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A NOTE ON THE MAXIMAL FUNCTION

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. We investigate the Hardy–Littlewood maximal function. A local condition for non-integrability is formulated. Moreover, a generalization of the Hardy–Littlewood's and Stein's integrability theorem is given. As special case, we get the classical statements.

1. Introduction

If $[a,b]$ is a compact interval on the real line and $f \in L^1[a,b]$ (i.e. the real valuable function f is defined on $[a,b]$ and it is integrable in Lebesgue's sense), then Mf (the *Hardy–Littlewood maximal function* of f) is defined as follows:

$$Mf(x) := \sup_I \frac{1}{|I|} \cdot \int_I |f| \qquad (x \in [a,b]),$$

where the supremum is taken over all $x \in I \subset [a,b]$ intervals (and $|A|$ stands for the Lebesgue measure of a set $A \subset \mathbb{R}$). It is well-known [2] that for the measurable function $Mf : [a,b] \to [0,+\infty]$ (in other words for the maximal operator M) the next properties hold: there is an absolute constant $C > 0$ such that

$$|\{Mf > y\}| \leq \frac{C}{y} \cdot \|f\|_1 \qquad (f \in L^1[a,b],\ y > 0)$$

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(the operator M is of *weak type* $(1,1)$). Moreover, the "strong" property

$$|\{Mf > y\}| \leq \frac{C}{y} \cdot \int_{\{|f|>y\}} |f| \qquad (f \in L^1[a,b],\ y > 0)$$

is also true. From this it is clear that

$$|\{Mf = +\infty\}| \leq |\{Mf > y\}| \leq \frac{C}{y} \cdot \|f\|_1 \to 0 \qquad (y \to +\infty),$$

i.e. $|\{Mf = +\infty\}| = 0$ and thus the maximal function is finite a.e. Furthermore [2], for all $1 < p \leq +\infty$ with a suitable constant $C_p > 0$ (depending only on p)

$$\|Mf\|_p \leq C_p \cdot \|f\|_p \qquad (f \in L^p[a,b])$$

(the operator M is of *type* (p,p)). Since $|f| \leq Mf$ a.e., we can say that for these exponents p the equivalence

$$\|Mf\|_p < +\infty \iff \|f\|_p < +\infty$$

holds. However, there is a function $f \in L^1[a,b]$ for which $\|Mf\|_1 = +\infty$ (for example see the function f_0 in the next section).

2. Non-integrability

If $f \in L^1[a,b]$ is given then let

$$F(t) := \int_a^t |f| \qquad (t \in [a,b]).$$

It is obvious that for all $a \leq d < y < b$

$$Mf(x) \geq \frac{1}{x-d} \cdot \int_d^x |f| \geq \frac{F(y) - F(d)}{x-d} \qquad (y \leq x \leq b).$$

Therefore

$$\|Mf\|_1 = \int_a^b Mf \geq \int_y^b Mf \geq (F(y) - F(d)) \cdot \int_y^b \frac{dx}{x-d} =$$

$$= (F(y) - F(d)) \cdot (\ln(b-d) - \ln(y-d)).$$

If we take into account that $F(y)-F(d)\to 0\ (y\to d)$, it follows

$$\|Mf\|_1 \ge \limsup_{y\to d+0}((F(y)-F(d))\cdot|\ln(y-d)|).$$

So we get the next

Proposition 1. *If* $f\in L^1[a,b]$ *and for some points* $a\le d<y<b$

$$\limsup_{y\to d+0}((F(y)-F(d))\cdot|\ln(y-d)|)=+\infty,$$

then $\|Mf\|_1=+\infty$.

Analogously, if $a<e\le b$, then

$$\limsup_{y\to e-0}((F(e)-F(y))\cdot|\ln(e-y)|)=+\infty \Longrightarrow \|Mf\|_1=+\infty,$$

or with an arbitrary point $z\in[a,b]$:

$$\limsup_{y\to z}(|F(y)-F(z)|\cdot|\ln(|y-z|)|)=+\infty \Longrightarrow \|Mf\|_1=+\infty.$$

(We see later that in this connection the assumption $\limsup\ldots>0$ is also enough.)

