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E.R. LOVE TYPE RIGHT SIDE FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. Motivated for the work of E.R. Love ([4], 1985) on integral inequalities we produce general right side direct and reverse integral inequalities. We apply these to ordinary and right side fractional integral inequalities. The last involves ordinary derivatives, right side Riemann-Liouville fractional integrals, right side Caputo fractional derivatives, and right side generalized fractional derivatives. These inequalities are of Opial type ([5]).

1. Introduction

This paper deals with ordinary and right side fractional integral inequalities. We are motivated by the following left side results:

Theorem 1.1. (Hardy's Inequality, integral version [3, Theorem 327].) *If f is a complex-valued function in $L^r(0,\infty)$, $\|\cdot\|$ is the $L^r(0,\infty)$ norm and $r>1$, then*

$$\left\| \frac{1}{x} \int_0^x f(t)\, dt \right\| \leq \frac{r}{r-1} \|f\| . \tag{1}$$

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Theorem 1.2. (E.R. Love, [4].) *If $s \geq r \geq 1$, $0 \leq a < b \leq \infty$, γ is real, $\omega(x)$ is decreasing and positive in (a,b), $f(x)$ and $H(x,y)$ are measurable and non-negative on (a,b), $H(x,y)$ is homogeneous of degree -1,*

$$(Hf)(x) = \int_a^x H(x,y) f(y)\, dy \tag{2}$$

and

$$\|f\|_r = \left(\int_a^b f(x)^r x^{\gamma-1} \omega(x)\, dx \right)^{\frac{1}{r}}, \tag{3}$$

then

$$\|Hf\|_r \leq C \|f\|_s, \tag{4}$$

where

$$C = \int_{\frac{a}{b}}^1 H(1,t)\, t^{-\frac{\gamma}{r}} \left(\int_a^{bt} x^{\gamma-1} \omega(x)\, dx \right)^{\frac{1}{r}-\frac{1}{s}} dt. \tag{5}$$

Here $\frac{a}{b}$ is to mean 0 if $a=0$ or $b=\infty$ or both; and bt is to mean ∞ if $b=\infty$.

We present direct and reverse integral inequalities.

2. Main results

2.1. Part I

First we give direct right side results. We present the following direct general right side result:

Theorem 2.1. *If $s \geq r \geq 1$, $0 \leq a < b \leq \infty$, $\gamma \in \mathbb{R}$, $\omega(x)$ is increasing and positive in (a,b), $f(x)$ and $H(z,x)$ are measurable and non-negative on (a,b), and $(a,b)^2$, respectively, $H(z,x)$ is homogeneous of degree -1,*

$$(Hf)(x) = \int_x^b H(z,x) f(z)\, dz, \tag{6}$$

and

$$\left\| f\right\|_{r}=\left(\int_{a}^{b} f^{r}\left(x\right) x^{\gamma -1}\omega \left(x\right) dx\right) ^{\frac{1}{r}}, \tag{7}$$

then

$$\left\| Hf\right\|_{r}\leq C\left\| f\right\|_{s}, \tag{8}$$

where

$$C:=\int_{1}^{\frac{b}{a}} H\left(t,1\right) t^{-\frac{\gamma}{r}}\left(\int_{at}^{b} x^{\gamma -1}\omega \left(x\right) dx\right) ^{\frac{1}{r}-\frac{1}{s}}dt. \tag{9}$$

Here $\frac{b}{a}$ is to mean ∞, if $a=0$ or $b=\infty$ or both; and at is to mean 0 if $a=0$.

Proof. (i) Let $x\in \left(a,b\right)$.

