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EXTENDED LOCAL CONVERGENCE ANALYSIS OF A THREE-STEP METHOD WITH A PARAMETER OF CONVERGENCE ORDER SIX

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Abstract. The convergence region of a Three Step Method (TSM) with a parameter of convergence order six for solving Banach space valued equations is extended by using conditions only on the first derivative and our idea of the restricted region.

1. Introduction

Let B_1, B_2 denote Banach spaces, $\Omega \subseteq B_1$ be convex and open. We are concerned with the problem of finding a locally unique solution x_* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \longrightarrow B_2$ is continuously Fréchet differentiable according to Fréchet. Solving (1.1) is of extreme importance, since many applications reduce to solving (1.1). We resort to iterative methods through which a sequence is generated converging to x_* under certain conditions on the initial data [1]–[16]. We study

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the local convergence of the three-step method defined for $x_0 \in \Omega$ and all $n = 0, 1, 2, \dots$ by

$$(1.2) \qquad \begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \alpha F'(y_n)^{-1}F(x_n), \\ x_{n+1} &= z_n - M_nF(z_n), \end{aligned}$$

$\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}, M_n = 2F'(y_n)^{-1} - F'(x_n)^{-1}$ developed in [15], when $X = Y = \mathbb{R}^i$. The order is six, if $\alpha = \frac{1}{2}$. The local convergence of (1.2) has been studied extensively under conditions on derivatives of F of order up to seven [15]. The convergence domain is not large, the error estimates on $\|x_n - x_*\|$ pessimistic and the information concerning the uniqueness of the solution not the best possible (in general). We address all these concerns and present a finer local convergence for TSM without additional conditions.

2. Convergence

We define some real functions and constants to be used in the analysis that follows. Set $S = [0, \infty)$.

Suppose that there exists continuous and nondecreasing function $w_0 : S \to S$ such that equation

$$(2.1) \qquad w_0(s) - 1 = 0$$

has a least positive solution denoted by ρ. Set $S_0 = [0, \rho)$.

Suppose that there exist continuous and nondecreasing functions $w : S_0 \to S$, $w_1 : S_0 \to S$ such that equation

$$(2.2) \qquad \varphi_1(s) := g_1(s) - 1 = 0$$

has a least solution $\rho_1 \in (0, \rho]$, where

$$g_1(s) = \frac{\int_0^1 w((1-\theta)s)d\theta + |1-\alpha| \int_0^1 w_1(\theta s)d\theta}{1 - w_0(s)}.$$

Suppose equation

$$(2.3) \qquad w_0(g_1(s)s) - 1 = 0$$

has a least solution $\rho_2 \in (0, \rho]$. Set $\rho_3 = \min\{\rho_1, \rho_2\}$.

Suppose equation

$$\varphi_2(s) := g_2(s) - 1 = 0 \tag{2.4}$$

has a least solution $\rho_4 \in (0, \rho_3)$, where

$$g_0(s) = \frac{\int_0^1 w((1-\theta)s)d\theta}{1 - w_0(s)}$$

and

$$g_2(s) = g_0(s) + \frac{(w_0(s) + w_0(g_1(s)s)) \int_0^1 w_1(\theta s)d\theta}{(1 - w_0(s))(1 - w_0(g_1(s)s))}.$$

Suppose equation

$$w_0(g_2(s)s) - 1 = 0 \tag{2.5}$$

has a least solution $\rho_5 \in (0, \rho_3)$. Set $\rho_6 = \min\{\rho_4, \rho_5\}$.

Suppose equation

$$\varphi_3(s) := g_3(s) - 1 = 0 \tag{2.6}$$

has a least solution $\rho_* \in (0, \rho_6)$, where

$$\begin{aligned} g_3(s) \quad = \quad & [g_0(g_2(s)s) + \left(\frac{w_0(g_2(s)s) + w_0(g_1(s)s)}{(1 - w_0(g_2(s)s))(1 - w_0(g_1(s)s))} + \right. \\ & \left. + \frac{w_0(g_2(s)s) + w_0(s)}{(1 - w_0(s))(1 - w_0(g_1(s)s))} \right) \int_0^1 w_1(\theta g_2(s)s)d\theta] g_2(s). \end{aligned}$$

It follows from these definitions that for all $s \in [0, \rho_*),\ i = 1, 2, 3$

$$0 \leq w_0(s) < 1, \tag{2.7}$$

$$0 \leq w_0(g_1(s)s) < 1, \tag{2.8}$$

$$0 \leq w_0(g_2(s)s) < 1 \tag{2.9}$$

and

$$0 \leq \varphi_i(s) < 1. \tag{2.10}$$

We shall show that ρ_* is a radius of convergence for (1.2).

