

# ABSTRACT FRACTIONAL LANDAU INEQUALITIES

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**Abstract.** We present uniform and  $L_p$  mixed Caputo–Bochner abstract fractional Landau inequalities over  $\mathbb{R}$  of fractional orders  $2 < \nu \leq 3$ . These estimate the size of first and second derivatives of a Banach space valued function over  $\mathbb{R}$ . We give applications when  $\nu = 2.5$ .

## 1. Introduction

Let  $p \in [1, \infty]$ ,  $I = \mathbb{R}_+$  or  $I = \mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  is twice differentiable with  $f, f'' \in L_p(I)$ , then  $f' \in L_p(I)$ . Moreover, there exists a constant  $C_p(I) > 0$  independent of  $f$ , such that

$$(1) \quad \|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}},$$

where  $\|\cdot\|_{p,I}$  is the  $p$ -norm on the interval  $I$ , see [1], [5].

The research on these inequalities started by E. Landau [11] in 1913. For the case of  $p = \infty$  he proved that

$$(2) \quad C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2},$$

are the best constants in (1).

In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for  $p = 2$ , with the best constants

$$(3) \quad C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1.$$

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In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants  $C_p(\mathbb{R}_+)$  in (1) satisfies the estimate

$$(4) \quad C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty),$$

which yields  $C_p(\mathbb{R}) \leq 2$  for  $p \in [1, \infty)$ .

In fact, in [6] and [9] was shown that  $C_p(\mathbb{R}) \leq \sqrt{2}$ .

In this work we are greatly inspired by interesting articles [10], [14].

We need the following concepts from abstract fractional calculus.

Our integrals next are of Bochner type [12].

We need

**Definition 1.** ([4], p. 150) Let  $[a, b] \subset \mathbb{R}$ ,  $X$  be a Banach space,  $\alpha > 0$ ;  $m = \lceil \alpha \rceil \in \mathbb{N}$ , ( $\lceil \cdot \rceil$  is the ceiling of the number),  $f : [a, b] \rightarrow X$ . We assume that  $f^{(m)} \in L_1([a, b], X)$ . We call the Caputo–Bochner left fractional derivative of order  $\alpha$ :

$$(5) \quad (D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad \forall x \in [a, b],$$

where  $\Gamma$  is the gamma function.

If  $\alpha \in \mathbb{N}$ , we set  $D_{*a}^\alpha f := f^{(m)}$  the ordinary  $X$ -valued derivative (defined similar to numerical one, see [13], p. 83), and also set  $D_{*a}^0 f := f$ .

By ([4], p. 2),  $(D_{*a}^\alpha f)(x)$  exists almost everywhere in  $x \in [a, b]$  and  $D_{*a}^\alpha f \in L_1([a, b], X)$ .

If  $\|f^{(m)}\|_{L_\infty([a,b],X)} < \infty$ , and  $\alpha \notin \mathbb{N}$ , then by ([4], p. 3),  $D_{*a}^\alpha f \in C([a, b], X)$ , hence  $\|D_{*a}^\alpha f\| \in C([a, b])$ .

We also need

**Definition 2.** ([4], p. 150) Let  $[a, b] \subset \mathbb{R}$ ,  $X$  be a Banach space,  $\alpha > 0$ ,  $m := \lceil \alpha \rceil$ . We assume that  $f^{(m)} \in L_1([a, b], X)$ , where  $f : [a, b] \rightarrow X$ . We call the Caputo–Bochner right fractional derivative of order  $\alpha$ :

$$(6) \quad (D_{b-}^\alpha f)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall x \in [a, b].$$

We observe that  $(D_{b-}^m f)(x) = (-1)^m f^{(m)}(x)$ , for  $m \in \mathbb{N}$ , and  $(D_{b-}^0 f)(x) = f(x)$ .

By ([4], p. 34),  $(D_{b-}^{\alpha} f)(x)$  exists almost everywhere on  $[a, b]$  and  $(D_{b-}^{\alpha} f) \in L_1([a, b], X)$ .

If  $\|f^{(m)}\|_{L_{\infty}([a, b], X)} < \infty$ , and  $\alpha \notin \mathbb{N}$ , by ([4], p. 37),  $D_{b-}^{\alpha} f \in C([a, b], X)$ , hence  $\|D_{b-}^{\alpha} f\| \in C([a, b])$ .

We mention the important:

**Corollary 3.** ([4], p. 157) *Let  $f \in C^m([a, b], X)$ ,  $m = [\alpha]$ ,  $\alpha > 0$ ,  $x, x_0 \in [a, b]$ . Then  $D_{*x_0}^{\alpha} f(x)$ ,  $D_{x_0-}^{\alpha} f(x)$  are jointly continuous functions in  $(x, x_0)$  from  $[a, b]^2$  into  $X$ ,  $X$  is a Banach space.*

By convention we suppose that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0,$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,$$

for all  $x, x_0 \in [a, b]$ .

