2204) & = & f(44)^2 + 3f(44)f(2) + f(2)^2 - f(22)^2 - \\ & & - 3f(22)f(20) - f(20)^2 = \\ & = & 44^2 + 3 \cdot 44 \cdot 2 + 2^2 - 22^2 - 3 \cdot 22 \cdot 20 - 20^2 = \\ & = & -(f(22) + 82)(f(22) - 22), \\ E(23111) & = & f(125)^2 + 3f(125)f(19) + f(19)^2 - f(121)^2 - \\ & & - 3f(121)f(22) - f(22)^2 = \\ & = & 125^2 + 3 \cdot 125 \cdot 19 + 19^2 - 121^2 - 3 \cdot 121 \cdot f(22) - f(22)^2 = \\ & = & -(f(22) + 385)(f(22) - 22). \end{array} \right.$$

This implies that $f(22) = 22$.

◦ **Proof of $f(23) = 23$.**

Since $[671, 23, 2, 19, 5] \in E$ and $[10469, 76, 19, 71, 23] \in E$, we have

$$\left\{ \begin{array}{lcl} E(671) & = & f(23)^2 + 3f(23)f(2) + f(2)^2 - f(19)^2 - \\ & & - 3f(19)f(5) - f(5)^2 = \\ & = & f(23)^2 + 3 \cdot 2f(23) + 4 - 19^2 - 3 \cdot 19 \cdot 5 - 5^2 = \\ & = & (f(23) + 29)(f(23) - 23), \\ E(10469) & = & f(76)^2 + 3f(76)f(19) + f(19)^2 - f(71)^2 - \\ & & - 3f(71)f(23) - f(23)^2 = \\ & = & 76^2 + 3 \cdot 76 \cdot 19 + 19^2 - f(71)^2 - 3 \cdot 71f(23) - f(23)^2 = \\ & = & -(f(23) + 236)(f(23) - 23). \end{array} \right.$$

This implies that $f(23) = 23$.

◦ **Proof of $f(28) = 28$.**

Since $[869, 28, 1, 20, 7] \in E$ and $[5909, 71, 4, 41, 28] \in E$, we have

$$\left\{ \begin{array}{lcl} E(869) & = & f(28)^2 + 3f(28)f(1) + f(1)^2 - f(20)^2 - \\ & & - 3f(20)f(7) - f(7)^2 = \\ & = & f(28)^2 + 3f(28) + 1 - 20^2 - 3 \cdot 7 \cdot 20 - 7^2 = \\ & = & (f(28) + 31)(f(28) - 28), \\ E(5909) & = & f(71)^2 + 3f(71)f(4) + f(4)^2 - f(41)^2 - \\ & & - 3f(41)f(28) - f(28)^2 = \\ & = & 71^2 + 3 \cdot 71 \cdot 4 + 4^2 - 41^2 - 3 \cdot 41f(28) - f(28)^2 = \\ & = & -(f(28) + 151)(f(28) - 28). \end{array} \right.$$

This implies that $f(28) = 28$.

◦ **Proof of $f(34) = 34$.**

Since $[1919, 34, 7, 29, 11] \in E$ and $[15455, 79, 34, 71, 41] \in E$, we have

$$\left\{ \begin{array}{lcl} E(1919) & = & f(34)^2 + 3f(34)f(7) + f(7)^2 - f(29)^2 - \\ & & - 3f(29)f(11) - f(11)^2 = \\ & = & f(34)^2 + 3 \cdot 7f(34) + 7^2 - 29^2 - 3 \cdot 29 \cdot 11 - 11^2 = \\ & = & (f(34) + 55)(f(34) - 34), \\ E(15455) & = & f(79)^2 + 3f(79)f(34) + f(34)^2 - f(71)^2 - \\ & & - 3f(71)f(41) - f(41)^2 = \\ & = & 79^2 + 3 \cdot 79 \cdot f(34) + f(34)^2 - 71^2 - 371 \cdot 41 - 41^2 = \\ & = & (f(34) + 271)(f(34) - 34). \end{array} \right.$$

This implies that $f(34) = 34$.