Here $x\leq z\leq b$ and $1\leq \frac{z}{x}\leq \frac{b}{x}$. set $t=\frac{z}{x}$, that is $z=tx$, and $1\leq t\leq \frac{b}{x}$. Infact $1\leq t<\frac{b}{a}$. By degree -1 homogenity of H we have

$$\left(Hf\right) \left(x\right) =\int_{1}^{\frac{b}{x}} H\left(tx,x\right) f\left(tx\right) xdt=$$

$$=\int_{1}^{\frac{b}{x}} x^{-1}H\left(t,1\right) f\left(tx\right) xdt=\int_{1}^{\frac{b}{x}} H\left(t,1\right) f\left(tx\right) dt.$$

That is

$$\left(Hf\right) \left(x\right) =\int_{1}^{\frac{b}{x}} H\left(t,1\right) f\left(tx\right) dt. \tag{10}$$

Using Minkowski's inequality at (11), and the increasing property of ω at (12), we get

$$\left\| Hf\right\|_{r}=\left(\int_{a}^{b}\left(\int_{1}^{\frac{b}{x}} H\left(t,1\right) f\left(tx\right) dt\right) ^{r}x^{\gamma -1}\omega \left(x\right) dx\right) ^{\frac{1}{r}}\leq$$

$$\leq \int_1^{\frac{b}{a}} \left(\int_a^{\frac{b}{t}} H(t,1)^r f(tx)^r x^{\gamma-1} \omega(x)\, dx \right)^{\frac{1}{r}} dt \leq \tag{11}$$

$$\leq \int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} \left(\int_a^{\frac{b}{t}} f(tx)^r (tx)^{\gamma-1} \omega(xt)\, t dx \right)^{\frac{1}{r}} dt = \tag{12}$$

$$= \int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} \left(\int_{at}^{b} f(z)^r z^{\gamma-1} \omega(z)\, dz \right)^{\frac{1}{r}} dt \leq \tag{13}$$

$$\leq \left(\int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} dt \right) \left(\int_{at}^{b} f(z)^r z^{\gamma-1} \omega(z)\, dz \right)^{\frac{1}{r}}. \tag{14}$$

If $s = r$, this is the required inequality.

(ii) Assume that $s > r$. Applying Hölder's inequality with indices $\frac{s}{r}$ and $\left(\frac{s}{s-r}\right)$ in the inside integral of (13) we get

$$\int_{at}^{b} f(z)^r z^{\gamma-1} \omega(z)\, dz = \int_{at}^{b} f(z)^r \left(z^{\gamma-1}\omega(z)\right)^{\frac{r}{s}} \left(z^{\gamma-1}\omega(z)\right)^{1-\frac{r}{s}} dz \leq$$

$$\leq \left(\int_{at}^{b} f(z)^s z^{\gamma-1} \omega(z)\, dz \right)^{\frac{r}{s}} \left(\int_{at}^{b} z^{\gamma-1} \omega(z)\, dz \right)^{1-\frac{r}{s}} \leq$$

$$\leq \left(\int_{a}^{b} f(z)^s z^{\gamma-1} \omega(z)\, dz \right)^{\frac{r}{s}} \left(\int_{at}^{b} z^{\gamma-1} \omega(z)\, dz \right)^{1-\frac{r}{s}} = \tag{15}$$

$$= \|f\|_s^r \left(\int_{at}^{b} z^{\gamma-1} \omega(z)\, dz \right)^{1-\frac{r}{s}}.$$

So we have (by use of (14) and (15))

$$
\begin{aligned}
\|Hf\|_r &\leq \int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} \left(\int_{at}^{b} f(z)^r z^{\gamma-1} \omega(z)\, dz \right)^{\frac{1}{r}} dt \leq \\
&\leq \|f\|_s \int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} \left(\int_{at}^{b} z^{\gamma-1} \omega(z)\, dz \right)^{\frac{1}{r}-\frac{1}{s}} dt = C \|f\|_s ,
\end{aligned} \tag{16}
$$

proving the claim. ■

Remark 2.1. If $r = s$, then (see (9))

$$
C := \int_1^{\frac{b}{a}} H(t,1)\, t^{-\frac{\gamma}{r}} dt, \tag{17}
$$

independent of ω.

We give

Corollary 2.1. *Let $r > 1$ and $f \in L^r(0,\infty)$. Then*

$$
\left\| \int_x^{\infty} \frac{f(t)}{t} dt \right\|_r \leq r \|f\|_r . \tag{18}
$$

Proof. Apply (8), with $a = 0$, $b = \infty$, $x \in \mathbb{R}_+$, $\gamma = 1$, $s = r > 1$, $\omega(x) = 1$, and $H(z,x) = \frac{1}{z}$. ■

We continue with a right side fractional result.