Let $U(v, a), \bar{U}(v, a)$ stand for the open and closed balls in B_1 with center $v \in B_1$ and of radius $a > 0$, respectively. Set $U_0 = \Omega \cap U(x_*, \rho)$. The following conditions (A) shall be used:

(a1) $F : \Omega \to B_2$ is differentiable and there exists x_* so that $F'(x_*)$ is invertible and $F(x_*) = 0$.

(a2) There exist continuous and nondecreasing function $w_0 : S \to S$ such that for each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|).$$

(a3) There exist continuous and nondecreasing functions $w : S_0 \to S$ and $w_1 : S_0 \to S$ such that for all $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \leq w_1(\|x - x_*\|).$$

(a4) $\bar{U}(x_*, \rho_*) \subset \Omega$, where ρ_* is defined previously.

(a5) There exists $\bar{\rho} \geq \rho_*$ such that

$$\int_0^1 w_0(\theta\bar{\rho})d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(x_*, \bar{\rho})$.

Next, the main local convergence result for (1.2) is provided using conditions (A) and the developed notation.

Theorem 2.1. *Suppose that the conditions (A) hold. Then, for $x_0 \in U(x_*, \rho_*)$ sequence $\{x_n\}$ generated by* (1.2) *is well defined, stays in $U(x_*, \rho_*)$ and $\lim_{n\to\infty} x_n = x_*$. Moreover, the following assertions hold for all $n = 0, 1, 2, \ldots$ and $e_n = \|x_n - x_*\|$*

$$\|y_n - x_*\| \leq g_1(e_n)e_n \leq e_n < \rho_*, \tag{2.11}$$

$$\|z_n - x_*\| \leq g_2(e_n)e_n \leq e_n \tag{2.12}$$

and

$$e_{n+1} \leq g_3(e_n)e_n \leq e_n. \tag{2.13}$$

Furthermore, ρ_* is the unique solution of equation $F(x) = 0$ in Ω_1 given in $(a5)$.

Proof. Let $u \in U(x_*, \rho_*)$. By the definition of ρ_*, (a1) and (a2), we have

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \le w_0(\|u - x_*\|) \le w_0(\rho_*) < 1, \tag{2.14}$$

so by a perturbation lemma due to Banach on inverses of operators [6] $F'(u)^{-1}$ exists with

$$\|F'(u)^{-1}F'(x_*)\| \le \frac{1}{1 - w_0(\|u - x_*\|)}. \tag{2.15}$$

In particular, for $u = x_0, y_0$ exists by the first substep of (1.2) (for $n = 0$). Then, we can write

$$y_0 - x_* = x_0 - x_* - F'(x_0)^{-1}F(x_0) + (1 - \alpha)F'(x_0)^{-1}F(x_0), \tag{2.16}$$

so by (a3) and (2.15) for $u = x_0$:

$$\begin{aligned}
\|y_0 - x_*\| &\le \|F'(x_0)^{-1}F'(x_*)\| \times \\
&\quad \times \Big\| \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\theta \Big\| + \\
&\quad + |1 - \alpha| \|F'(x_0)^{-1}F'(x_*)\| \times \\
&\quad \times \Big\| \int_0^1 F'(x_*)^{-1}F'(x_* + \theta(x_0 - x_*))(x_0 - x_*)d\theta \Big\| \le \\
&\le \frac{[\int_0^1 w((1 - \theta)e_0)d\theta + |1 - \alpha| \int_0^1 w_1(\theta e_0)d\theta]e_0}{1 - w_0(e_0)} \le \\
&\le g_1(e_0)e_0 \le e_0 < \rho_*, \qquad (2.17)
\end{aligned}$$

