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MINIMIZING NUMERICAL ERROR IN COMPUTING INTEGRALS OVER POLYGONAL DOMAINS

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Abstract. Integration of multivariable polynomials over polygons and polyhedra is an intensively studied topic of vector calculus. Physical properties of n-dimensional solids as volume, centroid and moment of inertia can be defined by this type of integrals. In many physical applications, it is essential to be able to quickly calculate integrals of polynomials. In this paper, we show a technique to calculate integrals of continuous functions over two-dimensional polygonal domains. Particularly, our results contain the known formulas for monomials and polynomials, and we will show how we can generate formulas that can be used to calculate integrals of arbitrary continuous functions. It becomes clear that a significant numerical error can occur, that depends on the direction of the edges of the polygon. We will show that a rotation by an angle θ can solve this problem. Moreover we will give the optimal rotation angle θ, where the numerical error acquired from the orientation of the polygon is minimal.

1. Introduction

The integration of simple functions, such as polynomials, over polyhedra is investigated by a lot of papers, since integrals of polynomials are the key to define physical properties of objects. Let us suppose that S is a polyhedron

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defining a physical object and $\varrho(\mathbf{x})$ is the spatial distribution function of mass within the body ($n \in \mathbb{N}^+, \mathbf{x} \in \mathbb{R}^n, \varrho : \mathbb{R}^n \to \mathbb{R}$). Then several physical properties of the body are defined by volume integrals of different scalar fields over S, for example

- $\iiint\limits_S \mathrm{d}\mathbf{x}$ is the volume,
- $\iiint\limits_S \varrho(\mathbf{x})\ \mathrm{d}\mathbf{x}$ is the mass,
- $\iiint\limits_S \varrho(\mathbf{x})x_i\ \mathrm{d}\mathbf{x}$ is the i-th coordinate of the centroid, and so on [19].

In most cases it is assumed that the physical object is homogeneous, i.e. the mass distribution function $\varrho(\mathbf{x}) = c \in \mathbb{R}$ is constant. In this case each of the above functions to be integrated is a polynomial. On the other hand, integrating polynomials is useful to approximate integrals of continuous functions, since we know by the Stone–Weierstrass theorem [6], that the space of three-dimensional polynomials is dense in the space of continuous real-valued functions defined on compact intervals.

Cattani and Paoluzzi gave a formula for integrals of polynomials over linear polyhedra in 1990 [5], one year later their result was generalized to higher dimension by Bernardini [2]. Another solution of the problem can be found in works of Rathod [21] [22] [24] and a generalization to n-dimension is also available. Monomials are the building blocks of polynomials, therefore a possible direction to give formulas for polynomials is to study integrals of the monomials. A good example for this approach the work of de Loera et al. [9]. Integrals of the simplest monomials are the moments. Soerjadi studied the moments of polygons already in 1968 [27]. On the other hand, monomials are homogeneous functions, which can also be studied to give formulas for polynomials. Homogeneous functions over convex polytopes were considered by Lasserre in [13]. His article is based on the Euler's theorem of homogeneous functions. He shows how to obtain a simpler task recursively reducing the dimension of the problem. A common technique is iteratively applying the Gauss–Ostrogradsky divergence theorem [4] to reduce the dimension, most of the generalizations into higher dimensions are based on this idea. For example, as we know, it is easy to deduce the volume integrals to surface integrals. At this point, there are two main approaches depending on whether we are looking for the exact or the approximate solution of the problem.

Giving approximate solutions is another widely studied topic. After the dimension reduction Dasgupta use Gaussian quadrature on line integrals [8]. Mousavi and Sukumar [20] use Lassere's results to integrate polynomials and some discontinuous homogeneous functions by Gaussian quadratures. In [7]

Chin, Lassere and Sukumar give a numerical method to integrate homogeneous functions over arbitrary polygons and polyhedra. Determining the integrals over polygons is easier, if a spatial partition of the domain is known, for instance a triangulation or a quadrangulation. Li et al. showed how we can define quadrature points if the domain of integration is a quadrangulated polygon [16] [17]. Gauss-Legendre quadrature points can be easily obtained after triangulating the polygonal domain, as Hossain shows in [12]. Sommariva and Vianello [28] define explicit formulas for points and weights of Gaussian quadrature, but in their solution the triangulation of the polygon is not needed. Sudhakar et al. [29] constructed a Gaussian quadrature on arbitrary polyhedron without deducing the problem into two dimensions.

Since general domains of the two-dimensional space can be approximated by polygons, there are some approaches that bring this problem to the fore. Rice [25] studied integrals over arbitrary two-dimensional domains which were approximated by polygons, Thiagarajan [30] showed an adaptive method to find optimal weights and points for arbitrary meshless domains.

Some other techniques were developed to calculate integrals on polyhedra, for example Schechter's inequality-based [26] and Liu's tensorial [18] approach, but the authors investigate most often only monomials, polynomials, homogeneous functions or approximate solutions using quadrature formulas. We mention the article of Gabard [11], which describes some formulas for polynomial-exponential products. The question arises, why do not we study (exact) integrals of larger class of functions?

In this paper, we investigate the general case of integrating continuous functions over polygons and polyhedra. Arguing the importance of this question, we notice, that if the mass distribution function is neither constant, nor polynomial, we need more general formulas to calculate the integrals. In the first part of this paper we discuss how we can calculate the *exact value* of integrals of scalar functions, taking advantage of the fact, that the domain of integration is special. As it turns out, the derived integral formula contains a term which causes numerical inaccuracy. In the second part of the paper we will show, that this inaccuracy can be decreased using a rotation of the coordinate system. Moreover, we give a method to calculate the optimal rotation angle, where the inaccuracy is minimal.

2. Preliminaries

First we give some notations and definitions. Let S be an arbitrary set. Denote by $\mathrm{int}S, \partial S, \overline{S}$ the interior, the boundary and the closure of S, respectively. Similarly to the one-dimensional intervals, we will denote the segments from $\mathbf{v} \in \mathbb{R}^n$ to $\mathbf{w} \in \mathbb{R}^n$ by $[\mathbf{v}, \mathbf{w}]$ $(n \in \mathbb{N}^+)$, i.e.

$$[\mathbf{v}, \mathbf{w}] := \Big\{(1-t)\mathbf{v} + t\mathbf{w} \mid t \in [0,1]\Big\}.$$

The one-dimensional discrete interval from a to b is denoted by $[a..b]$, i.e. $[a..b] := [a, b] \cap \mathbb{N}$.