◦ **Proof of $f(35) = 35$.**

Since $[1661, 35, 4, 31, 7] \in E$ and $[7781, 61, 20, 44, 35] \in E$, we have

$$\left\{ \begin{array}{lcl} E(1661) & = & f(35)^2 + 3f(35)f(4) + f(4)^2 - f(31)^2 - \\ & & - 3f(31)f(7) - f(7)^2 = \\ & = & f(35)^2 + 3 \cdot 4 \cdot f(35) + 4^2 - 31^2 - 3 \cdot 31 \cdot 7 - 7^2 = \\ & = & (f(35) + 47)(f(35) - 35), \\ E(7781) & = & f(61)^2 + 3f(61)f(20) + f(20)^2 - f(44)^2 - \\ & & - 3f(44)f(35) - f(35)^2 = \\ & = & 61^2 + 361 \cdot 20 + 20^2 - 44^2 - 3 \cdot 44 \cdot f(35) - f(35)^2 = \\ & = & -(f(35) + 167)(f(35) - 35). \end{array} \right.$$

This implies that $f(35) = 35$.

◦ **Proof of $f(37) = 37$.**

Since $[3839, 59, 2, 37, 19] \in E$ and $[16379, 125, 2, 79, 37] \in E$, we have

$$\left\{ \begin{array}{lcl} E(3839) & = & f(59)^2 + 3f(59)f(2) + f(2)^2 - f(37)^2 - \\ & & - 3f(37)f(19) - f(19)^2 = \\ & = & 59^2 + 3 \cdot 59 \cdot 2 + 2^2 - f(37)^2 - 3 \cdot 19 \cdot f(37) - 19^2 = \\ & = & -(f(37) + 94)(f(37) - 37), \\ E(16379) & = & f(125)^2 + 3f(125)f(2) + f(2)^2 - f(79)^2 - \\ & & - 3f(79)f(37) - f(37)^2 = \\ & = & 125^2 + 3 \cdot 125 \cdot 2 + 2^2 - 79^2 - 3 \cdot 79 \cdot f(37) - f(37)^2 = \\ & = & -(f(37) + 274)(f(37) - 37). \end{array} \right.$$

This implies that $f(37) = 37$.

Lemma 6 is proved. ■

Lemma 7. *Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ satisfying (3.2). Then*

$$f(n) = n \quad \text{for every } n \leq 44.$$

Proof. By using Lemma 4 and Lemma 6 we assume that

$$\{1, 2, 3, 4, 5, 6, 7, 8, 11, 15, 16, 19, 20, 22, 23, 28, 29, 31, 34, 35, 37, 41, 44\} \subseteq I.$$

Similar as in the proof of Lemma 6, we shall find those elements of the form $[N, x, y, u, v] \in E$, that is

$$N = x^2 + 3xy + y^2 = u^2 + 3uv + v^2$$

and

$$E(N) = f(x)^2 + 3f(x)f(y) + f(y)^2 - f(u)^2 - 3f(u)f(v) - f(v)^2 \in \mathbb{Z}[n]$$

is a polynomial of $f(n)$ with a suitable number $n \in \mathbb{N}$. We give a solutions in the following table.

Table 1

$[N, x, y, u, v]$	$E(N) \in \mathbb{Z}[n], E(N) = 0$	$f(n) = n$
[319, 15, 2, 9, 7] [34655, 139, 34, 109, 59]	$-(f(9) - 9)(f(9) + 30)$ $3(f(9) - 9)(2f(9)^2 + 4f(9) + 77)$	$f(9) = 9$
[551, 22, 1, 11, 10] [21605, 131, 11, 89, 44]	$-(f(10) - 10)(f(10) + 43)$ $(f(10) - 10)(f(10) + 13)(f(10)^2 + 3f(10) + 165)$	$f(10) = 10$
[209, 13, 1, 8, 5] [979, 23, 6, 15, 13]	$(f(13) + 16)(f(13) - 13)$ $-(f(13) + 58)(f(13) - 13)$	$f(13) = 13$
[1691, 34, 5, 23, 14] [6479, 61, 14, 37, 35]	$-(f(14) + 83)(f(14) - 14)$ $(f(14) + 197)(f(14) - 14)$	$f(14) = 14$