Theorem 2.2. *If $p > 0$, $q > 0$, $p + q = r \geq 1$, $0 < a < b < \infty$, $\gamma > 0$, $\omega(x)$ is increasing and positive in (a,b), $f(x)$ is measurable and non-negative on (a,b), I_{b-}^{α} is the right Riemann-Liouville operator of fractional integration defined by*

$$
I_{b-}^{\alpha} f(x) = \int_x^b \frac{(z-x)^{\alpha-1}}{\Gamma(\alpha)} f(z)\, dz, \tag{19}
$$

where Γ is the gamma function, $\alpha > 0$,

$$
I_{b-}^{0} f(x) = f(x),
$$

and $I_{b-}^{\beta} f$ *is defined similarly for* $\beta \geq 0$, *then*

$$\int_a^b \left(I_{b-}^{\alpha} f(x)\right)^p \left(I_{b-}^{\beta} f(x)\right)^q x^{\gamma-\alpha p-\beta q-1} \omega(x)\, dx \leq \tag{20}$$
$$\leq C^* \int_a^b f^r(x)\, x^{\gamma-1} \omega(x)\, dx,$$

where

$$C^* = \left(\frac{b-a}{a}\right)^{(\alpha p+\beta q)} \frac{1}{\Gamma^r(\alpha+1)}. \tag{21}$$

Proof. Consider $H(z,x) = \frac{(z-x)^{\alpha-1}}{x^{\alpha}\Gamma(\alpha)}$, $a < x < z < b$. Notice $H(tz,tx) = t^{-1}H(z,x)$, so that H is homogeneous of degree -1. Here we consider

$$(Hf)(x) = \frac{1}{x^{\alpha}} \int_x^b \frac{(z-x)^{\alpha-1}}{\Gamma(\alpha)} f(z)\, dz = x^{-\alpha} I_{b-}^{\alpha} f(x),$$

and by (8) we have

$$\left\| x^{-\alpha} I_{b-}^{\alpha} f(x) \right\|_r \leq A \|f\|_r, \tag{22}$$

where

$$A = \frac{1}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} (t-1)^{\alpha-1} t^{-\frac{\gamma}{r}}\, dt.$$

We notice that

$$\int_1^{\frac{b}{a}} (t-1)^{\alpha-1} t^{-\frac{\gamma}{r}}\, dt \leq \int_1^{\frac{b}{a}} (t-1)^{\alpha-1}\, dt = \frac{(b-a)^{\alpha}}{a^{\alpha}\alpha}. \tag{23}$$

That is

$$A \leq \frac{(b-a)^{\alpha}}{a^{\alpha}\Gamma(\alpha+1)}. \tag{24}$$

Therefore it holds

$$\left\| x^{-\alpha} I_{b-}^{\alpha} f(x) \right\|_r \leq \frac{(b-a)^{\alpha}}{a^{\alpha}\Gamma(\alpha+1)} \|f\|_r, \tag{25}$$

and

$$\left\| x^{-\beta} I_{b-}^{\beta} f(x) \right\|_r \leq \frac{(b-a)^{\beta}}{a^{\beta}\Gamma(\beta+1)} \|f\|_r . \tag{26}$$

Using Hölder's inequality with indices $\frac{r}{p}$ and $\frac{r}{q}$, we get

$$\int_a^b \left(x^{-\alpha} I_{b-}^{\alpha} f(x)\right)^p \left(x^{-\beta} I_{b-}^{\beta} f(x)\right)^q x^{\gamma-1}\omega(x)\,dx \leq$$

$$\leq \left(\int_a^b \left(x^{-\alpha} I_{b-}^{\alpha} f(x)\right)^r x^{\gamma-1}\omega(x)\,dx\right)^{\frac{p}{r}} \left(\int_a^b \left(x^{-\beta} I_{b-}^{\beta} f(x)\right)^r x^{\gamma-1}\omega(x)\,dx\right)^{\frac{q}{r}} =$$