The vertices of a polygon can be stored in a matrix. Let $L \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_{L+1} \in \mathbb{R}^2$ be vertices of the plane $\mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{R}^2$, and

$$\mathbf{X} := \begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_{L+1}^T \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_{L+1} & y_{L+1} \end{bmatrix} \in \mathbb{R}^{(L+1)\times 2}.$$

As we can see, the i-th row of $\mathbf{X}$ represents the i-th vertex. Denote the polygonal line defined by $\mathbf{X}$ by $\mathcal{L}(\mathbf{X})$, i.e.

$$\mathcal{L}(\mathbf{X}) := \bigcup_{i=1}^{L} [\mathbf{x}_i, \mathbf{x}_{i+1}].$$

Definition 2.1. Let $L \in \mathbb{N}$, $L \geq 2$, and let $\mathbf{X} \in \mathbb{R}^{(L+1)\times 2}$ be a matrix containing the vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{L+1}$, for which $\mathbf{x}_1 = \mathbf{x}_{L+1}$ and

$$\forall i, j \in [1..L] \;:\; i \neq j \implies \mathbf{x}_i \neq \mathbf{x}_j.$$

Then we say $\mathcal{L}(\mathbf{X})$ is the edge loop defined by $\mathbf{X}$.

Definition 2.2. Let $\mathcal{L}(\mathbf{X})$ be an edge loop defined by $\mathbf{X}$. If $\mathcal{L}(\mathbf{X})$ is non-self-intersecting, i.e.

$$\forall i, j \in [1..L] \;:\; i \neq j \implies \operatorname{int}[\mathbf{x}_i, \mathbf{x}_{i+1}] \cap \operatorname{int}[\mathbf{x}_j, \mathbf{x}_{j+1}] = \emptyset,$$

then we say $\mathcal{L}(\mathbf{X})$ is simple.

It is easy to see that if $\mathcal{L}(\mathbf{X})$ is a simple edge loop, then it is a simple closed plane curve. The Jordan curve theorem implies that $\mathcal{L}(\mathbf{X})$ divides the plane into exactly two connected components. One of the components is bounded, the other is unbounded. This fact allows us to define the polygon properly.

Definition 2.3. Let $\mathcal{L}(\mathbf{X})$ be a simple edge loop defined by $\mathbf{X}$. Then there exists a bounded set $P \subset \mathbb{R}^2$, for which $\partial P = \mathcal{L}(\mathbf{X})$. We say $\mathcal{P}(\mathbf{X}) := \overline{P}$ is the (simple) polygon defined by $\mathbf{X}$.

Let us introduce the following differences:

$$\Delta x_i := x_{i+1} - x_i, \qquad \Delta y_i := y_{i+1} - y_i, \qquad \Delta \mathbf{x}_i := \mathbf{x}_{i+1} - \mathbf{x}_i.$$

Using our notations, we can determine a parametrization of the i-th edge of the polygon $\mathcal{P}(\mathbf{X})$:

$$\begin{aligned} \mathbf{e}_i(t) &= (1-t)\mathbf{x}_i + t\mathbf{x}_{i+1} = \mathbf{x}_i + t\Delta\mathbf{x}_i = \\ &= \begin{pmatrix} x_i \\ y_i \end{pmatrix} + t \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} \qquad (t \in [0,1], i \in [1..L]). \end{aligned}$$

It is easy to see that $\mathbf{e}_i$ is a bijection from $[0,1]$ to $[\mathbf{x}_i, \mathbf{x}_{i+1}]$.

The existence of orientation of $\mathcal{L}(\mathbf{X})$ is also the consequence of Jordan curve theorem. Namely by travelling along a simple plane curve, the bounded region enclosed by the curve always lies on either the right side or the left side. In the first case, we say the curve is clockwise (CW) oriented, in the second case we say it is counterclockwise (CCW) oriented. In our case it is enough to investigate a single edge and its neighbourhood to decide the orientation of $\mathcal{L}(\mathbf{X})$.

Corollary 2.1. *Let us define*

$$\boldsymbol{n}_i := \begin{pmatrix} y_{i+1} - y_i \\ -x_{i+1} + x_i \end{pmatrix},$$

a normal vector of the i-th edge, which is obtained by rotating the $\boldsymbol{x}_{i+1} - \boldsymbol{x}_i$ vector by 90 degrees clockwise. Then $\mathcal{L}(\boldsymbol{X})$ is CW oriented if and only if

$$\forall t \in (0,1) \quad \exists T \in \mathbb{R} \quad \forall s \in (0,T) \ : \ \boldsymbol{e}_i(t) + \boldsymbol{n}_i s \in \mathcal{P}(\boldsymbol{X}),$$

and CCW oriented if and only if

$$\forall t \in (0,1) \quad \exists T \in \mathbb{R} \quad \forall s \in (0,T) \ : \ \boldsymbol{e}_i(t) - \boldsymbol{n}_i s \in \mathcal{P}(\boldsymbol{X}).$$

Now we are ready to formulate our statements.

3. Integrals of continuous functions over polygons

In this section we give a successive integral formula for integrating a continuous function $f : \mathcal{P}(\mathbf{X}) \to \mathbb{R}$ over the $\mathcal{P}(\mathbf{X})$ polygonal domain.