The author has already done an extensive body of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]) and the derived inequalities were for small fractional orders, i.e.  $\alpha \in (0, 1)$ . Usually there the domains where  $[A, +\infty)$  or  $(-\infty, B]$ , with  $A, B \in \mathbb{R}$  and in one mixed case the domain was all of  $\mathbb{R}$ .

In this work with less assumptions we establish uniform and  $L_p$  type mixed Caputo–Bochner abstract fractional Landau inequalities over  $\mathbb{R}$  for fractional orders  $2 < \nu \leq 3$ . The method of proving is based on left and right Caputo–Bochner fractional Taylor’s formulae with integral remainders, see [4], pp. 151 – 152.

We give also applications when  $\nu = 2.5$ . Regardless to say that we are also inspired by [3], [4].

## 2. Main results

We present the following abstract mixed fractional Landau inequalities over  $\mathbb{R}$ .

**Theorem 4.** *Let  $2 < \nu \leq 3$  and  $f \in C^3(\mathbb{R}, X)$ , where  $(X, \|\cdot\|)$  is a Banach space. We assume that  $\|f\|_{\infty, \mathbb{R}} < \infty$ , and that*

$$(7) \quad K := \max \left\{ \| \|D_{*a}^{\nu} f(z)\| \|_{\infty, \mathbb{R}^2}, \| \|D_{a-}^{\nu} f(z)\| \|_{\infty, \mathbb{R}^2} \right\} < \infty,$$

where  $(a, z) \in \mathbb{R}^2$ .

Then

$$(8) \quad |||f'||||_{\infty, \mathbb{R}} \leq \nu \left( \frac{K}{\Gamma(\nu+1)} \right)^{\frac{1}{\nu}} \left( \frac{|||f|||_{\infty, \mathbb{R}}}{\nu-1} \right)^{\frac{\nu-1}{\nu}},$$

and

$$(9) \quad |||f''|||_{\infty, \mathbb{R}} \leq \nu \left( \frac{K}{\Gamma(\nu+1)} \right)^{\frac{2}{\nu}} \left( \frac{4|||f|||_{\infty, \mathbb{R}}}{\nu-2} \right)^{\frac{\nu-2}{\nu}}.$$

That is  $|||f'||||_{\infty, \mathbb{R}}, |||f''|||_{\infty, \mathbb{R}} < \infty$ .

**Proof.** Here  $2 < \nu \leq 3$ , i.e.  $[\nu] = 3$ . Let  $f \in C^3(\mathbb{R}, X)$ , where  $(X, \|\cdot\|)$  is a Banach space,  $a \in \mathbb{R}$ .

We need the following abstract fractional Taylor formulae.

By Theorem 5.8, p. 151 ([4]) we get,

$$(10) \quad \begin{aligned} f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2}f''(a) = \\ = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^\nu f)(z) dz, \quad \forall x \geq a. \end{aligned}$$

And by Theorem 5.9, p. 152 ([4]), see also Definition 5.10, p. 152 ([4]):

$$(11) \quad \begin{aligned} f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2}f''(a) = \\ = \frac{1}{\Gamma(\nu)} \int_x^a (z-x)^{\nu-1} (D_{a-}^\nu f)(z) dz, \quad \forall x \leq a. \end{aligned}$$

The remainders are continuous functions.

Let  $x_1 > a$ , then

$$(12) \quad \begin{aligned} f(x_1) - f(a) - (x_1-a)f'(a) - \frac{(x_1-a)^2}{2}f''(a) = \\ = \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1-z)^{\nu-1} (D_{*a}^\nu f)(z) dz, \end{aligned}$$

and for  $x_2 < a$ , we get

$$(13) \quad \begin{aligned} f(x_2) - f(a) - (x_2-a)f'(a) - \frac{(x_2-a)^2}{2}f''(a) = \\ = \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z-x_2)^{\nu-1} (D_{a-}^\nu f)(z) dz. \end{aligned}$$

That is, we have the system of two equations with unknowns  $f'(a), f''(a)$ :

$$(14) \quad \begin{aligned} & (x_1 - a) f'(a) + \frac{(x_1 - a)^2}{2} f''(a) = \\ & = f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^\nu f)(z) dz =: A, \end{aligned}$$

and

$$(15) \quad \begin{aligned} & (x_2 - a) f'(a) + \frac{(x_2 - a)^2}{2} f''(a) = \\ & = f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^\nu f)(z) dz =: B. \end{aligned}$$

The coefficient determinant is  $D = \frac{1}{2} (x_1 - a) (x_2 - a) (x_2 - x_1)$ . It is  $D \neq 0$ .

