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EXTENDING THE APPLICABILITY OF ITERATIVE METHODS FREE OF DERIVATIVES BY MEANS OF MEMORY

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Abstract. Iterative methods free of derivatives are useful for generating a sequence approximating a solution of nonlinear equations involving Banach space valued operators. The high convergence order of these methods is important too. The aim of this article is to extend the convergence of these methods in cases not covered before. Numerical experiments test our theoretical results.

1. Introduction

A plethora of applications from various disciplines results to finding a solution x_* of the equation

$$F(x) = 0, \tag{1.1}$$

with $F : D \subset B_1 \to B_2$ is being differentiable as by Frechét, where B_1, B_2 stand for Banach spaces and D being an open and convex set. The solution x_* is elusive in many cases as a closed form. Therefore, researchers and practitioners develop iterative methods producing sequences converging to the solution under some sufficient semilocal criteria on the initial data. Some popular and well studied methods are:

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Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

Traub's-Steffensen's method:

$$x_{n+1} = x_n - [w_n, x_n; F]^{-1}F(x_n), \tag{1.3}$$

where $x_0 \in D$, $n = 0, 1, 2, \ldots$, $w_n = x_n + bF(x_n)$, $b \in \mathbb{R}$ and $[\cdot, \cdot; F] : D \times D \to \mathcal{L}(B_1, B_2)$ is the standard divided difference of order one [2, 12]. If $b = 1$, (1.3) reduces to the popular Steffensen's method [4, 11, 12, 13].

Shamanskii in [18] (when $B_1 = B_2 = \mathbb{R}^k$) showed that method (1.3) is of order two if we start close enough to the solution. Notice that method (1.3) is not using derivatives (as in the case of methods (1.2) also of order two) which are expensive to compute in general. Many attempts have been made to increase the order of convergence for iterative methods [6, 7, 8, 11].

Recently, in particular the local convergence of the following methods has been studied in [5] when $B_1 = B_2 = \mathbb{R}^k$:

$$\begin{aligned} y_n &= x_n - A_n^{-1}F(x_n), \\ x_{n+1} &= y_n - B_n^{-1}F(y_n), \end{aligned} \tag{1.4}$$

and the Kurchatov-type method with memory

$$\begin{aligned} w_n &= x_n - C_n^{-1}F(x_n), \\ y_n &= x_n - A_n^{-1}F(x_n), \\ x_{n+1} &= y_n - B_n^{-1}F(y_n), \end{aligned} \tag{1.5}$$

where $A_n = [w_n, x_n; F], B_n = [w_n, y_n; F]$ and $C_n = [2x_n - x_{n-1}, x_{n-1}; F]$, $x_0, x_{-1} \in D$, $n = 0,1,2,\ldots$.

Method (1.4) is shown to be of order three using hypotheses up to the fifth derivative, whereas method (1.5) is of order fifth. But these high order derivatives are not on these methods.

Note that Kurchatov's method

$$x_{n+1} = x_n - C_n^{-1}F(x_n),\ x_0, x_{-1} \in D,\ n = 0,1,2,\ldots \tag{1.6}$$

was proposed in [10]. It was later explored in [1, 2, 15, 16, 17]. Other modifications of Kurchatov's method are discussed in [3, 14].

These hypotheses limit the applicability of the methods. Indeed, consider the academic and motivational example, when $B_1 = B_2 = \mathbb{R}$ and $D = [-\frac{1}{2}, \frac{3}{2}]$ given by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, $f'''(x)$ is unbounded on D. Hence, the results in [5] do not guarantee the convergence of method (1.4) or (1.5) to $x_* = 1$ (although these methods may converge). Other problems with the analysis in [5] include no computable bounds on $\|x_n - x_*\|$, results on the uniqueness of x_* in D or how the initial point x_0 is chosen. Based on the above it is clear that a weaker convergence analysis is needed that also addresses all these problems. The novelty of our paper lies in the fact that we only use hypotheses on the derivative of order one. Our idea can be used to extend the usage of other methods using inverses in a similar fashion [1] – [18].