showing (2.11) for $n = 0$, $y_0 \in U(x_*, \rho_*)$ and z_0, x_1 are well defined (see (2.15) for $u = y_0$). Then, we can also write

$$\begin{aligned}
z_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) + (F'(x_0)^{-1} - F'(y_0)^{-1})F(x_0) = \\
&= (x_0 - x_* - F'(x_0)^{-1}F(x_0)) + \\
&\quad + F'(x_0)^{-1}[(F'(y_0) - F'(x_*)) + (F'(x_*) - F'(x_0))]F'(y_0)^{-1}F(x_0). \qquad (2.18)
\end{aligned}$$

Hence, as in (2.17) but using (2.18), we get in turn that

$$\begin{aligned}
\|z_0 - x_*\| &\leq \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| + \\
&\quad + \|F'(x_0)^{-1}F'(x_*)\| \, \|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| + \\
&\quad + \|F'(x_*)^{-1}(F'(x_*) - F'(x_0))\| \, \|F'(y_0)^{-1}F'(x_*)\| \times \\
&\quad \times \|F'(x_*)^{-1}F(x_0)\| \leq \\
&\leq [g_0(e_0) + \frac{(w_0(e_0) + w_0(\|y_0 - x_*\|)) \int_0^1 w_1(\theta e_0) d\theta}{(1 - w_0(e_0))(1 - w_0(\|y_0 - x_*\|))}] e_0 \leq \\
&\leq g_2(e_0) e_0 \leq e_0, \qquad (2.19)
\end{aligned}$$

showing (2.12) for $n = 0$ and $z_0 \in U(x_*, \rho_*)$. Then, by the last substep of (1.2) for $n = 0$, we first have

$$\begin{aligned}
x_1 - x_* &= z_0 - x_* - (2F'(y_0)^{-1} - F'(x_0)^{-1})F(z_0) = \\
&= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + \\
&\quad + [(F'(z_0)^{-1} - F'(y_0)^{-1}) + (F'(x_0)^{-1} - F'(y_0)^{-1})]F(z_0), \qquad (2.20)
\end{aligned}$$

so as in (2.19)

$$\begin{aligned}
e_1 &\leq [g_0(\|z_0 - x_*\|) + \Bigg(\frac{w_0(\|z_0 - x_*\|) + w_0(\|y_0 - x_*\|)}{(1 - w_0(\|z_0 - x_*\|))(1 - w_0(\|y_0 - x_*\|))} + \\
&\quad + \frac{w_0(\|z_0 - x_*\|) + w_0(e_0)}{(1 - w_0(e_0))(1 - w_0(\|y_0 - x_*\|))} \Bigg) \int_0^1 w_1(\theta \|z_0 - x_*\|) d\theta] \|z_0 - x_*\| \leq \qquad (2.21) \\
&\leq g_3(e_0) e_0 \leq e_0,
\end{aligned}$$

which completes the induction for estimates (2.11)–(2.13) for $n = 0$. Suppose these estimates hold for $m = 0, 1, 2, \ldots, n$. Then, by switching x_0, y_0, z_0, x_1 with x_m, y_m, z_m, x_{m+1}, respectively in the preceding calculations, we terminate the induction for estimates (2.11)–(2.13) for all n. Then, for the estimate

$$\|x_{m+1} - x_*\| \leq r\|x_n - x_*\| < \rho_*, \; r = g_3(e_0) \in [0, 1), \qquad (2.22)$$

we deduce $\lim_{m \to \infty} x_m = x_*$ and $x_{m+1} \in U(x_*, \rho_*)$. Concerning the uniqueness of the solution part of the proof, let $x_{**} \in \Omega_1$ with $F(x_{**}) = 0$ and define