$$= \left\| x^{-\alpha} I_{b-}^{\alpha} f(x) \right\|_r^p \left\| x^{-\beta} I_{b-}^{\beta} f(x) \right\|_r^q \leq$$

$$\leq \left(\frac{(b-a)^{\alpha}}{a^{\alpha}\Gamma(\alpha+1)}\right)^p \left(\frac{(b-a)^{\beta}}{a^{\beta}\Gamma(\beta+1)}\right)^q \|f\|_r^p \|f\|_r^q =$$

$$= \frac{(b-a)^{\alpha p+\beta q}}{a^{\alpha p+\beta q}\Gamma(\alpha+1)^r} \|f\|_r^r , \tag{27}$$

proving the claim. ■

We continue with

Corollary 2.2. *If $p > 0$, $q > 0$, $p + q = r \geq 1$, $0 < a < b < \infty$, $\gamma > 0$, $\omega(x)$ is increasing and positive in (a, b), $f(x)$ is measurable and non-negative on (a, b), I_{b-}^{α} is the right Riemann-Liouville fractional integral, $\alpha > 0$, then*

$$\begin{aligned} &\int_a^b \left(I_{b-}^{\alpha} f(x)\right)^p f(x)^q x^{\gamma-\alpha p-1}\omega(x)\,dx \leq \\ &\leq \left(\frac{b-a}{a}\right)^{\alpha p} \frac{1}{\Gamma^r(\alpha+1)} \int_a^b f^r(x)\, x^{\gamma-1}\omega(x)\,dx. \end{aligned} \tag{28}$$

Proof. By Theorem 2.2 for $\beta = 0$. ■

We also give

Corollary 2.3. *Here $p, q > 0$, $p + q = r \geq 1$, $0 < a < b < \infty$, $\gamma > 0$, $\omega(x)$ is increasing and positive in (a, b). Let $f \in AC^n([a, b])$ (i.e. $f^{(n-1)}$ is absolutely continuous), n is even and $f^{(n)} \geq 0$. Assume further that $f^{(k)}(b) = 0$, $k = 0, 1, ..., n-1$. Then*

$$\int_a^b f^p(x)\left(f^{(n)}(x)\right)^q x^{\gamma-np-1}\omega(x)\,dx \leq \tag{29}$$
$$\leq \left(\frac{b-a}{a}\right)^{np}\frac{1}{(n!)^r}\int_a^b \left(f^{(n)}(x)\right)^r x^{\gamma-1}\omega(x)\,dx.$$

If $\gamma = 1$, $\omega(x) = 1$, *we have*

$$\int_a^b f^p(x)\left(f^{(n)}(x)\right)^q x^{-np}dx \leq \left(\frac{b-a}{a}\right)^{np}\frac{1}{(n!)^r}\int_a^b \left(f^{(n)}(x)\right)^r dx. \tag{30}$$

Proof. Since $f \in AC^n([a,b])$, $x \in [a,b]$, we have that (Taylor's formula)

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{(n-1)!}\int_b^x (x-t)^{n-1} f^{(n)}(t)\,dt. \tag{31}$$

By the assumptions we get that

$$f(x) = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f^{(n)}(t)\,dt = I_{b-}^n f^{(n)}(x). \tag{32}$$

Here we have that there exists $f^{(n)}$ almost everywhere and $f^{(n)} \in L_1([a,b])$. Direct application of (28) produces (29). ■

Remark 2.2. Since $a \leq x \leq b$, by (30) we easily get that

$$\int_a^b f^p(x)\left(f^{(n)}(x)\right)^q dx \leq \left(\frac{b(b-a)}{a}\right)^{np}\frac{1}{(n!)^r}\int_a^b \left(f^{(n)}(x)\right)^r dx. \tag{33}$$

Setting $p = q = 1$, we have

$$\int_a^b f(x) f^{(n)}(x)\,dx \leq \left(\frac{b(b-a)}{a}\right)^{n}\frac{1}{(n!)^2}\int_a^b \left(f^{(n)}(x)\right)^2 dx, \tag{34}$$

which is a ride side Opial's type inequality.