2. Convergence for method (1.4)

It is convenient to introduce some scalar functions and constants to be used in the local convergence. Let $\omega_0 : [0,\infty)\times[0,\infty) \to [0,\infty)$ be a continuous and increasing function satisfying $\omega_0(0,0) = 0$ and let $\alpha \geq 0$. Suppose equation

$$\omega_0(\alpha t, t) = 1 \tag{2.1}$$

has a minimal positive root ρ_1. Let $\omega : [0,\rho_1)\times[0,\rho_1) \to [0,\infty)$ be a continuous and increasing function satisfying $\omega(0,0) = 0$. Define functions g_1 and g_2 on the interval $[0,\rho_1)$ by

$$g_1(t) = \frac{\omega((1+\alpha)t, t)}{1-\omega_0(\alpha t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

By these definitions $h_1(0) = -1$ and $h_1(t) \to \infty$ as $t \to \rho_1^-$. Denote by r_1 the minimal root of equation $h_1(t) = 0$ in $(0,\rho_1)$ assured to exist by the intermediate value theorem.

Suppose that equation

$$\omega_0(\alpha t, g_1(t)t) = 1 \tag{2.2}$$

has a minimal positive root ρ_2. Set $\rho = \min\{\rho_1, \rho_2\}$. Define functions g_2 and h_2 on the interval $[0,\rho)$ by

$$g_2(t) = \frac{\omega(\alpha t + g_1(t)t, g_1(t)t)g_1(t)}{1-\omega_0(\alpha t, g_1(t)t)}$$

and

$$h_2(t) = g_2(t) - 1.$$

We get again $h_2(0) = -1$ and $h_2(t) \to \infty$ as $t \to \rho_2^-$. Denote by r_2 the minimal root of equation $h_2(t) = 0$ in $(0, \rho)$. Define a radius of convergence r by

$$r = \min\{r_1, r_2\}. \tag{2.3}$$

Then, for all $t \in [0, r)$ we obtain

$$0 \le \omega_0(\alpha t, t) < 1, \tag{2.4}$$

$$0 \le \omega_0(\alpha t, g_1(t)t) < 1, \tag{2.5}$$

$$0 \le g_1(t) < 1 \tag{2.6}$$

and

$$0 \le g_2(t) < 1. \tag{2.7}$$

we denote by $U(v, s)$ and $\bar{U}(v, s)$ the open and closed balls in B_1 of center $v \in D$ and with radius $s > 0$.

The following conditions (A) are utilized in our analysis:

(A_1) $F : D \to B_2$ is differentiable, $[\cdot, \cdot; F] : D \times D \to \mathcal{L}(B_1, B_2)$ is a standard divided difference of order one, and $x_* \in D$ exists with $F(x_*) = 0$ which is simple.

(A_2) $\omega_0 : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous and increasing function with $\omega_0(0, 0) = 0$ such that for all $x, y \in D$

$$\|F'(x_*)^{-1}([x, y; F] - F'(x_*))\| \le \omega_0(\|x - x_*\|, \|y - x_*\|)$$

and

$$\|I + b[x, x_*; F]\| \le \alpha \|x - x_*\|$$

for some $\alpha \ge 0$.

Set $D_0 = D \bigcap U(x_*, \rho_1)$.

(A_3) $\omega : [0, \rho_1) \times [0, \rho_1) \to [0, \infty)$ is a continuous and increasing function with $\omega(0, 0) = 0$ such that for all $x, y \in D_0$

$$\|F'(x_*)^{-1}([x, y; F] - [y, x_*; F])\| \le \omega(\|x - y\|, \|y - x_*\|).$$

(A_4) $U(x_*, r_*) \subset D$, $r_* = \max\{r, \alpha r\}$, ρ_1, ρ_2 given in (2.1) and (2.2) exist and r is defined in (2.3).

(A_5) There exist $\bar{r} \ge r$ such that

$$\omega_0(0, \bar{r}) < 1 \quad \text{or} \quad \omega_0(\bar{\mathrm{r}}, 0) < 1.$$

Set $D_1 = D \bigcap \bar{U}(x_*, r_*)$.

Next, the conditions (A) together with the preceding notation are utilized in the local convergence analysis of method (1.4).

Theorem 2.1. *Assume that conditions* (A) *hold and choose starting point* $x_0 \in U(x_*, r) - \{x_*\}$. *Then, the following items hold*

$$x_n \in U(x_*, r), \tag{2.8}$$

$$\lim_{n\to\infty} x_n = x_*, \tag{2.9}$$

$$\|w_n - x_*\| \leq \alpha \|x_n - x_*\| \leq \alpha r, \tag{2.10}$$

$$\|y_n - x_*\| \leq g_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \leq r, \tag{2.11}$$

$$\|x_{n+1} - x_*\| \leq g_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.12}$$

and x_* *is the only solution of equation* $F(x) = 0$ *in the set* D_1 *given in* (A_5).