Theorem 3.1. *Let $\boldsymbol{X} \in \mathbb{R}^{(L+1)\times 2}$, $\mathcal{L}(\boldsymbol{X})$ a CCW oriented simple edge loop, furthermore $f : \mathcal{P}(\boldsymbol{X}) \to \mathbb{R}$ a continuous function, and*

$$\forall i \in [1..L] \ : \ \Delta y_i \neq 0.$$

Then

$$\mathcal{I}_{\mathcal{P}(\boldsymbol{X})}(f) := \iint\limits_{\mathcal{P}(\boldsymbol{X})} f(x,y)\ \mathrm{d}x\mathrm{d}y = \sum_{i=1}^{L} \int\limits_{y_i}^{y_{i+1}} \int\limits_{0}^{\alpha_i\eta+\beta_i} f(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta,$$

where

$$\alpha_i = \frac{\Delta x_i}{\Delta y_i}, \qquad \beta_i = \frac{x_i\Delta y_i - y_i\Delta x_i}{\Delta y_i} = x_i - \alpha_i y_i.$$

Proof. Let us suppose that $\psi, \varphi : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions on $\mathcal{P}(\mathbf{X})$. Then Green's theorem states

$$\oint\limits_{\partial\mathcal{P}(\mathbf{X})} \psi\ \mathrm{d}x\ +\ \varphi\ \mathrm{d}y = \iint\limits_{\mathcal{P}(\mathbf{X})} \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial y}\ \mathrm{d}x\mathrm{d}y.$$

Consider a continuous function f, and let

$$\varphi(x,y) := \int\limits_{a}^{x} f(\xi,y)\ \mathrm{d}\xi, \qquad \psi(x,y) := 0.$$

Using Green's theorem, we get

$$\oint\limits_{\partial\mathcal{P}(\mathbf{X})} \varphi\ \mathrm{d}y = \iint\limits_{\mathcal{P}(\mathbf{X})} \frac{\partial\varphi}{\partial x}\ \mathrm{d}x\mathrm{d}y = \iint\limits_{\mathcal{P}(\mathbf{X})} f\ \mathrm{d}x\mathrm{d}y.$$

Let us determine a parametrization of a single edge of the edge loop enclosing the $\mathcal{P}(\mathbf{X})$ polygon. Consider $\mathbf{e}_i$, the parametrization of the i-th edge. Obviously

$$\mathbf{e}_i'(t) = \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} \implies \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} \mathrm{d}t.$$

Consequently

$$\int\limits_{\mathbf{x}_i}^{\mathbf{x}_{i+1}} \varphi(x,y)\ \mathrm{d}y = \int\limits_{0}^{1} (\varphi\circ\mathbf{e}_i)(t)\ \Delta y_i\ \mathrm{d}t =$$

$$= \int\limits_{0}^{1} \varphi(x_i + t\Delta x_i, y_i + t\Delta y_i)\ \Delta y_i \mathrm{d}t.$$

Using the substitutions

$$\eta := y_i + t\Delta y_i, \qquad t = (\eta - y_i)\tfrac{1}{\Delta y_i}, \qquad \mathrm{d}t = \tfrac{1}{\Delta y_i}\mathrm{d}\eta,$$

the integral above can be written in the following form:

$$\int_{y_i}^{y_{i+1}} \varphi\left(x_i + \frac{\eta - y_i}{\Delta y_i}\Delta x_i,\ \eta\right)\Delta y_i \frac{1}{\Delta y_i}\mathrm{d}\eta = \int_{y_i}^{y_{i+1}} \varphi(\alpha_i\eta + \beta_i)\mathrm{d}\eta,$$

where

$$\alpha_i = \frac{\Delta x_i}{\Delta y_i}, \qquad \beta_i = \frac{x_i\Delta y_i - y_i\Delta x_i}{\Delta y_i}.$$

Using the definition of φ and the fact that

$$\oint_{\partial\mathcal{P}(\mathbf{X})} \varphi(x,y)\ \mathrm{d}y = \oint_{\mathcal{L}(\mathbf{X})} \varphi(x,y)\ \mathrm{d}y = \sum_{i=1}^{L}\int_{\mathbf{x}_i}^{\mathbf{x}_{i+1}} \varphi(x,y)\ \mathrm{d}y,$$

we obtain the proof of the statement, i.e.

$$\iint_{\mathcal{P}(\mathbf{X})} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \sum_{i=1}^{L}\int_{y_i}^{y_{i+1}} \varphi(\alpha_i\eta + \beta_i)\ \mathrm{d}\eta =$$

$$= \sum_{i=1}^{L}\int_{y_i}^{y_{i+1}}\int_{0}^{\alpha_i\eta+\beta_i} f(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta.$$

■

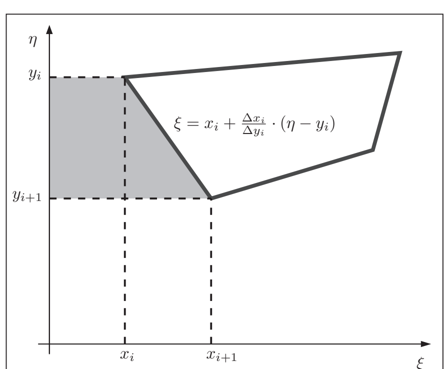

Figure 1. Integrating over two-dimensional polygons can be calculated by the sum defined in Theorem 3.1. The i-th term of the sum is determined by an integral over quadrilateral, which is defined by the lines $\eta = y_i$, $\eta = y_{i+1}$, $\xi = 0$ and $\xi = x_i + \frac{\Delta x_i}{\Delta y_i}(\eta - y_i)$.

Note that if $\Delta y_i = 0$, then

$$\int\limits_{\mathbf{x}_i}^{\mathbf{x}_{i+1}} \varphi(x,y)\ \mathrm{d}y = 0,$$

therefore the last theorem can be strengthened as follows.

Corollary 3.1. *Let $\boldsymbol{X} \in \mathbb{R}^{(L+1)\times 2}$, $\mathcal{L}(\boldsymbol{X})$ a CCW oriented simple edge loop, furthermore $f : \mathcal{P}(\boldsymbol{X}) \to \mathbb{R}$ a continuous function. Then*

$$\mathcal{I}_{\mathcal{P}(\boldsymbol{X})}(f) = \sum_{\substack{i=1\\ \Delta y_i \neq 0}}^{L} \int\limits_{y_i}^{y_{i+1}} \int\limits_{0}^{\alpha_i\eta+\beta_i} f(\xi,\eta)\ \mathrm{d}\xi \mathrm{d}\eta.$$

Although the formula in Corollary 3.1 can be evaluated for an arbitrary $\mathbf{X}$ that defines an $\mathcal{L}(\mathbf{X})$ simple edge loop, it may cause difficulties when it comes to numerical evaluation. In numerical applications the condition $\Delta y_i \neq 0$ is often replaced by $|\Delta y_i| \geq \varepsilon$ for some fixed $\varepsilon > 0$ parameter. In this case in the sum above we simply ignore the terms for which $|\Delta y_i| < \varepsilon$. As we will see later, this is not a robust solution. If for some index $k \in [0..L]$ $|\Delta y_k| = \varepsilon + \delta$ a significant numerical error may occur when $\delta \to 0+0$, since the k-th member of the sum can not be ignored, but the computation of α_k, β_k becomes inaccurate.