We have the system:

$$(16) \quad \begin{aligned} & (x_1 - a) f'(a) + \frac{(x_1 - a)^2}{2} f''(a) = A, \\ & (x_2 - a) f'(a) + \frac{(x_2 - a)^2}{2} f''(a) = B. \end{aligned}$$

Solving the system (16), it has a unique non-trivial solution, we get:

$$(17) \quad f'(a) = \frac{\begin{vmatrix} A & \frac{(x_1 - a)^2}{2} \\ B & \frac{(x_2 - a)^2}{2} \end{vmatrix}}{\frac{1}{2} (x_1 - a) (x_2 - a) (x_2 - x_1)} = \frac{A (x_2 - a)^2 - B (x_1 - a)^2}{(x_1 - a) (x_2 - a) (x_2 - x_1)},$$

and

$$(18) \quad f''(a) = \frac{\begin{vmatrix} x_1 - a & A \\ x_2 - a & B \end{vmatrix}}{\frac{1}{2} (x_1 - a) (x_2 - a) (x_2 - x_1)} = \frac{2 [B (x_1 - a) - A (x_2 - a)]}{(x_1 - a) (x_2 - a) (x_2 - x_1)}.$$

We derive that

$$(19) \quad \begin{aligned} f'(a) = & \left[ \left[ f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^\nu f)(z) dz \right] (x_2 - a)^2 - \right. \\ & \left. - \left[ f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^\nu f)(z) dz \right] (x_1 - a)^2 \right] \\ & \cdot \frac{1}{(x_1 - a) (x_2 - a) (x_2 - x_1)}, \end{aligned}$$

and

$$\begin{aligned}
 f''(a) = 2 & \left[ \left[ f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right] (x_1 - a) - \right. \\
 (20) \quad & \left. - \left[ f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right] (x_2 - a) \right] \cdot \\
 & \cdot \frac{1}{(x_1 - a)(x_2 - a)(x_2 - x_1)}.
 \end{aligned}$$

Simplifying things we have

$$\begin{aligned}
 f'(a) = & \frac{\left[ f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right] (x_2 - a)}{(x_1 - a)(x_2 - x_1)} - \\
 (21) \quad & - \frac{\left[ f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right] (x_1 - a)}{(x_2 - a)(x_2 - x_1)},
 \end{aligned}$$

and

$$\begin{aligned}
 f''(a) = & \frac{2 \left[ f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right]}{(x_2 - a)(x_2 - x_1)} - \\
 (22) \quad & - \frac{2 \left[ f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right]}{(x_1 - a)(x_2 - x_1)}.
 \end{aligned}$$

Rewriting above solutions we have

$$\begin{aligned}
 f'(a) = & \frac{(f(x_1) - f(a))(x_2 - a)}{(x_1 - a)(x_2 - x_1)} - \frac{(f(x_2) - f(a))(x_1 - a)}{(x_2 - a)(x_2 - x_1)} - \\
 (23) \quad & - \frac{\left( \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right) (x_2 - a)}{\Gamma(\nu)(x_1 - a)(x_2 - x_1)} + \\
 & + \frac{\left( \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right) (x_1 - a)}{\Gamma(\nu)(x_2 - a)(x_2 - x_1)},
 \end{aligned}$$

and

$$(24) \quad f''(a) = \frac{2(f(x_2) - f(a))}{(x_2 - a)(x_2 - x_1)} - \frac{2(f(x_1) - f(a))}{(x_1 - a)(x_2 - x_1)} - \\ - \frac{2 \left( \int_{x_2}^a (z - x_2)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right)}{\Gamma(\nu)(x_2 - a)(x_2 - x_1)} + \frac{2 \left( \int_a^{x_1} (x_1 - z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right)}{\Gamma(\nu)(x_1 - a)(x_2 - x_1)}.$$

Let now  $h > 0$ , and choose  $x_1 = a + h$  and  $x_2 = a - h$ . That is  $x_1 - a = h$ ,  $x_2 - a = -h$ , and  $x_2 - x_1 = -2h$ .

We derive the following:

$$(25) \quad f'(a) = \frac{(f(a+h) - f(a))}{2h} - \frac{(f(a-h) - f(a))}{2h} - \\ - \frac{\left( \int_a^{a+h} (a+h-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right)}{2h\Gamma(\nu)} + \frac{\left( \int_{a-h}^a (z-a+h)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right)}{2h\Gamma(\nu)},$$

and similarly,

$$(26) \quad f''(a) = \frac{(f(a-h) - f(a))}{h^2} - \frac{(f(a+h) - f(a))}{h^2} - \\ - \frac{\left( \int_{a-h}^a (z-a+h)^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right)}{h^2\Gamma(\nu)} - \frac{\left( \int_a^{a+h} (a+h-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz \right)}{h^2\Gamma(\nu)}.$$