We would like to formalize the last result.

Corollary 2.4. *Let* $0 < a < b < \infty$, $f \in AC^n([a,b])$, *n is even and* $f^{(n)} \geq 0$. *Assume that* $f^{(k)}(b) = 0$, $k = 0, 1, ..., n-1$. *Then*

$$\int_a^b f(x) f^{(n)}(x)\, dx \leq \left(\frac{b(b-a)}{a}\right)^n \frac{1}{(n!)^2} \int_a^b \left(f^{(n)}(x)\right)^2 dx. \tag{35}$$

We specialize (35) as follows:

Corollary 2.5. *Let* $f \in AC^2([1,2])$, *with* $f^{(2)} \geq 0$. *Assume that* $f(2) = = f'(2) = 0$. *Then*

$$\int_1^2 f(x) f^{(2)}(x)\, dx \leq \int_1^2 \left(f^{(2)}(x)\right)^2 dx. \tag{36}$$

We need

Definition 2.1. ([1], p. 336.) *Let* $f \in AC^m([a,b])$, $m \in \mathbb{N}$, *where* $m = = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ *the ceiling of number*). *We define the right Caputo fractional derivative of order* α, *by*

$$D_{b-}^{\alpha} f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \quad \forall\, x \in [a,b], \tag{37}$$

that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{38}$$

We mention the right Caputo fractional Taylor formula:

Theorem 2.3. ([1], p. 341.) *Let* $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. *Then*

1)

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz, \tag{39}$$

2) *in case of* $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, *we have*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz = \left(I_{b-}^{\alpha}\left(D_{b-}^{\alpha} f\right)\right)(x), \tag{40}$$

$\forall\, x \in [a,b]$.

We present

Corollary 2.6. *Let* $p,q>0$, $p+q=r\geq 1$, $0<a<b<\infty$, $\gamma>0$, $\omega(x)$ *is increasing and positive in* (a,b), $\alpha>0$, $m=\lceil\alpha\rceil$. *Assume that* $f\in AC^{m}([a,b])$ *such that* $f^{(k)}(b)=0$, $k=0,1,...,m-1$ *and* $D_{b-}^{\alpha}f\geq 0$ *over* (a,b). *Then*

$$\int_a^b (f(x))^p \left(\left(D_{b-}^{\alpha}f\right)(x)\right)^q x^{\gamma-\alpha p-1}\omega(x)\,dx \leq \tag{41}$$

$$\leq \left(\frac{b-a}{a}\right)^{\alpha p}\frac{1}{\Gamma^r(\alpha+1)}\int_a^b \left(\left(D_{b-}^{\alpha}f\right)(x)\right)^r x^{\gamma-1}\omega(x)\,dx.$$

Proof. Apply (28) and (40). ■

We continue with

Definition 2.2. ([1], p. 345.) *Let* $\nu>0$, $n:=[\nu]$ ($[\cdot]$ *integral part*), $\alpha=\nu-n$, $0<\alpha<1$, $f\in C([a,b])$. *Consider the subspace of functions*

$$C_{b-}^{\nu}([a,b]):=\left\{f\in C^n([a,b]):I_{b-}^{1-\alpha}f^{(n)}\in C^1([a,b])\right\}. \tag{42}$$

We define the right generalized ν*-fractional derivative of* f *over* $[a,b]$ *as*

$$\overline{D}_{b-}^{\nu}f=(-1)^{n-1}\left(I_{b-}^{1-\alpha}f^{(n)}\right)'. \tag{43}$$

That is

$$\left(\overline{D}_{b-}^{\nu}f\right)(x)=\frac{(-1)^{n-1}}{\Gamma(n-\nu+1)}\frac{d}{dx}\int_x^b (z-x)^{n-\nu}f^{(n)}(z)\,dz, \tag{44}$$

$\forall\ x\in[a,b]$.

We need the following right fractional Taylor formula.

Theorem 2.4. ([1], p. 348.) *Let* $f\in C_{b-}^{\nu}([a,b])$, $\nu>0$, $n=[\nu]$.