In the next sections we will assume that $\Delta y_i \neq 0\ \ (i \in [1..L])$, but keep in mind, that the results are true even when this last condition is violated, since the problematic terms can be omitted from the sum (see Corollary 3.1).

3.1. Integral of polynomials over polygons

Using Theorem 3.1 a successive integral formula can be constructed for exact evaluation of integrals of two-dimensional polynomials over polygonal domains. Similar results can be found in [27], [5], [21] and [11]. Although the formula to be derived is known, we give a novel proof based on Theorem 3.1.

Theorem 3.2. *Let $\boldsymbol{X} \in \mathbb{R}^{(L+1)\times 2}$, $\mathcal{L}(\boldsymbol{X})$ a CCW oriented simple edge loop, and for all $i \in [1..L]\ :\ \Delta y_i \neq 0$. Moreover let $a_{jk} \in \mathbb{R}$ be arbitrary real numbers $(j \in [0..J],\ k \in [0..K],\ J, K \in \mathbb{N})$ and*

$$f(x,y) := \sum_{j=0}^{J}\sum_{k=0}^{K} a_{jk} x^j y^k.$$

Then

$$\mathcal{I}_{\mathcal{P}(\boldsymbol{X})}(f) = \sum_{i=1}^{L}\sum_{j=0}^{J} \frac{1}{j+1} \sum_{k=0}^{K} a_{jk} \sum_{l=0}^{j+1} \frac{\alpha_i^l \beta_i^{j+1-l}}{k+l+1} \binom{j+1}{l} \left(y_{i+1}^{k+l+1} - y_i^{k+l+1}\right).$$

Proof. It is clear that f is continuous, therefore Theorem 3.1 can be used for computing the integral:

$$\mathcal{I}_{\mathcal{P}(\mathbf{x})}(f) = \sum_{i=1}^{L} \int_{y_i}^{y_{i+1}} \int_{0}^{\alpha_i\eta+\beta_i} f(\xi,\eta)\ \mathrm{d}\xi \mathrm{d}\eta =$$

$$= \sum_{i=1}^{L} \int_{y_i}^{y_{i+1}} \int_{0}^{\alpha_i\eta+\beta_i} \sum_{j=0}^{J} \sum_{k=0}^{K} a_{jk}\xi^j\eta^k\ \mathrm{d}\xi \mathrm{d}\eta =$$

$$= \sum_{i=1}^{L} \sum_{j=0}^{J} \sum_{k=0}^{K} a_{jk} \int_{y_i}^{y_{i+1}} \eta^k \left[\frac{\xi^{j+1}}{j+1}\right]_0^{\alpha_i\eta+\beta_i} \mathrm{d}\eta =$$

$$= \sum_{i=1}^{L} \sum_{j=0}^{J} \sum_{k=0}^{K} \frac{a_{jk}}{j+1} \int_{y_i}^{y_{i+1}} \eta^k(\alpha_i\eta+\beta_i)^{j+1}\ \mathrm{d}\eta.$$

Using the binomial theorem we have:

$$(\alpha_i\eta+\beta_i)^{j+1} = \sum_{l=0}^{j+1} \binom{j+1}{l} \alpha_i^l \eta^l \beta_i^{j+1-l}.$$

Therefore

$$\mathcal{I}_{\mathcal{P}(\mathbf{x})}(f) = \sum_{i=1}^{L} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{l=0}^{j+1} \frac{a_{jk}}{j+1} \binom{j+1}{l} \int_{y_i}^{y_{i+1}} \alpha_i^l \eta^{k+l} \beta_i^{j+1-l}\ \mathrm{d}\eta =$$

$$= \sum_{i=1}^{L} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{l=0}^{j+1} \frac{a_{jk}}{j+1} \binom{j+1}{l} \alpha_i^l \beta_i^{j+1-l} \left[\frac{\eta^{k+l+1}}{k+l+1}\right]_{y_i}^{y_{i+1}} =$$

$$= \sum_{i=1}^{L} \sum_{j=0}^{J} \frac{1}{j+1} \sum_{k=0}^{K} a_{jk} \sum_{l=0}^{j+1} \frac{\alpha_i^l \beta_i^{j+1-l}}{k+l+1} \binom{j+1}{l} \left(y_{i+1}^{k+l+1} - y_i^{k+l+1}\right).$$

■

3.2. Integral of a non-polynomial function

We also give a non-polynomial example. Let $f(x,y)=\cos(x)e^y$. Applying Theorem 3.1 we have

$$\begin{aligned}
\mathcal{I}_{\mathcal{P}(\mathbf{x})}(f) &= \sum_{i=1}^{L}\int\limits_{y_i}^{y_{i+1}}\int\limits_{0}^{\alpha_i\eta+\beta_i} f(\xi,\eta,0)\,\mathrm{d}\xi\mathrm{d}\eta = \\
&= \sum_{i=1}^{L}\int\limits_{y_i}^{y_{i+1}}\int\limits_{0}^{\alpha_i\eta+\beta_i} \cos(\xi)e^{\eta}\,\mathrm{d}\xi\mathrm{d}\eta = \\
&= \sum_{i=1}^{L}\int\limits_{y_i}^{y_{i+1}}\Big[e^{\eta}\sin(\xi)\Big]_0^{\alpha_i\eta+\beta_i}\mathrm{d}\eta = \sum_{i=1}^{L}\int\limits_{y_i}^{y_{i+1}} e^{\eta}\sin(\alpha_i\eta+\beta_i)\mathrm{d}\eta = \\
&= \sum_{i=1}^{L}\left[\frac{e^{\eta}(\sin(\alpha_i\eta+\beta_i)-\alpha_i\cos(\alpha_i\eta+\beta_i))}{\alpha_i^2+1}\right]_{y_i}^{y_{i+1}} = \\
&= \sum_{i=1}^{L}\frac{1}{\alpha_i^2+1}\Bigg[e^{y_{i+1}}\Big(\sin(\alpha_i y_{i+1}+\beta_i)-\alpha_i\cos(\alpha_i y_{i+1}+\beta_i)\Big)- \\
&\qquad\qquad - e^{y_i}\Big(\sin(\alpha_i y_i+\beta_i)-\alpha_i\cos(\alpha_i y_i+\beta_i)\Big)\Bigg].
\end{aligned}$$

As the last example shows, if we are able to determine the primitive functions of the integrands, we can give a finite sum using only the coordinates (x_i, y_i) of the vertices of the polygonal domain for calculating the exact value of integral of a non-polynomial function.