Putting things together we derive

$$(27) \quad f'(a) = \frac{(f(a+h) - f(a-h))}{2h} - \frac{1}{2h\Gamma(\nu)} \cdot \\ \cdot \left[ \int_a^{a+h} (a+h-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz - \int_{a-h}^a (z-(a-h))^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right],$$

and

$$(28) \quad f''(a) = \frac{(f(a+h) + f(a-h) - 2f(a))}{h^2} - \frac{1}{h^2\Gamma(\nu)} \cdot \\ \cdot \left[ \int_a^{a+h} (a+h-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz + \int_{a-h}^a (z-(a-h))^{\nu-1} (D_{a-}^{\nu} f)(z) dz \right].$$

By Corollary 3 we obtain that  $(D_{*a}^\nu f)(z)$ ,  $(D_{a-}^\nu f)(z)$  are jointly continuous functions in  $(z, a)$  from  $\mathbb{R}^2$  into  $X$ , where  $X$  is a Banach space. We assumed here that  $|||D_{*a}^\nu f(z)|||_{\infty, \mathbb{R}^2}$ ,  $|||D_{a-}^\nu f(z)|||_{\infty, \mathbb{R}^2} < \infty$ .

By (27) we get that

$$\begin{aligned}
 (29) \quad & \|f'(a)\| \leq \frac{\|f(a+h) - f(a-h)\|}{2h} + \\
 & + \frac{1}{2h\Gamma(\nu)} \left[ \int_a^{a+h} (a+h-z)^{\nu-1} \|(D_{*a}^\nu f)(z)\| dz + \right. \\
 & \left. + \int_{a-h}^a (z-(a-h))^{\nu-1} \|(D_{a-}^\nu f)(z)\| dz \right] \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \\
 & + \frac{K}{2h\Gamma(\nu)} \left[ \int_a^{a+h} (a+h-z)^{\nu-1} dz + \int_{a-h}^a (z-(a-h))^{\nu-1} dz \right] = \\
 & = \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{K}{2h\Gamma(\nu)} \left[ \frac{2h^\nu}{\nu} \right] = \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{Kh^{\nu-1}}{\Gamma(\nu+1)}.
 \end{aligned}$$

That is

$$(30) \quad \|f'(a)\| \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{Kh^{\nu-1}}{\Gamma(\nu+1)},$$

$\forall a \in \mathbb{R}, \forall h > 0$ .

I.e. it holds

$$(31) \quad |||f' |||_{\infty, \mathbb{R}} \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{Kh^{\nu-1}}{\Gamma(\nu+1)},$$

$\forall h > 0, 2 < \nu \leq 3$ .

Call

$$\begin{aligned}
 (32) \quad & \mu := |||f|||_{\infty, \mathbb{R}}, \\
 & \theta = \frac{K}{\Gamma(\nu+1)},
 \end{aligned}$$

both are greater than zero.

Set also  $\rho := \nu - 1 > 1$ .

We consider the function

$$(33) \quad y(h) := \frac{\mu}{h} + \theta h^\rho, \quad \forall h > 0.$$

We have

$$(34) \quad y'(h) = -\frac{\mu}{h^2} + \rho\theta h^{\rho-1} = 0,$$

then

$$\rho\theta h^{\rho-1} = \frac{\mu}{h^2},$$

and

$$\rho\theta h^{\rho+1} = \mu,$$

with a unique solution

$$(35) \quad h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}.$$

We have that

$$y'(h) = -\mu h^{-2} + \rho\theta h^{\rho-1},$$

and

$$(36) \quad y''(h) = 2\mu h^{-3} + \rho(\rho-1)\theta h^{\rho-2}.$$

We see that

$$\begin{aligned} (37) \quad y''(h_0) &= 2\mu \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} + \rho(\rho-1)\theta \left(\frac{\mu}{\rho\theta}\right)^{\frac{((\rho+1)-3)}{\rho+1}} = \\ &= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} \left[2\mu + \rho(\rho-1)\theta \left(\frac{\mu}{\rho\theta}\right)\right] = \\ &= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} [2\mu + \mu(\rho-1)] = \mu \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} (\rho+1) > 0. \end{aligned}$$