1) *If* $\nu\geq 1$, *then*

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k+\left(I_{b-}^{\nu}\overline{D}_{b-}^{\nu}f\right)(x), \tag{45}$$

$\forall\ x\in[a,b]$.

2) *If* $0<\nu<1$, *then*

$$f(x)=I_{b-}^{\nu}\overline{D}_{b-}^{\nu}f(x), \tag{46}$$

$\forall\ x\in[a,b]$.

We make

Remark 2.3. By Theorem 16 we have: for $f \in C_{b-}^{\nu}([a,b])$, $\nu > 0$, $n = [\nu]$, we get that $f(x) = I_{b-}^{\nu} \overline{D}_{b-}^{\nu} f(x)$, $\forall\, x \in [a,b]$, given that $0 < \nu < 1$, or $\nu \geq 1$ and $f^{(k)}(b) = 0$, $k = 0, 1, ..., n-1$.

We present

Corollary 2.7. *Let $p, q > 0$, $p + q = r \geq 1$, $0 < a < b < \infty$, $\gamma > 0$, $\omega(x)$ is increasing and positive in (a,b), $\nu > 0$, $n = [\nu]$. Assume that $f \in C_{b-}^{\nu}([a,b])$, in case of $\nu \geq 1$ we have that $f^{(k)}(b) = 0$, $k = 0, 1, ..., n-1$ and $\overline{D}_{b-}^{\nu} f \geq 0$ over (a,b). Then*

$$\int_a^b (f(x))^p \left(\left(\overline{D}_{b-}^{\nu} f\right)(x)\right)^q x^{\gamma - \nu p - 1} \omega(x)\, dx \leq$$

$$\leq \left(\frac{b-a}{a}\right)^{\nu p} \frac{1}{\Gamma^r(\nu+1)} \int_a^b \left(\left(\overline{D}_{b-}^{\nu} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx. \tag{47}$$

Proof. Apply (28) and Remark 2.3. ■

2.2 Part II

Next we get reverse right side results.

We present

Theorem 2.5. *Let $0 < r < 1$, $0 < a < b < \infty$, $f(x)$ and $H(z,x)$ are measurable and non-negative on (a,b), $(a,b)^2$, respectively, $H(z,x)$ is homogeneous of degree -1,*

$$(Hf)(x) = \int_x^b H(z,x) f(z)\, dz, \tag{48}$$

and

$$\|f\|_{r,[a,b]} = \left(\int_a^b f^r(x)\, dx\right)^{\frac{1}{r}}. \tag{49}$$

We suppose that $\|Hf\|_{r,[a,b]} < \infty$ and $H(t,1) f(tx) > 0$, for almost all $t \in \left[1, \frac{b}{x}\right]$, for almost all $x \in [a,b]$, and $\frac{H(t,1)}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} < \infty$, for almost all $t \in \left[1, \frac{b}{a}\right]$.

Then

$$\|Hf\|_{r,[a,b]} \geq \int_1^{\frac{b}{a}} \frac{H(t,1)}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} \, dt. \tag{50}$$

Proof. For $a < x < b$, the homogeneity of degree -1 of H gives

$$(Hf)(x) = \int_1^{\frac{b}{x}} H(t,1) f(tx) \, dt,$$

where $x \leq z \leq b$, $1 \leq \frac{z}{x} \leq \frac{b}{x}$, setting $t := \frac{z}{x}$ then $z = tx$ and $1 \leq t \leq \frac{b}{x}$. Infact $1 \leq t < \frac{b}{a}$. Using the reverse Minkowski integral inequality ([2]) we obtain

$$\|Hf\|_{r,[a,b]} = \left(\int_a^b \left(\int_1^{\frac{b}{x}} H(t,1) f(tx) \, dt \right)^r dx \right)^{\frac{1}{r}} \geq$$