Proof. By (A_2), (2.1), (2.3), (2.4) and choice $x_0 \in U(x_*, r) - \{x_*\}$, we have

$$\begin{aligned} \|w_0 - x_*\| &= \|(I + b[x_0, x_*; F])(x_0 - x_*)\| \leq \\ &\leq \|(I + b[x_0, x_*; F])\|\|x_0 - x_*\| \leq \alpha\|x_0 - x_*\| \leq \alpha r \end{aligned} \tag{2.13}$$

and

$$\|F'(x_*)^{-1}(A_0 - F'(x_*))\| \leq \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|) \leq \omega_0(\alpha r, r) < 1. \tag{2.14}$$

Consequently, by Lemma of Banach for invertible operator [9] and (2.14), we get A_0 is invertible and

$$\|A_0^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)}. \tag{2.15}$$

Hence, y_0 is well defined by the method (1.4) for $n = 0$. Using (2.3), (A_3), (2.15) and for first substep of method (1.4) for $n = 0$, we obtain

$$\begin{aligned} \|y_0 - x_*\| &= \|x_0 - x_* - A_0^{-1}F(x_0)\| = \\ &= \|[A_0^{-1}F'(x_*)][F'(x_*)^{-1}(A_0 - [x_0, x_*; F])\|\|x_0 - x_*\|)\| \leq \\ &\leq \|A_0^{-1}F'(x_*)\|\|F'(x_*)^{-1}(A_0 - [x_0, x_*; F])\|\|x_0 - x_*\|)\| \leq \\ &\leq \frac{\omega(\|w_0 - x_*\|, \|x_0 - x_*\|)\|x_0 - x_*\|}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)} \leq \\ &\leq \frac{\omega((1+\alpha)\|x_0 - x_*\|, \|x_0 - x_*\|)}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)}\|x_0 - x_*\| \leq \\ &\leq g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{2.16}$$

so $y_0 \in U(x_*, r)$ and (2.10),(2.11) hold for $n = 0$. Moreover, we have by the second substep of method (1.4), (2.3),(2.15) and (2.16) that

$$\begin{aligned} \|x_1 - x_*\| &= \|[B_0^{-1}F'(x_*)][F'(x_*)^{-1}(B_0 - [y_0, x_*; F])(y_0 - x_*)\| \leq \\ &\leq \|B_0^{-1}F'(x_*)\|\|F'(x_*)^{-1}(B_0 - [y_0, x_*; F])\|\|y_0 - x_*\| \leq \\ &\leq \frac{\omega(\|w_0 - y_0\|, \|y_0 - x_*\|)}{1 - \omega_0(\alpha\|x_0 - x_*\|, \|y_0 - x_*\|)}\|y_0 - x_*\| \leq \\ &\leq g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{2.17}$$

showing $x_1 \in U(x_*, r)$ and (2.12) for $n = 0$.

These computations can be repeated with x_i, w_i, y_i, x_{i+1} replacing x_0, w_0, y_0, x_1 in the preceding computions to end the induction for (2.8)–(2.12). Furthermore, in view of the estimate

$$\|x_{i+1} - x_*\| \leq c\|x_i - x_*\| < r, \tag{2.18}$$

where $c = g_2(\|x_0 - x_*\|) \in [0, 1)$, we conclude $x_{i+1} \in U(x_*, r)$ and $\lim_{i\to\infty} x_i = x_*$. Finally, let $Q = [x_*, y_*; F]$ for $y_* \in D_1$ with $F(y_*) = 0$. Then, using (A_2) and (A_5), we get in turn that

$$\|F'(x_*)^{-1}(Q - F'(x_*))\| \leq \omega_0(\|x_* - x_*\|, \|y_* - x_*\|) \leq \omega(0, \bar{r}) < 1, \tag{2.19}$$

so Q is invertible. Therefore, $x_* = y_*$ is obtained from

$$0 = F(x_*) - F(y_*) = [x_*, y_*; F](x_* - y_*) = Q(x_* - y_*). \qquad \blacksquare$$

3. Convergence for method (1.5)

As in Section 2, suppose equation

$$\omega_0(3t, t) = 1 \tag{3.1}$$

has a minimal positive solution ρ_0. Define functions g_0 and h_0 on the interval $[0, \rho_0)$ by

$$g_0(t) = \frac{\omega(2t, t)}{1 - \omega_0(3t, t)}$$

and

$$h_0(t) = g_0(t) - 1.$$

Denote by r_0 the minimal solution of equation $h_0(t) = 0$ in $(0, \rho_0)$.