$G = \int_0^1 F'(x_{**} + \theta(x_* - x_{**}))d\theta$. Then, by (a2) and (a5), we obtain

$$\begin{aligned} \|F'(x_*)^{-1}(G - F'(x_*))\| &\leq \int_0^1 w_0((1-\theta)\|x_* - x_{**}\|)d\theta \leq \\ &\leq \int_0^1 w_0(\theta\bar{\rho})d\theta < 1, \end{aligned} \tag{2.23}$$

so G^{-1} exists and $x_* = x_{**}$ follows from $0 = F(x_*) - F(x_{**}) = G(x_* - x_{**})$. ■

Remarks. 1. By (a2), and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \leq \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

second condition in (a3) can be dropped, and w_1 be defined as

$$w_1(t) = 1 + w_0(t).$$

Notice that, if $w_1(t) < 1 + w_0(t)$, then R_1 can be larger (see Example 3.1).

2. The results obtained here can be used for operators G satisfying autonomous differential equations [1]–[5] of the form

$$F'(x) = T(F(x))$$

where T is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.

3. The local results obtained here can be used for projection algorithms such as the Arnoldi's algorithm , the generalized minimum residual algorithm (GMRES), the generalized conjugate algorithm (GCR) for combined Newton/finite projection algorithms and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [1]–[5].

4. Let $w_0(t) = L_0 t$, and $w(t) = Lt$. The parameter $r_A = \frac{2}{2L_0 + L}$ was shown by us to be the convergence radius of Newton's algorithm [2]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ \text{ for each } \ n = 0, 1, 2, \cdots \tag{2.24}$$

under the conditions (a1)–(a3) (w_1 is not used). It follows that the convergence radius R of algorithm (1.2) cannot be larger than the convergence radius r_A of the second order Newton's algorithm (2.24). As already noted in [3] r_A is at least as large as the convergence ball given by Rheinboldt [1]

$$r_{TR} = \frac{2}{3L_1},$$

where L_1 is the Lipschitz constant on $\Omega, L_0 \leq L_1$ and $L \leq L_1$. In particular, for $L_0 < L_1$ or $L < L_1$, we have that

$$r_{TR} < r_A \quad \text{and} \quad \frac{r_{TR}}{r_A} \to \frac{1}{3} \;\; as \;\; \frac{L_0}{L_1} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_{TR} was given by Traub [1].

5. It is worth noticing that solver (1.2) is not changing, when we use the conditions (A) of Theorem 2.1 instead of the stronger conditions used in [15]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the existence of the seventh Fréchet derivative for operator F.

3. Numerical examples

In all the examples we have taken $\alpha = \frac{1}{2*L1}$.

Example 3.1. Let $B_1 = B_2 = \Omega = \mathbb{R}$. Define $F(x) = \sin x$. Then, we get that $x^* = 0,\ \omega_0(s) = \omega(s) = s$ and $\omega_1(s) = 1$. Then, we have

Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.3333	0.3333
r_2	0.2845	0.2613
r_3	0.2469	0.2221

Table 1. Radius for Example 3.1

Example 3.2. Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \ \text{ for each } \ \xi \in \Omega.$$

Then, we get that $x_* = 0,\ \omega_0(s) = \omega_1(s) = 7.5s, \omega_1(s) = 2$. This way, we have that

Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0444	0.0444
r_2	0.0253	0.0400
r_3	0.0211	0.0329

Table 2. Radius for Example 3.2

Example 3.3. Let $B_1 = B_2 = \mathbb{R}^3$, $\Omega = U(0,1), x_* = (0,0,0)^T$, and define F on Ω by

$$F(x) = F(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{3.2}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $G'(x_*) = diag(1,1,1)$, we get by conditions (A) $\omega_0(s) = (e-1)s$, $\omega(s) = e^{\frac{1}{e-1}}s$, and $\omega_1(s) = e^{\frac{1}{e-1}}$. Then, we have

Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.2488	0.2488
r_2	0.2496	0.1370
r_3	0.2240	0.1217

Table 3. Radius for Example 3.3

Example 3.4. Returning back to the motivational example at the introduction of this study, we have $\omega_0(s) = \omega(s) = 96.662907s$, $\omega_1(s) = 1.0631$. Then, we have

Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0034	0.0034
r_2	0.0028	0.0027
r_3	0.0025	0.0023

Table 4. Radius for Example 3.4

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