$$\geq \int_1^{\frac{b}{a}} \left(\int_a^{\frac{b}{t}} H(t,1)^r f(tx)^r \, dx \right)^{\frac{1}{r}} dt = \int_1^{\frac{b}{a}} \frac{H(t,1)}{t^{\frac{1}{r}}} \left(\int_a^{\frac{b}{t}} f(tx)^r \, t dx \right)^{\frac{1}{r}} dt = \tag{51}$$

$$= \int_1^{\frac{b}{a}} \frac{H(t,1)}{t^{\frac{1}{r}}} \left(= \int_{at}^{b} f(y)^r \, dy \right)^{\frac{1}{r}} dt = \int_1^{\frac{b}{a}} \frac{H(t,1)}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} \, dt,$$

proving (50). ■

Next we apply Theorem 2.5.

We give

Proposition 2.6. *Let* $0 < r < 1$, $0 < a < b < \infty$, $\alpha > 0$, $f(x)$ *is measurable and non-negative on* (a,b). *Assume that* $\left\|x^{-\alpha} I_{b-}^{\alpha} f(x)\right\|_{r,[a,b]} < \infty$; $(t-1)^{\alpha-1} f(tx) > 0$, *for almost all* $t \in \left[1, \frac{b}{x}\right]$, *for almost all* $x \in [a,b]$ *and* $\frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} < \infty$, *for almost all* $t \in \left[1, \frac{b}{a}\right]$. *Then*

$$\left\|x^{-\alpha} I_{b-}^{\alpha} f(x)\right\|_{r,[a,b]} \geq \frac{1}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} \, dt. \tag{52}$$

Proof. Consider here $H(z,x) = \frac{(z-x)^{\alpha-1}}{x^{\alpha}\Gamma(\alpha)}$, $a < x < z < b$, then $(Hf)(x) = x^{-\alpha} I^{\alpha}_{b-} f(x)$. Finally we apply Theorem 2.5. ■

We give a reverse right side fractional integral inequality.

Theorem 2.7. *Let* $0 < r < p$, *with* $r < 1$, $0 < a < b < \infty$, f *is measurable and non-negative on* (a,b) *such that* $f(x) > 0$ *almost everywhere on* $[a,b]$, $\alpha > 0$. *Assume that* $\left\| x^{-\alpha} I^{\alpha}_{b-} f(x) \right\|_{r,[a,b]} < \infty$, *and* $\|f\|_{r,[at,b]} < \infty$, *for almost all* $t \in \left[1, \frac{b}{a}\right]$. *Then*

$$\int_a^b \left(I^{\alpha}_{b-} f(x)\right)^p (f(x))^{r-p} x^{-\alpha p} dx \geq \tag{53}$$
$$\geq \frac{1}{\Gamma^p(\alpha)} \left(\int_1^{\frac{b}{a}} \frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} \, dt \right)^p \|f\|^{r-p}_{r,[a,b]} .$$

Proof. We will use Proposition 2.6.

Here $0 < r < p$, hence $0 < \frac{r}{p} < 1$, also $\frac{r}{r-p} < 0$, and $f(x) > 0$ almost everywhere in $[a,b]$. Next we apply the reverse Hölder's inequality:

$$\int_a^b \left(x^{-\alpha} \left(I^{\alpha}_{b-} f\right)(x)\right)^p (f(x))^{r-p} \, dx \geq$$
$$\geq \left(\int_a^b \left(x^{-\alpha} \left(I^{\alpha}_{b-} f\right)(x)\right)^r dx \right)^{\frac{p}{r}} \left(\int_a^b (f(x))^r \, dx \right)^{\frac{r-p}{r}} = \tag{54}$$
$$= \left\| x^{-\alpha} \left(I^{\alpha}_{b-} f\right)(x) \right\|^p_{r,[a,b]} \|f\|^{r-p}_{r,[a,b]} \geq$$
$$\geq \frac{1}{\Gamma^p(\alpha)} \left(\int_1^{\frac{b}{a}} \frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[at,b]} \, dt \right)^p \|f\|^{r-p}_{r,[a,b]} ,$$

proving the claim. ■

Next we give some reverse right side Opial type inequalities.

We start with an ordinary derivatives result.