Therefore  $y$  has a global minimum at  $h_0 = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}$ , which is

$$\begin{aligned} (38) \quad y(h_0) &= \frac{\mu}{\left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}} + \theta \left(\frac{\mu}{\rho\theta}\right)^{\frac{\rho}{\rho+1}} = \\ &= (\rho\theta)^{\frac{1}{\rho+1}} \frac{\mu}{\mu^{\frac{1}{\rho+1}}} + \frac{\theta\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}\theta^{\frac{\rho}{\rho+1}}} = (\rho\theta)^{\frac{1}{\rho+1}} \mu^{1-\frac{1}{\rho+1}} + \frac{\theta^{1-\frac{\rho}{\rho+1}}\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}} = \\ &= \rho^{\frac{1}{\rho+1}}\theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}} + \theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}}\rho^{-\frac{\rho}{\rho+1}} = \theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}} \left[\rho^{\frac{1}{\rho+1}} + \rho^{-\frac{\rho}{\rho+1}}\right] = \\ &= (\theta\mu^\rho)^{\frac{1}{\rho+1}} \left(\rho^{\frac{1}{\rho+1}} + \frac{1}{\rho^{\frac{\rho}{\rho+1}}}\right) = (\theta\mu^\rho)^{\frac{1}{\rho+1}} \left(\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\right) = \\ &= (\theta\mu^\rho)^{\frac{1}{\rho+1}} (\rho+1) \rho^{-\frac{\rho}{\rho+1}}. \end{aligned}$$

That is

$$(39) \quad y(h_0) = (\theta\mu^\rho)^{\frac{1}{\rho+1}} (\rho+1) \rho^{-\frac{\rho}{\rho+1}}.$$

Consequently

$$(40) \quad y(h_0) = \left( \frac{K}{\Gamma(\nu+1)} \| \| f \| \|_{\infty, \mathbb{R}}^{\nu-1} \right)^{\frac{1}{\nu}} \nu (\nu-1)^{-\left(\frac{\nu-1}{\nu}\right)}.$$

We have proved that

$$(41) \quad \| \| f' \| \|_{\infty, \mathbb{R}} \leq \left( \frac{K}{\Gamma(\nu+1)} \| \| f \| \|_{\infty, \mathbb{R}}^{\nu-1} \right)^{\frac{1}{\nu}} \nu (\nu-1)^{-\left(\frac{\nu-1}{\nu}\right)}.$$

From (28) we derive

$$(42) \quad \begin{aligned} \| f''(a) \| &\leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \\ &+ \frac{K}{h^2 \Gamma(\nu)} \left[ \int_a^{a+h} (a+h-z)^{\nu-1} dz + \int_{a-h}^a (z-(a-h))^{\nu-1} dz \right] = \\ &= \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{K}{h^2 \Gamma(\nu)} \left[ \frac{2h^\nu}{\nu} \right] = \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{2Kh^{\nu-2}}{\Gamma(\nu+1)}. \end{aligned}$$

That is

$$(43) \quad \| f''(a) \| \leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{2Kh^{\nu-2}}{\Gamma(\nu+1)},$$

$\forall a \in \mathbb{R}, \forall h > 0$ .

Hence it holds

$$(44) \quad \| \| f'' \| \|_{\infty, \mathbb{R}} \leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{2Kh^{\nu-2}}{\Gamma(\nu+1)},$$

$\forall h > 0, 2 < \nu \leq 3$ .

Call

$$(45) \quad \begin{aligned} \xi &:= 4 \| \| f \| \|_{\infty, \mathbb{R}}, \\ \psi &= \frac{2K}{\Gamma(\nu+1)}, \end{aligned}$$

both are greater than zero.

Set also  $\varphi := \nu - 2 > 0$ .

We consider the function

$$(46) \quad \gamma(h) := \frac{\xi}{h^2} + \psi h^\varphi = \xi h^{-2} + \psi h^\varphi, \quad \forall h > 0.$$

We have

$$(47) \quad \gamma'(h) = -2\xi h^{-3} + \varphi \psi h^{\varphi-1} = 0,$$

then

$$\varphi \psi h^{\varphi-1} = 2\xi h^{-3},$$

and

$$\varphi \psi h^{\varphi+2} = 2\xi,$$

with a unique solution

$$(48) \quad h_0 := h_{crit.no} = \left( \frac{2\xi}{\varphi \psi} \right)^{\frac{1}{\varphi+2}}.$$

We have that

$$(49) \quad \gamma''(h) = 6\xi h^{-4} + \varphi(\varphi-1)\psi h^{\varphi-2}.$$

We see that

$$(50) \quad \begin{aligned} \gamma''(h_0) &= 6\xi \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{4}{\varphi+2}} + \varphi(\varphi-1)\psi \left( \frac{2\xi}{\varphi \psi} \right)^{\frac{(\varphi+2)-4}{\varphi+2}} = \\ &= \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{4}{\varphi+2}} \left[ 6\xi + \varphi(\varphi-1)\psi \frac{2\xi}{\varphi \psi} \right] = \\ &= \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{4}{\varphi+2}} [6\xi + (\varphi-1)2\xi] = 2\xi \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{4}{\varphi+2}} (\varphi+2) > 0. \end{aligned}$$