4. Rotation of the polygonal domain

As we have seen in the preceding sections, the condition $\Delta y_i \neq 0$ should be satisfied if we want to use our formulas unchanged. As the polygon $\mathcal{P}(\mathbf{X})$ is defined by finitely many vertices, there exists an angle θ, so that rotating the points by θ the condition can be satisfied. Consider the matrix of CCW rotation by an angle θ:

$$\mathbf{R}(\theta) := \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}.$$

Consider a vertex $\mathbf{x}$. Denote by $\widetilde{\mathbf{x}}$ the transformed* vertex:

$$\widetilde{\mathbf{x}} := \mathbf{R}(-\theta)\mathbf{x},$$

*We define the transformation as a CW rotation, this choice makes our formulas simpler.

i.e.

$$\begin{pmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Notice that $\mathbf{R}(-\theta)$ is an orthogonal matrix satisfying the following identities:

$$\mathbf{R}^{-1}(-\theta) = \mathbf{R}^T(-\theta) = \mathbf{R}(\theta).$$

The transformation of all the vertices of an $\mathbf{X} \in \mathbb{R}^{(L+1)\times 2}$ can be determined as follows:

$$\widetilde{\mathbf{x}}^T = \Big(\mathbf{R}(-\theta)\mathbf{x}\Big)^T = \mathbf{x}^T\mathbf{R}^T(-\theta) = \mathbf{x}^T\mathbf{R}(\theta),$$

so

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \widetilde{\mathbf{x}}_1 \\ \vdots \\ \widetilde{\mathbf{x}}_{L+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^T\mathbf{R}(\theta) \\ \vdots \\ \mathbf{x}_{L+1}^T\mathbf{R}(\theta) \end{bmatrix} = \mathbf{X}\mathbf{R}(\theta).$$

Then the following theorem holds.

Theorem 4.1. *Using the preceding conditions and notations:*

$$\mathcal{I}_{\mathcal{P}(\boldsymbol{X})}(f) = \iint\limits_{\mathcal{P}(\boldsymbol{X})} f(\boldsymbol{x}) \;\; \mathrm{d}\boldsymbol{x} = \iint\limits_{\mathcal{P}(\widetilde{\boldsymbol{X}})} f(\boldsymbol{R}(\theta)\boldsymbol{x}) \; \mathrm{d}\boldsymbol{x}.$$

Proof. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be the following function:

$$\sigma(\mathbf{x}) := \mathbf{R}(\theta)\mathbf{x}.$$

σ is injective, and obviously

$$\sigma^{-1}(\mathbf{x}) = \mathbf{R}(-\theta)\mathbf{x} = \widetilde{\mathbf{x}},$$

i.e.

$$\sigma^{-1}[\mathcal{P}(\mathbf{X})] = \mathcal{P}(\mathbf{X}\mathbf{R}(\theta)) = \mathcal{P}(\widetilde{\mathbf{X}}).$$

Moreover σ is differentiable, and

$$\sigma'(\mathbf{x}) = \mathbf{R}(\theta) \implies |\det \sigma'(\mathbf{x})| = 1.$$

Consequently, we can use the theorem of integration by substitution:

$$\begin{aligned}
\mathcal{I}_{\mathcal{P}(\mathbf{X})}(f) &= \iint\limits_{\mathcal{P}(\mathbf{X})} f(\mathbf{x}) \;\; \mathrm{d}\mathbf{x} = \\
&= \iint\limits_{\sigma^{-1}[\mathcal{P}(\mathbf{X})]} f(\sigma(\mathbf{x}))\big|\det \sigma'(\mathbf{x})\big| \;\mathrm{d}\mathbf{x} = \\
&= \iint\limits_{\mathcal{P}(\widetilde{\mathbf{X}})} f\Big(\mathbf{R}(\theta)\mathbf{x}\Big)\cdot 1 \;\; \mathrm{d}\mathbf{x}.
\end{aligned}$$

■

It was previously shown, that if $f(x,y)$ is a two-dimensional polynomial, i.e.

$$f(x,y) := \sum_{j=0}^{J}\sum_{k=0}^{K} a_{jk}x^j y^k,$$

then $\mathcal{I}_{\mathcal{P}(\mathbf{X})}(f)$ can be computed by a closed formula (see Theorem 3.2). If the coordinate system is rotated by an angle θ, then the resulting function $f\circ\mathbf{R}(\theta)$ is a polynomial, as well. The degree and the coefficients of the transformed polynomial may be changed by the rotation, our next task is to compute the changed coefficients. As we observed above

$$\widetilde{\mathbf{x}} = \mathbf{R}(-\theta)\mathbf{x} \quad\Longleftrightarrow\quad \mathbf{x} = \mathbf{R}(\theta)\widetilde{\mathbf{x}},$$

i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix}.$$

Now we are looking for those degrees $P, Q \in \mathbb{N}$ and coefficents $\widetilde{a}_{pq}$, for which

$$\sum_{j=0}^{J}\sum_{k=0}^{K} a_{jk}x^j y^k = f(\mathbf{x}) = f\big(\mathbf{R}(\theta)\widetilde{\mathbf{x}}\big) = \sum_{p=0}^{P}\sum_{q=0}^{Q} \widetilde{a}_{pq}\widetilde{x}^p\widetilde{y}^q.$$

For a monomial we have:

$$x^j y^k = \Big(\cos(\theta)\widetilde{x} - \sin(\theta)\widetilde{y}\Big)^j \Big(\sin(\theta)\widetilde{x} + \cos(\theta)\widetilde{y}\Big)^k.$$

Using the binomial theorem, we get

$$\Big(\cos(\theta)\widetilde{x} - \sin(\theta)\widetilde{y}\Big)^j = \sum_{j'=0}^{j}\binom{j}{j'}\Big(\cos(\theta)\widetilde{x}\Big)^{j'}\Big(-\sin(\theta)\widetilde{y}\Big)^{j-j'}$$

and

$$\Big(\sin(\theta)\widetilde{x} + \cos(\theta)\widetilde{y}\Big)^k = \sum_{k'=0}^{k}\binom{k}{k'}\Big(\sin(\theta)\widetilde{x}\Big)^{k'}\Big(\cos(\theta)\widetilde{y}\Big)^{k-k'}.$$