Suppose that equation

$$\omega_0(g_0(t)t, t) = 1 \tag{3.2}$$

has a minimal positive solution ρ_1.

Define functions g_1 and h_1 on the interval $[0, \min\{\rho_0, \rho_1\})$ by

$$g_1(t) = \frac{\omega((1 + g_0(t))t, t)}{1 - \omega_0(g_0(t)t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Denote by r_1 the minimal solution of equation $h_1(t) = 0$ in $[0, \min\{\rho_0, \rho_1\})$. Suppose that equation

$$\omega_0(g_0(t)t, g_1(t)t) = 1 \tag{3.3}$$

has a minimal positive solution ρ_2. Set $\rho = \min\{\rho_0, \rho_1, \rho_2\}$. Define functions g_2 and h_2 on interval $[0, \rho)$ by

$$g_2(t) = \frac{\omega(g_0(t)t + g_1(t)t, g_1(t)t)g_1(t)}{1 - \omega_0(g_0(t)t, g_1(t)t)}$$

and

$$h_2(t) = g_2(t) - 1.$$

Denote by r_2 the minimal solution of equation $h_2(t) = 0$ in $(0, \rho)$. Define a radius of convergence r by

$$r = \min\{r_0, r_1, r_2\}. \tag{3.4}$$

Then, for all $t \in [0, r)$

$$0 \leq \omega_0(3t, t) < 1, \tag{3.5}$$

$$0 \leq \omega_0(g_0(t)t, t) < 1, \tag{3.6}$$

$$0 \leq \omega_0(g_0(t)t, g_1(t)t) < 1, \tag{3.7}$$

$$0 \leq g_0(t) < 1, \tag{3.8}$$

$$0 \leq g_1(t) < 1 \tag{3.9}$$

and

$$0 \leq g_2(t) < 1. \tag{3.10}$$

We shall use conditions (H):

$(H_1) = (A_1)$.

(H_2) $\omega_0 : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and increasing function with $\omega_0(0,0) = 0$ such that for all $x, y \in D$

$$\|F'(x_*)^{-1}([x,y;F] - F'(x_*))\| \leq \omega_0(\|x - x_*\|, \|y - x_*\|)$$

and

$$\|I + b[x, x_*; F]\| \leq \alpha \|x - x_*\|.$$

Set $D_0 = D \bigcap U(x_*, \rho_0)$.

$(H_3) = (A_3)$.

(H_4) $U(x_*, 3r) \subset D$, where r is defined in (3.3) and ρ_0, ρ_1, ρ_2 exist and are given by (3.1), (3.2) and (3.3), respectively.

$(H_5) = (A_5)$.

Next, the conditions (H) and the preceding notation give the local convergence analysis of method (1.5).

Theorem 3.1. *Assume that conditions (H) hold and choose starting points $x_{-1}, x_0 \in U(x_*, r) - \{x_*\}$. Then, the following items hold*

$$x_n \in U(x_*, r),$$

$$\lim_{n\to\infty} x_n = x_*,$$

$$\begin{aligned} \|w_n - x_*\| &\leq g_0(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \\ \|y_n - x_*\| &\leq g_1(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r \end{aligned}$$

and

$$\|x_{n+1} - x_*\| \leq g_2(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r,$$

and x_* is the only solution of equation $F(x) = 0$ in the set D_1 given in (H_5).