Therefore  $\gamma$  has a global minimum at  $h_0 = \left( \frac{2\xi}{\varphi \psi} \right)^{\frac{1}{\varphi+2}}$ , which is

$$(51) \quad \begin{aligned} \gamma(h_0) &= \xi \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{2}{\varphi+2}} + \psi \left( \frac{2\xi}{\varphi \psi} \right)^{\frac{\varphi+2-2}{\varphi+2}} = \\ &= \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{2}{\varphi+2}} \left[ \xi + \psi \frac{2\xi}{\varphi \psi} \right] = \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{2}{\varphi+2}} \left[ 1 + \frac{2}{\varphi} \right] = \\ &= \frac{\xi}{\varphi} \left( \frac{2\xi}{\varphi \psi} \right)^{-\frac{2}{\varphi+2}} (\varphi+2) = \frac{\xi}{\varphi} \left( \frac{\xi}{\varphi} \right)^{-\frac{2}{\varphi+2}} \left( \frac{2}{\psi} \right)^{-\frac{2}{\varphi+2}} (\varphi+2) = \\ &= \left( \frac{\xi}{\varphi} \right)^{\frac{\varphi}{\varphi+2}} \left( \frac{\psi}{2} \right)^{\frac{2}{\varphi+2}} (\varphi+2). \end{aligned}$$

That is

$$(52) \quad \gamma(h_0) = \left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}} \left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}} (\varphi+2).$$

Consequently

$$(53) \quad \gamma(h_0) = \left(\frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{\nu-2}\right)^{\frac{\nu-2}{\nu}} \left(\frac{K}{\Gamma(\nu+1)}\right)^{\frac{2}{\nu}} \nu.$$

We have proved that

$$(54) \quad \| \| f'' \| \|_{\infty, \mathbb{R}} \leq \left(\frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{\nu-2}\right)^{\frac{\nu-2}{\nu}} \left(\frac{K}{\Gamma(\nu+1)}\right)^{\frac{2}{\nu}} \nu.$$

The theorem is proved. ■

We also give an  $L_p$  analog of a fractional Landau inequality

**Theorem 5.** *Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ ,  $2 < \nu \leq 3$  and  $f \in C^3(\mathbb{R}, X)$ , where  $(X, \|\cdot\|)$  is a Banach space. We assume that  $\| \| f \| \|_{\infty, \mathbb{R}} < \infty$ , and that*

$$(55) \quad M := \max \left\{ \sup_{a \in \mathbb{R}} \| \| D_{*a}^\nu f \| \|_{p, \mathbb{R}}, \sup_{a \in \mathbb{R}} \| \| D_{a-}^\nu f \| \|_{p, \mathbb{R}} \right\} < \infty.$$

Then

1)

$$(56) \quad \begin{aligned} & \| \| f' \| \|_{\infty, \mathbb{R}} \leq \\ & \leq \left(\nu - \frac{1}{p}\right) \left(\frac{M}{\Gamma(\nu) (q(\nu-1) + 1)^{\frac{1}{q}}}\right)^{\frac{1}{(\nu-\frac{1}{p})}} \left(\| \| f \| \|_{\infty, \mathbb{R}}\right)^{\left(\frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}}\right)}, \end{aligned}$$

and

2) under the additional assumption  $2 + \frac{1}{p} < \nu \leq 3$ , we have

$$(57) \quad \begin{aligned} & \| \| f'' \| \|_{\infty, \mathbb{R}} \leq \\ & \leq \left(\nu - \frac{1}{p}\right) \left(\frac{M}{\Gamma(\nu) (q(\nu-1) + 1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\nu-\frac{1}{p}}\right)} \left(\frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{\nu-2-\frac{1}{p}}\right)^{\left(\frac{\nu-2-\frac{1}{p}}{\nu-\frac{1}{p}}\right)}. \end{aligned}$$

That is  $\| \| f' \| \|_{\infty, \mathbb{R}}, \| \| f'' \| \|_{\infty, \mathbb{R}} < \infty$ .

**Proof.** From (27) we get

$$\begin{aligned}
 \|f'(a)\| &\leq \frac{\|f(a+h) - f(a-h)\|}{2h} + \\
 &+ \frac{1}{2h\Gamma(\nu)} \left[ \int_a^{a+h} (a+h-z)^{\nu-1} \|(D_{*a}^\nu f)(z)\| dz + \right. \\
 &+ \left. \int_{a-h}^a (z-(a-h))^{\nu-1} \|(D_{a-}^\nu f)(z)\| dz \right] \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \\
 (58) \quad &+ \frac{1}{2h\Gamma(\nu)} \left[ \left( \int_a^{a+h} (a+h-z)^{q(\nu-1)} dz \right)^{\frac{1}{q}} \left( \int_a^{a+h} \|(D_{*a}^\nu f)(z)\|^p dz \right)^{\frac{1}{p}} + \right. \\
 &+ \left. \left( \int_{a-h}^a (z-(a-h))^{q(\nu-1)} dz \right)^{\frac{1}{q}} \left( \int_{a-h}^a \|(D_{a-}^\nu f)(z)\|^p dz \right)^{\frac{1}{p}} \right] \leq \\
 &\leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{1}{2h\Gamma(\nu)} \frac{h^{\nu-\frac{1}{p}}}{(q(\nu-1)+1)^{\frac{1}{q}}} \cdot \\
 &\cdot \left( \sup_{a \in \mathbb{R}} \|D_{*a}^\nu f\|_{p, \mathbb{R}} + \sup_{a \in \mathbb{R}} \|D_{a-}^\nu f\|_{p, \mathbb{R}} \right) \leq \\
 &\leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{Mh^{(\nu-1-\frac{1}{p})}}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}.
 \end{aligned}$$