Let us define the following quantity:

$$C_{j,k,j',k'} := \binom{j}{j'}\binom{k}{k'}(-1)^{j-j'}\cos(\theta)^{k-k'+j'}(\sin(\theta))^{j-j'+k'}. \tag{4.1}$$

Thus we obtain the following formula for an $x^j y^k$ monomial:

$$x^j y^k = \sum_{j'=0}^{j}\sum_{k'=0}^{k} C_{j,k,j',k'}\widetilde{x}^{j'+k'}\widetilde{y}^{j+k-(j'+k')}.$$

Finally, the $f(x,y)$ polynomial can be written in the following form:

$$\sum_{j=0}^{J}\sum_{k=0}^{K} a_{jk}x^j y^k = \sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{j'=0}^{j}\sum_{k'=0}^{k} a_{jk}C_{j,k,j',k'}\widetilde{x}^{j'+k'}\widetilde{y}^{j+k-(j'+k')},$$

i.e.

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{j'=0}^{j}\sum_{k'=0}^{k} a_{jk}C_{j,k,j',k'}\widetilde{x}^{j'+k'}\widetilde{y}^{j+k-(j'+k')} = \sum_{p=0}^{P}\sum_{q=0}^{Q}\widetilde{a}_{pq}\widetilde{x}^p\widetilde{y}^q.$$

Because of the linear independence of the $\widetilde{x}^p\widetilde{y}^q$ monomials, their coefficients are necessarily the same on the two sides of the equation above, thus

$$\widetilde{a}_{pq} = \sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{j'=0}^{j}\sum_{k'=0}^{k} \delta_{j'+k',p}\delta_{j+k,p+q}a_{jk}C_{j,k,j',k'}, \tag{4.2}$$

where δ denotes the Kronecker-symbol. Now let us determine the P,Q values. Since

$$\left\{\begin{array}{c} 0\le j\le J \\ 0\le k\le K \\ 0\le j'\le j \\ 0\le k'\le k \end{array}\right\} \implies 0\le j'+k'\le j+k\le J+K,$$

and $p=j'+k'$

$$0\le p\le j+k\le J+K =: P.$$

On the other hand

$$j+k=p+q \iff q=j+k-p \implies 0\le q\le J+K-p =: Q.$$

We can summarize our results in the following theorem.

Theorem 4.2. *Let $f(x,y)=\sum\limits_{j=0}^{J}\sum\limits_{k=0}^{K} a_{jk}x^jy^k$, and $\theta\in\mathbb{R}$ arbitrary angle. Then*

$$f(x,y) = \sum_{p=0}^{J+K}\sum_{q=0}^{J+K-p}\widetilde{a}_{pq}\widetilde{x}^p\widetilde{y}^q,$$

where $\widetilde{a}_{pq}$ coefficients can be computed by formulas (4.2) and (4.1).

5. Occurrence of the numerical error

Let us consider an example. Let f be the following polynomial:

$$f(x,y) := 1 + y + y^2 + 2x + 2xy + 2xy^2 - x^2 - 3x^2y^2$$

and

$$X := \begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T \in \mathbb{R}^{5\times 2}.$$

In this case the $\mathcal{P}(\mathbf{X})$ polynomial is the $[0,2]\times[0,1]$ rectangle. It can easily

Figure 2. The example integral to be calculated. The values of the $f(x,y)$ polynomial imaged by a heatmap.

be checked that

$$\mathcal{I}_{\mathcal{P}(\mathbf{X})}(f) = \frac{17}{3}.$$

The integral can be computed also using Theorem 3.2 and Corollary 3.1, i.e.

$$\mathcal{I}_{\mathcal{P}(\mathbf{X})}(f) \approx \sum_{\substack{i=1\\ |\Delta y_i|\geq\varepsilon}}^{L} \sum_{j=0}^{J} \frac{1}{j+1} \sum_{k=0}^{K} a_{jk} \sum_{l=0}^{j+1} \frac{\alpha_i^l \beta_i^{j+1-l}}{k+l+1} \binom{j+1}{l} \left(y_{i+1}^{k+l+1} - y_i^{k+l+1}\right).$$

That means, if an edge is approximately parallel to the x-axis, it will be ignored. But it is worth to examine, what happens, if $|\Delta y_i| \geq \varepsilon$, but value of $|\Delta y_i|$ is very close to ε. Let us define a rotation angle θ, and try to compute the integral in the rotated coordinate system. It can be done using the results of Theorem 4.2 and Lemma 4.1. Obviously $\min\limits_{i\in[1..L]} |\Delta y_i|$ changes with θ, and as the minimum approaches ε from above, the numerical error of the integral formula greatly increases. This error occurs because the numerical unstable terms of the sum

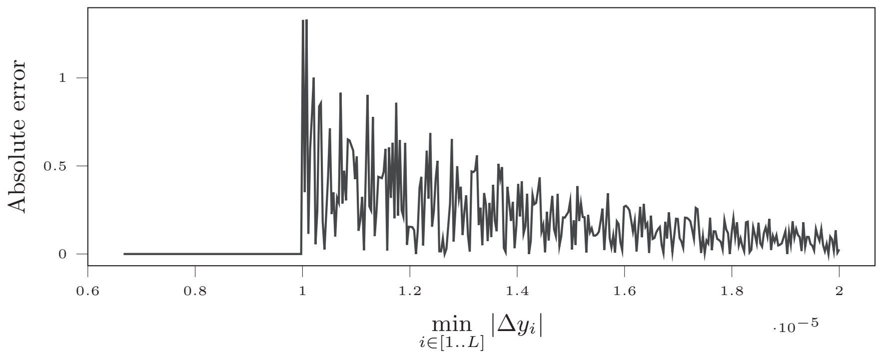

Figure 3. In our experiment $\varepsilon = 10^{-5}$ was chosen. On the x-axis one can see $\min_{i\in[1..L]} |\Delta y_i|$, on the y-axis we measured the absolute error of the integral formula compared to the exact value of the integral. If $\min_{i\in[1..L]} |\Delta y_i| \approx \varepsilon = 10^{-5}$ then the error function has a large jump.

are not ignored, since $\min_{i\in[1..L]} |\Delta y_i| \geq 0$. It is clear that the problem can not be solved by redefining ε, this phenomenon more or less always impairs the numerical accuracy of the integral formula.