Proof. As in the proof of Theorem 2.1, we use mathematical induction to obtain

$$\begin{aligned} \|w_n - x_*\| &= \|[C_n^{-1}F'(x_*)][F'(x_*)^{-1}(C_n - [x_n, x_*; F])(x_n - x_*)]\| \leq \\ &\leq \frac{\omega(\|2x_n - x_{n-1} - x_n\|, \|x_{n-1} - x_*\|)\|x_n - x_*\|}{1 - \omega_0(\|2x_n - x_{n-1} - x_*\|, \|x_{n-1} - x_*\|)} \leq \\ &\leq \frac{\omega(\|x_n - x_*\| + \|x_{n-1} - x_*\|, \|x_{n-1} - x_*\|)\|x_n - x_*\|}{1 - \omega_0(2\|x_n - x_*\| + \|x_{n-1} - x_*\|, \|x_{n-1} - x_*\|)} \leq \\ &\leq g_0(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \end{aligned}$$

$\|2x_n - x_{n-1} - x_*\| \leq \|x_n - x_*\| + \|x_{n-1} - x_n\| \leq \leq 2\|x_n - x_*\| + \|x_{n-1} - x_*\| \leq 3r,$

and as in (2.16), (2.17), respectively

$$\begin{aligned} \|y_n - x_*\| &\leq g_1(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \\ \|x_{n+1} - x_*\| &\leq g_2(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r. \end{aligned}$$ ■

4. Numerical examples

We use $[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta$ and $b = 1$ in all examples.

Example 4.1. Define $D = U(0, 1)$, $v = (v_1, v_2, v_3)^T$, function $F : D \to \mathbb{R}^3$ by

$$F(v) = \left(e^{v_1} - 1, \frac{e-1}{2}v_2^2 + v_2, v_3\right)^T.$$

Hence, we get

$$F'(v) = \begin{pmatrix} e^{v_1} & 0 & 0 \\ 0 & (e-1)v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have for $x_* = (0, 0, 0)^T$ that $\omega_0(s, t) = \frac{e-1}{2}(s + t)$, $\omega(s, t) = (e - 1)s + e^{\frac{1}{e-1}}t$ and $\alpha = \frac{e+3}{2}$.

Example 4.2. Let us choose $B_1 = B_2 = C_{[0,1]}$. Set $D = U(0, 1)$, and define function F on D by

$$F(\Gamma)(x) = \psi(x) - \int_0^1 x\theta\Gamma(\theta)^3 d\theta$$

and

$$F'(\Gamma(\xi))(x) = \xi(x) - 3\int_0^1 x\theta\Gamma(\theta)^2\xi(\theta)d\theta.$$

Then, we have for $x_* = 0$ that $\omega_0(s, t) = \frac{3}{2}(s + t)$, $\omega(s, t) = 3(s + t)$ and $\alpha = 5$.

Example 4.3. For the academic problem that we considered in the introduction we have for $x_* = 0$, $\omega_0(s, t) = \omega(s, t) = \frac{96.662907}{2}(s + t)$ and $\alpha = 52$.

Example	Method (1.4)	Method (1.5)
4.1	7.873962171772243e-02	1.315565965112603e-01
4.2	2.973980668120351e-02	7.599937578041070e-02
4.3	1.500342940298857e-04	3.448409981435105e-03

Table 1. Values of convergence radii

In Table 1 there are the convergence radii of the considered methods. The Kurchatov-type method with memory (1.5) has a wider convergence region than method (1.4).

Starting points	Kurchatov's method (1.6)	Method (1.4)	Method (1.5)
(0.1, 0.1, 0.1)	4	4	3
(3, 3, 3)	9	5	5
(1, 2, 0.5)	7	6	4
(1, 5, 5)	8	9	4
(2, 10, 5)	9	19	5

Table 2. Count of iterations for Example 4.1

In Table 2 we give a count of iterations, which need to solve a system of nonlinear equations from Example 4.1. Results are obtained for $\varepsilon = 10^{-8}$. Additional starting point is chosen as $x_{-1} = x_0 + 0.0001$. We can see that the method (1.5) is faster than method (1.4) and the Kurchatov's method.

5. Conclusion

Problems from a plethora of areas can be solved by considering the corresponding equation given on a suitable abstract space by Mathematical Modeling. A sequence is then constructed by some developed method whose limit solves the equation. Looking at this direction together with the need for the development of faster methods and a wider range of initial guesses, we studied method (1.4) and (1.5) (given in [5] for $B_1 = B_2 = \mathbb{R}^k$) in the more general case of a space according to Banach. Our conditions involve the first order derivative appearing in the method whereas the work in [5] used higher order derivatives not on these methods. Our analysis produces error estimates, a radius of convergence and results on the location for the solution that are computable. This was not done in [5]. Hence, we expand the usage of the methods. Our idea can be used on other methods with the same benefits. Numerical experiments testing our theory and the performance of the methods terminate this article.

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