$[N, x, y, u, v]$	$E(N) \in \mathbb{Z}[n], E(N) = 0$	$f(n) = n$
[2501, 35, 11, 28, 17] [4061, 41, 17, 29, 28]	$-(f(17) + 101)(f(17) - 17)$ $(f(17) + 140)(f(17) - 17)$	$f(17) = 17$
[1804, 34, 6, 20, 18] [1711, 37, 3, 19, 18]	$-(f(18) + 78)(f(18) - 18)$ $-(f(18) + 75)(f(18) - 18)$	$f(18) = 18$
[4141, 44, 15, 37, 21] [11005, 99, 4, 76, 21]	$-(f(21) + 132)(f(21) - 21)$ $-(f(21) + 249)(f(21) - 21)$	$f(21) = 21$
[1111, 29, 3, 25, 6] [3509, 44, 11, 28, 25]	$-(f(25) + 43)(f(25) - 25)$ $-(f(25) + 109)(f(25) - 25)$	$f(25) = 25$
[2071, 37, 6, 26, 15] [1364, 34, 2, 26, 8]	$-(f(26) + 71)(f(26) - 26)$ $-(f(26) + 50)(f(26) - 26)$	$f(26) = 26$
[1529, 32, 5, 19, 16] [2201, 32, 11, 23, 19]	$(f(32) + 47)f(32) - 32$ $(f(32) + 65)(f(32) - 32)$	$f(32) = 32$
[3751, 41, 15, 33, 22] [2299, 45, 2, 33, 11]	$-(f(33) + 99)(f(33) - 33)$ $-(f(33) + 66)(f(33) - 33)$	$f(33) = 33$
[2489, 40, 7, 29, 16] [5921, 55, 16, 40, 29]	$(f(40) + 61)(f(40) - 40)$ $-(f(40) + 127)(f(40) - 40)$	$f(40) = 40$
[4661, 61, 5, 43, 19] [6809, 71, 8, 43, 31]	$-(f(43) + 100)(f(43) - 43)$ $-(f(43) + 136)(f(43) - 43)$	$f(43) = 43$

Thus we have proved that

$$(3.13) \quad \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, \\ 21, 22, 23, 25, 26, 28, 29, 31, 32, 33, 34, 35, 37, 40, 41, 43, 44\} \subseteq I$$

and we complete the proof of Lemma 7, it suffices to show

$$(3.14) \quad \{12, 24, 27, 30, 36, 38, 39, 42\} \subseteq I.$$

By using (3.13), we give the solutions of (3.14) in the following Table 2.

Table 2

$[N, x, y, u, v]$	$E(N) \in \mathbb{Z}[n], E(N) = 0$	$f(n) = n$
$[781, 21, 5, 13, 12]$	$-(f(12) + 51)(f(12) - 12)$	$f(12) = 12$
$[1045, 28, 3, 17, 12]$	$-(f(12) + 63)(f(12) - 12)$	
$[649, 24, 1, 15, 8]$	$(f(24) + 27)(f(24) - 24)$	$f(24) = 24$
$[2761, 40, 9, 24, 23]$	$-(f(24) + 93)(f(24) - 24)$	
$[1159, 27, 5, 22, 9]$	$(f(27) + 42)(f(27) - 27)$	$f(27) = 27$
$[1441, 27, 8, 19, 15]$	$(f(27) + 51)(f(27) - 27)$	
$[1276, 30, 4, 18, 14]$	$(f(30) + 42)(f(30) - 30)$	$f(30) = 30$
$[2356, 40, 6, 30, 14]$	$-(f(30) + 72)(f(30) - 30)$	
$[2101, 36, 7, 21, 20]$	$(f(36) + 57)(f(36) - 36)$	$f(36) = 36$
$[2869, 41, 9, 36, 13]$	$-(f(36) + 75)(f(36) - 36)$	
$[N, x, y, u, v]$	$E(N) \in \mathbb{Z}[n], E(N) = 0$	$f(n) = n$
$[2291, 38, 7, 26, 17]$	$(f(38) + 59)(f(38) - 38)$	$f(38) = 38$
$[2420, 38, 8, 22, 22]$	$(f(38) + 62)(f(38) - 38)$	
$[2929, 39, 11, 33, 16]$	$(f(39) + 72)(f(39) - 39)$	$f(39) = 39$
$[3355, 39, 14, 31, 21]$	$(f(39) + 81)(f(39) - 39)$	
$[1891, 42, 1, 35, 6]$	$(f(42) + 45)(f(42) - 42)$	$f(42) = 42$
$[2419, 42, 5, 23, 21]$	$(f(42) + 57)(f(42) - 42)$	

Thus, (3.13) and Table 2 prove Lemma 7.

Lemma 7 is proved. ■

4. Proof of Theorem 1 for $D = 3$

Assume that $f : \mathbb{N} \rightarrow \mathbb{C}$ with $f(1) = 1$ and

$$f(n^2 + 3nm + m^2) = f^2(n) + 3f(n)f(m) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Applying Lemma 3 with $D = 3$, we obtain that $f(n) = n$ for every $n \in \mathbb{N}$ if

$$f(k) = k \quad \text{for every } k \leq 8D + 20 = 8 \cdot 3 + 20 = 44.$$

This assertion is proved in Lemma 7. Theorem 1 is proved. ■

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