The question arises how we can stabilize our integral formula? We found that, the larger the $\min_{i\in[1..L]} |\Delta y_i|$ quantity, the smaller the numerical error. In the next section we show, how can one compute an "optimal rotation angle", for which $\min_{i\in[1..L]} |\Delta \widetilde{y}_i|$ as large as possible.

6. Determining the optimal rotation angle

In this section we show, how to find the θ optimal rotation angle, where the smallest $|\Delta \widetilde{y}_i|$ value is maximal, that is $\theta^* \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ value†, for which

$$\theta^* = \arg \max_{\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]} \min_{i\in[1..L]} |\Delta \widetilde{y}_i|.$$

Let us deal with the quantity to be minimized:

$$\begin{pmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_i \cos(\theta) - y_i \sin(\theta) \\ x_i \sin(\theta) + y_i \cos(\theta) \end{pmatrix}.$$

†It is not necessary to examine the entire $[-\pi, \pi]$ interval, since rotating by π angle, all the $\Delta \widetilde{y}_i$ values are changing to $-\Delta \widetilde{y}_i$, but this transformation has no effect to the value of the objective function.

Therefore

$$\widetilde{y}_i = x_i \sin(\theta) + y_i \cos(\theta),$$

moreover

$$|\Delta\widetilde{y}_i| = |\widetilde{y}_{i+1} - \widetilde{y}_i| = \Big|x_{i+1}\sin(\theta) + y_{i+1}\cos(\theta) - x_i\sin(\theta) - y_i\cos(\theta)\Big| =$$
$$= \Big|(x_{i+1} - x_i)\sin(\theta) + (y_{i+1} - y_i)\cos(\theta)\Big| = \Big|\Delta x_i \sin(\theta) + \Delta y_i \cos(\theta)\Big|.$$

It can be seen, that $|\Delta\widetilde{y}_i|$ is a function of the θ rotation angle, therefore we introduce the following notations:

$$e_i(\theta) := |\Delta\widetilde{y}_i| = |\Delta x_i \sin(\theta) + \Delta y_i \cos(\theta)|, \qquad E(\theta) := \min_{i \in [1..L]} e_i(\theta).$$

It is clear that θ^* is the absolute maximum point of E on the $[-\frac{\pi}{2}, \frac{\pi}{2}]$ interval. On the other hand, E is a lower envelope of the e_i functions. Consider two functions: $e_i, e_j \ \ (i \neq j)$. If $\Delta x_i = \Delta x_j$ and $\Delta y_i = \Delta y_j$, then $e_i - e_j \equiv 0$, therefore one of the functions can be ignored by computing the lower envelope. Because of this, we will assume, that $\Delta x_i \neq \Delta x_j$ or $\Delta y_i \neq \Delta y_j$. As a consequence of the continuity of trigonometric functions and the absolute value function, $e_i - e_j$ is continuous. Bolzano's theorem implies that if function $e_i - e_j$ changes sign on an interval $[a, b]$, then there exists $\theta \in [a, b]$ where $e_i(\theta) - e_j(\theta) = 0$. Let us find the solutions of the $e_i(\theta) - e_j(\theta) = 0$ equation:

$$\begin{aligned}
|\Delta x_i \sin(\theta) + \Delta y_i \cos(\theta)| &= |\Delta x_j \sin(\theta) + \Delta y_j \cos(\theta)| \\
\Delta x_i \sin(\theta) + \Delta y_i \cos(\theta) &= \pm\left(\Delta x_j \sin(\theta) + \Delta y_j \cos(\theta)\right) \\
(\Delta x_i \mp \Delta x_j)\sin(\theta) &= (-\Delta y_i \pm \Delta y_j)\cos(\theta) \\
(\Delta x_i \mp \Delta x_j)\sin(\theta) &= -\left(\Delta y_i \mp \Delta y_j\right)\cos(\theta).
\end{aligned}$$

Consequently, if $\Delta x_i \mp \Delta x_j \neq 0$, then

$$\tan(\theta) = -\frac{\Delta y_i \mp \Delta y_j}{\Delta x_i \mp \Delta x_j} = -\frac{\Delta y_j \pm \Delta y_i}{\Delta x_j \pm \Delta x_i} \implies \theta = -\arctan\left(\frac{\Delta y_j \pm \Delta y_i}{\Delta x_j \pm \Delta x_i}\right).$$

On the other hand, if $\Delta x_i \mp \Delta x_j = 0$, then necessarily $\cos(\theta) = 0$, i.e. $\theta = \pm\frac{\pi}{2}$, since $\Delta y_i \pm \Delta y_j = 0$ can not be satisfied at the same time. Remember, the solutions $\pm\frac{\pi}{2}$ are equivalent in the sense that $E(\frac{\pi}{2}) = E(-\frac{\pi}{2})$, therefore we can use the convention, that $\Delta x_i \mp \Delta x_j = 0$ implies $\theta = \mp\frac{\pi}{2}$. It can be seen, the $e_i(\theta) - e_j(\theta) = 0$ equation always has exactly 2 solutions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$$\theta_{i,j}^{(1)} := \begin{cases} -\arctan\left(\frac{\Delta y_j + \Delta y_i}{\Delta x_j + \Delta x_i}\right) & \Delta x_j + \Delta x_i \neq 0 \\ \frac{\pi}{2} & \Delta x_j + \Delta x_i = 0, \end{cases}$$

$$\theta_{i,j}^{(2)} := \begin{cases} -\arctan\left(\frac{\Delta y_j - \Delta y_i}{\Delta x_j - \Delta x_i}\right) & \Delta x_j - \Delta x_i \neq 0 \\ -\frac{\pi}{2} & \Delta x_j - \Delta x_i = 0. \end{cases}$$

Consider the following set:

$$\Theta := \left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\} \cup \bigcup_{i=2}^{L} \bigcup_{j=1}^{i-1} \left\{ \theta_{i,j}^{(1)}, \theta_{i,j}^{(2)} \right\}.$$