That is

$$(59) \quad \|f'(a)\| \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} h^{(\nu-1-\frac{1}{p})},$$

$\forall a \in \mathbb{R}, \forall h > 0$ .

I.e. it holds

$$(60) \quad |||f' |||_{\infty, \mathbb{R}} \leq \frac{|||f|||_{\infty, \mathbb{R}}}{h} + \frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} h^{(\nu-1-\frac{1}{p})},$$

$\forall h > 0, 2 < \nu \leq 3$ .

Call

$$\begin{aligned}
 (61) \quad \mu &:= |||f|||_{\infty, \mathbb{R}}, \\
 \theta &= \frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}},
 \end{aligned}$$

both are greater than zero.

Set also  $\rho := \nu - 1 - \frac{1}{p} > \frac{1}{q} > 0$ .

We consider the function

$$(62) \quad y(h) := \frac{\mu}{h} + \theta h^\rho, \quad \forall h > 0.$$

As in the proof of Theorem 4 it has only one critical number

$$(63) \quad h_0 := h_{crit.no} = \left( \frac{\mu}{\rho\theta} \right)^{\frac{1}{\rho+1}},$$

and a global minimum

$$(64) \quad y(h_0) = \theta^{\frac{1}{\rho+1}} \mu^{\frac{\rho}{\rho+1}} (\rho + 1) \rho^{-\frac{\rho}{\rho+1}}.$$

Consequently

$$(65) \quad y(h_0) = \left( \frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} \right)^{\frac{1}{(\nu-\frac{1}{p})}} \left( |||f|||_{\infty, \mathbb{R}} \right)^{\left( \frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}} \right)} \cdot \left( \nu - \frac{1}{p} \right) \left( \nu - 1 - \frac{1}{p} \right)^{-\left( \frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}} \right)}.$$

We have proved that

$$(66) \quad |||f'|||_{\infty, \mathbb{R}} \leq \left( \frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} \right)^{\frac{1}{(\nu-\frac{1}{p})}} \left( \frac{|||f|||_{\infty, \mathbb{R}}}{\nu-1-\frac{1}{p}} \right)^{\left( \frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}} \right)} \left( \nu - \frac{1}{p} \right).$$

From (28) we derive

$$(67) \quad \begin{aligned} \|f''(a)\| &\leq \frac{4 |||f|||_{\infty, \mathbb{R}}}{h^2} + \\ &+ \frac{1}{h^2 \Gamma(\nu)} \left[ \int_a^{a+h} (a+h-z)^{\nu-1} \|(D_{*a}^\nu f)(z)\| dz + \right. \\ &+ \left. \int_{a-h}^a (z-(a-h))^{\nu-1} \|(D_{a-}^\nu f)(z)\| dz \right] \leq \frac{4 |||f|||_{\infty, \mathbb{R}}}{h^2} + \\ &+ \frac{1}{h^2 \Gamma(\nu)} \left[ \left( \int_a^{a+h} (a+h-z)^{q(\nu-1)} dz \right)^{\frac{1}{q}} \sup_{a \in \mathbb{R}} |||D_{*a}^\nu f|||_{p, \mathbb{R}} + \right. \end{aligned}$$

$$\begin{aligned}
(67) \quad & + \left( \int_{a-h}^a (z - (a-h))^{q(\nu-1)} dz \right)^{\frac{1}{q}} \sup_{a \in \mathbb{R}} \| \| D_{a-}^{\nu} f \| \|_{p, \mathbb{R}} \Big] \leq \\
& \leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{M}{h^2 \Gamma(\nu)} \left( \frac{2h^{\nu-\frac{1}{p}}}{(q(\nu-1)+1)^{\frac{1}{q}}} \right) = \\
& = \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \frac{2Mh^{\nu-2-\frac{1}{p}}}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}.
\end{aligned}$$

That is

$$(68) \quad \| f''(a) \| \leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \left( \frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} \right) h^{\nu-2-\frac{1}{p}},$$

$\forall a \in \mathbb{R}, \forall h > 0$ .