Corollary 2.8. *Let* $0 < r < p$, *with* $r < 1$, $0 < a < b < \infty$, $f \in AC^n([a,b])$, n *is even and* $f^{(n)} \geq 0$ *on* (a,b) *with* $f^{(n)} > 0$ *almost everywhere on* $[a,b]$.

Assume that $f^{(k)}(b)=0$, $k=0,1,...,n-1$, *and* $\|x^{-\alpha}f(x)\|_{r,[a,b]}<\infty$, *and* $\|f^{(n)}\|_{r,[at,b]}<\infty$, *for almost all* $t\in\left[1,\frac{b}{a}\right]$. *Then*

$$\int_a^b (f(x))^p\left(f^{(n)}(x)\right)^{r-p}x^{-np}dx\geq$$

(55)

$$\geq\frac{1}{((n-1)!)^p}\left(\int_1^{\frac{b}{a}}\frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}}\left\|f^{(n)}\right\|_{r,[at,b]}dt\right)^p\left\|f^{(n)}\right\|_{r,[a,b]}^{r-p}.$$

Proof. Apply Theorem 2.7 and see the proof Corollary 2.3. ■

We continue with fractional reverse results.

Corollary 2.9. *Let* $0<r<p$, *with* $r<1$, $0<a<b<\infty$, $\alpha>0$, $m=\lceil\alpha\rceil$, $f\in AC^m([a,b])$, *such that* $f^{(k)}(b)=0$, $k=0,1,...,m-1$, *and* $D_{b-}^{\alpha}f\geq0$ *over* (a,b) *with* $D_{b-}^{\alpha}f>0$ *almost everywhere on* $[a,b]$. *Assume that* $\|x^{-\alpha}f(x)\|_{r,[a,b]}<\infty$, *and* $\|D_{b-}^{\alpha}f\|_{r,[at,b]}<\infty$, *for almost all* $t\in\left[1,\frac{b}{a}\right]$. *Then*

$$\int_a^b (f(x))^p\left(D_{b-}^{\alpha}f(x)\right)^{r-p}x^{-\alpha p}dx\geq$$

(56)

$$\geq\frac{1}{\Gamma^p(\alpha)}\left(\int_1^{\frac{b}{a}}\frac{(t-1)^{\alpha-1}}{t^{\frac{1}{r}}}\left\|D_{b-}^{\alpha}f\right\|_{r,[at,b]}dt\right)^p\left\|D_{b-}^{\alpha}f\right\|_{r,[a,b]}^{r-p}.$$

Proof. Use of Theorem 2.3 and Theorem 2.7. ■

We finish with

Corollary 2.10. *Let* $0<r<p$, *with* $r<1$, $0<a<b<\infty$, $\nu>0$, $n=[\nu]$, $f\in C_{b-}^{\nu}([a,b])$, *for either* $0<\nu<1$, *or* $\nu\geq1$ *and* $f^{(k)}(b)=0$, $k=0,1,...,n-1$. *Suppose that* $\overline{D}_{b-}^{\nu}f\geq0$ *over* (a,b) *with* $\overline{D}_{b-}^{\nu}f>0$ *almost everywhere on* $[a,b]$. *Assume further that* $\|x^{-\nu}f(x)\|_{r,[a,b]}<\infty$, *and* $\left\|\overline{D}_{b-}^{\nu}f\right\|_{r,[at,b]}<\infty$, *for almost all* $t\in\left[1,\frac{b}{a}\right]$. *Then*

$$\int_a^b (f(x))^p\left(\overline{D}_{b-}^{\nu}f(x)\right)^{r-p}x^{-\nu p}dx\geq$$

(57)

$$\geq\frac{1}{\Gamma^p(\nu)}\left(\int_1^{\frac{b}{a}}\frac{(t-1)^{\nu-1}}{t^{\frac{1}{r}}}\left\|\overline{D}_{b-}^{\nu}f\right\|_{r,[at,b]}dt\right)^p\left\|\overline{D}_{b-}^{\nu}f\right\|_{r,[a,b]}^{r-p}.$$

Proof. Use of Remark 2.3 and Theorem 2.7. ■

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