Let $\theta_1, \ldots, \theta_M$ be the values of Θ sorted in ascending order:

$$M = |\Theta|, \qquad -\frac{\pi}{2} = \theta_1 < \theta_2 < \cdots < \theta_M = \frac{\pi}{2}, \qquad \bigcup_{i=1}^{M} \theta_i = \Theta.$$

Then the θ_m values divide the $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ interval into subintervals, and the restriction of E to a subinterval equals with one of the e_j functions:

$$\forall m \in [1..M-1] \quad \exists j \in [1..L] \quad : \quad E|_{(\theta_m, \theta_{m+1})} = e_j|_{(\theta_m, \theta_{m+1})}.$$

This is a consequence of the fact, that the $e_i - e_j$ functions have no roots on the (θ_m, θ_{m+1}) open intervals. Since E definition may change from one (θ_m, θ_{m+1}) to the other, each θ_m is a possible absolute maximum point. It can be noticed, that E can attain its maximum also inside the (θ_m, θ_{m+1}) intervals. Namely, if $E|_{(\theta_m, \theta_{m+1})} = e_j|_{(\theta_m, \theta_{m+1})}$ and e_j has a local extremum point inside $\theta \in (\theta_m, \theta_{m+1})$, then θ is also a possible absolute maximum point. It is easy to see, that if $e_j(\theta) = 0$, then θ is a local minimum of $|e_j|$, and if θ is a local extremum of e_j, it is a local maximum of $|e_j|$. Since e_j is differentiable function, we can find its possible extremum points by solving the $e_j'(\theta) = 0$ equation:

$$\begin{aligned} \Big(\Delta x_j \sin(\theta) + \Delta y_j \cos(\theta)\Big)' &= 0 \\ \Delta x_j \cos(\theta) - \Delta y_j \sin(\theta) &= 0 \\ \Delta x_j \cos(\theta) &= \Delta y_j \sin(\theta). \end{aligned}$$

If $\Delta y_j \neq 0$, then

$$\tan(\theta) = \frac{\Delta x_j}{\Delta y_j} \implies \theta = \arctan\left(\frac{\Delta x_j}{\Delta y_j}\right),$$

If $\Delta y_j = 0$, then

$$\cos(\theta) = 0 \implies \theta = \pm\frac{\pi}{2},$$

since $\Delta y_j = 0$ and $\Delta x_j = 0$ can not be satisfied at the same time. An e_j function has a single point that can be its extremum point on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

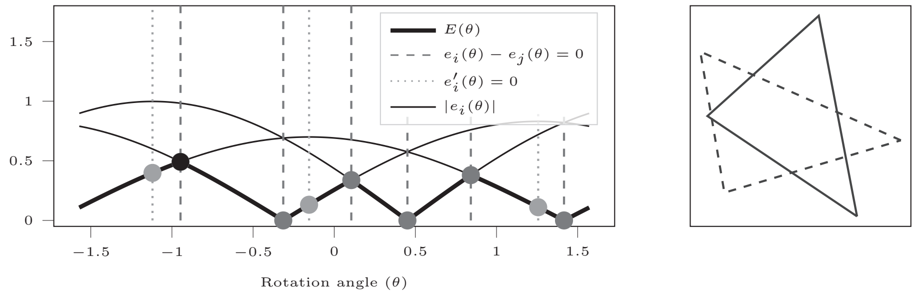

Figure 4. On the right side there is a triangle (dashed line), and the same triangle rotated by the optimal θ angle (solid line). On the left side we can see the $E(\theta)$ function (thick line), the $|e_i(\theta)|$ functions (solid lines) and the possible absolute maximum points of E: the local extrema of the e_i functions (dotted lines) and the intersection points of some e_i, e_j functions (dashed lines).

this is the following:

$$\theta'_j := \begin{cases} -\arctan\left(\frac{\Delta x_i}{\Delta y_i}\right) & \Delta y_i \neq 0 \\ \frac{\pi}{2} & \Delta y_i = 0. \end{cases}$$

Let us define the following set:

$$\Theta' := \bigcup_{j=1}^{L} \left\{\theta'_j\right\}.$$

It is obvious, that the absolute maximum point of E is one of the followings:

- It is an element of Θ, i.e. an intersection point of some e_i, e_j functions, or
- it is an element of Θ', i.e. an extremum point of some e_j function, where the extremum is inside the interval, where $e_j \equiv E$.

In summary, we showed, that the absolute maximum point of E can be determined by finite evaluation of the objective function. This statement can be formulated as follows.

Theorem 6.1. *Using the notations introduced in this subsection:*

$$\theta^* = \arg \max_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} E(\theta) = \arg \max_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \min_{i \in [1..L]} e_i(\theta) =$$
$$= \arg \max_{\theta \in \Theta \cup \Theta'} \min_{i \in [1..L]} e_i(\theta).$$

As the e_i, e_j function can intersect at most in $\frac{L(L+1)}{2}$ point (including $\pm\frac{\pi}{2}$), we can estimate the cardinality of $|\Theta|$:

$$|\Theta| \leq \frac{L(L+1)}{2} + 2 = \frac{L^2 + L + 4}{2}.$$

On the other hand, we can also estimate the number of possible local extremum points:

$$|\Theta'| \leq L.$$

Consequently

$$|\Theta \cup \Theta'| \leq \frac{L^2 + L + 4}{2} + L = \frac{L^2 + 3L + 4}{2} = O(L^2).$$

By the most simple (naive) approach, we have to evaluate all e_i functions in each points of $\Theta \cup \Theta'$, therefore θ^* can be determined in $O(L^3)$ time.

7. Summary

In this paper we dealt with integrals of continuous functions over two-dimensional polygonal domains. In Theorem 3.1 we gave a formula for computing integrals of this type. This successive integral formula is a generalization of some known results. In Theorem 3.2 it was shown, that the known formula for integrating polynomials over polygonal domains is a particular case of Theorem 3.1. In Lemma 4.1 and Theorem 4.2 we examined, how a polynomial integrand transforms, if we rotate the coordinate system. Thereafter, we showed that the numerical computation of our integral formula can be problematic, if the domain of integration contains edges that are approximately parallel to the x-axis. Finally, we showed that there exists the optimal rotation angle θ^*, for which this type of numerical error is minimal. Moreover θ^* can be computed in $O(L^3)$ time, where L is the number of edges of the polygon.

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