I.e. it holds

$$(69) \quad \| \| f'' \| \|_{\infty, \mathbb{R}} \leq \frac{4 \| \| f \| \|_{\infty, \mathbb{R}}}{h^2} + \left( \frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}} \right) h^{\nu-2-\frac{1}{p}},$$

$\forall h > 0$ , under  $2 + \frac{1}{p} < \nu \leq 3$ .

Call

$$\begin{aligned}
(70) \quad & \xi := 4 \| \| f \| \|_{\infty, \mathbb{R}}, \\
& \psi := \frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}},
\end{aligned}$$

both are greater than zero.

Set also  $\varphi := \nu - 2 - \frac{1}{p} > 0$ .

We consider the function

$$(71) \quad \gamma(h) := \frac{\xi}{h^2} + \psi h^{\varphi}, \quad \forall h > 0.$$

We have as in the proof of Theorem 4 that  $\gamma$  has a global minimum at

$$(72) \quad h_0 = \left( \frac{2\xi}{\varphi\psi} \right)^{\frac{1}{\varphi+2}},$$

which is

$$(73) \quad \gamma(h_0) = \left( \frac{\xi}{\varphi} \right)^{\frac{\varphi}{\varphi+2}} \left( \frac{\psi}{2} \right)^{\frac{2}{\varphi+2}} (\varphi + 2).$$

Consequently

$$(74) \quad \gamma(h_0) = \left( \frac{4 |||f|||_{\infty, \mathbb{R}}}{\nu - 2 - \frac{1}{p}} \right)^{\left( \frac{\nu - 2 - \frac{1}{p}}{\nu - \frac{1}{p}} \right)} \left( \frac{M}{\Gamma(\nu)(q(\nu - 1) + 1)^{\frac{1}{q}}} \right)^{\left( \frac{2}{\nu - \frac{1}{p}} \right)} \left( \nu - \frac{1}{p} \right).$$

We have proved that

$$(75) \quad |||f''|||_{\infty, \mathbb{R}} \leq \left( \frac{4 |||f|||_{\infty, \mathbb{R}}}{\nu - 2 - \frac{1}{p}} \right)^{\left( \frac{\nu - 2 - \frac{1}{p}}{\nu - \frac{1}{p}} \right)} \left( \frac{M}{\Gamma(\nu)(q(\nu - 1) + 1)^{\frac{1}{q}}} \right)^{\left( \frac{2}{\nu - \frac{1}{p}} \right)} \left( \nu - \frac{1}{p} \right).$$

The theorem is proved. ■

We give

**Corollary 6.** (case of  $\nu = 2.5$ ) Let  $f \in C^3(\mathbb{R}, X)$ , where  $(X, \|\cdot\|)$  is a Banach space. We assume that  $|||f|||_{\infty, \mathbb{R}} < \infty$ , and that

$$(76) \quad K_{2.5} := \max \left\{ |||D_{*a}^{2.5} f(z)|||_{\infty, \mathbb{R}^2}, |||D_{a-}^{2.5} f(z)|||_{\infty, \mathbb{R}^2} \right\} < \infty,$$

where  $(a, z) \in \mathbb{R}^2$ .

Then

$$(77) \quad |||f'||||_{\infty, \mathbb{R}} \leq 1.21136 (K_{2.5})^{0.4} \left( |||f|||_{\infty, \mathbb{R}} \right)^{0.6},$$

and

$$(78) \quad |||f''|||_{\infty, \mathbb{R}} \leq 1.44713 (K_{2.5})^{0.8} \left( |||f|||_{\infty, \mathbb{R}} \right)^{0.2}.$$

That is  $|||f'||||_{\infty, \mathbb{R}}, |||f''|||_{\infty, \mathbb{R}} < \infty$ .

**Proof.** By Theorem 4. ■

We finish with

**Corollary 7.** (case of  $\nu = 2.5$ ,  $p = q = 2$ ) Let  $f \in C^3(\mathbb{R}, X)$ , where  $(X, \|\cdot\|)$  is a Banach space. We assume that  $|||f|||_{\infty, \mathbb{R}} < \infty$ , and that

$$(79) \quad M_{2.5} := \max \left\{ \sup_{a \in \mathbb{R}} |||D_{*a}^{2.5} f|||_{2, \mathbb{R}}, \sup_{a \in \mathbb{R}} |||D_{a-}^{2.5} f|||_{2, \mathbb{R}} \right\} < \infty.$$

Then

$$(80) \quad |||f'| |||_{\infty, \mathbb{R}} \leq 1.226583057 \sqrt{M_{2.5}} |||f|||_{\infty, \mathbb{R}}.$$

That is  $|||f'| |||_{\infty, \mathbb{R}} < \infty$ .

**Proof.** By Theorem 5. ■

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