Since

$$(F(y)-F(d))\cdot|\ln(y-d)|=(y-d)\cdot|\ln(y-d)|\cdot\frac{1}{y-d}\cdot\int_d^y|f|\le$$
$$\le Mf(d)\cdot(y-d)\cdot|\ln(y-d)|$$

and

$$(y-d)\cdot\ln(y-d)\to 0 \qquad (y\to d+0),$$

it follows (assumed $Mf(d)<+\infty$) that

$$\lim_{y\to d+0}((F(y)-F(d))\cdot\ln(y-d))=0,$$

or

$$\lim_{y\to e-0}((F(e)-F(y))\cdot\ln(e-y))=0,$$

if $Mf(e)<+\infty$. We know that Mf is finite a.e., therefore

$$\lim_{y\to z}((F(y)-F(z))\cdot\ln(|y-z|))=0 \qquad (\text{a.e. } z\in[a,b]).$$

In other words for such points $z \in [a,b]$ we have

$$\lim_{n\to\infty} |I_n| = 0 \Longrightarrow \lim_{n\to\infty} \Big(\ln(|I_n|) \cdot \int_{I_n} |f| \Big) = 0 \tag{1}$$

with every sequence (I_n) of intervals $z \in I_n \subset [a,b]$ $(n \in \mathbb{N})$.

We recall that F is differentiable a.e. and if this is true for a point z then

$$\frac{1}{y-z} \cdot \int_z^y |f| = \frac{F(y)-F(z)}{y-z} \to F'(z) = |f(z)| \qquad (y \to z)$$

and (1) follows.

For example, we consider the function

$$f(x) := f_0(x) := \begin{cases} \dfrac{1}{(x-a)\cdot \ln^2(x-a)} & (a < x \le c) \\ 0 & (x \in [a,b] \setminus (a,c]) \end{cases}$$

(where $c \in (a,b)$ and $c - a < 1$),. It is not hard to see that $f_0 \in L^1[a,b]$ and

$$F(y) \cdot |\ln(y-a)| = 1 \qquad (y \in (a,c)).$$

On the other hand, if

$$f(x) := \begin{cases} \dfrac{1}{(x-a)\cdot |\ln(x-a)|^{3/2}} & (a < x \le c) \\ 0 & (x \in [a,b] \setminus (a,c]), \end{cases}$$

then $f \in L^1[a,b]$, but

$$F(y) \cdot |\ln(y-a)| = 2 \cdot \sqrt{|\ln(y-a)|} \to +\infty \qquad (y \to a).$$

The first statement on the integrability of Mf reads as follows [2]:

$$f \in L \log^+ L[a,b] \Longrightarrow \|Mf\|_1 < \infty.$$

This means by Proposition 1 that for all $z \in [a,b]$ the implication

$$f \in L \log^+ L[a,b] \Longrightarrow \limsup_{y\to z} (|F(y) - F(z)| \cdot |\ln(|y-z|)|) < +\infty \tag{2}$$

is true. Moreover, it is easy to show

Proposition 2. *Let* $f \in L\log^+ L[a,b]$, *then for all* $z \in [a,b]$ *we have*

$$\limsup_{y\to z}(|F(y)-F(z)|\cdot|\ln(|y-z|)|)=0.$$

Proof. For example, take the points $a < z = d < y < b$ and $y - d < 1$. Then

$$F(y)-F(d)=\int_d^y |f| = \int_{[d,y]\cap\{|f|<1/\sqrt{y-d}\}} |f| + \int_{[d,y]\cap\{|f|\geq 1/\sqrt{y-d}\}} |f| \leq$$

$$\leq \sqrt{y-d} + \int_{[d,y]\cap\{f^2\geq 1/(y-d)\}} |f|.$$

Therefore

$$(F(y)-F(d))\cdot|\ln(y-d)| \leq$$

$$\leq \sqrt{y-d}\cdot|\ln(y-d)| + 2\cdot \int_{[d,y]\cap\{f^2\geq 1/(y-d)\}} |f(x)|\cdot\ln(|f(x)|)\,dx \leq$$

$$\leq \sqrt{y-d}\cdot|\ln(y-d)| + 2\cdot\int_d^y |f|\cdot(\log^+\circ|f|)$$

and here

$$\sqrt{y-d}\cdot|\ln(y-d)| \to 0 \qquad (y\to d+0).$$

If $f \in L\log^+ L[a,b]$, then

$$\int_d^y |f|\cdot(\log^+\circ|f|) \to 0 \qquad (y\to d+0),$$

which leads to

$$(F(y)-F(d))\cdot|\ln(y-d)| \to 0 \qquad (y\to d+0). \qquad \blacksquare$$

The function f_0 shows that the converse of (2) is not true, in contrast to the above statement on the integrability of Mf. Namely, Stein [3] proved that for a function $f \in L^1[a,b]$, we get

$$\|Mf\|_1 < +\infty \Longrightarrow f \in L\log^+ L[a,b].$$

From this it follows immediately that for all $f \in L^1[a,b]$ the next equivalence is true:

$$\|Mf\|_1 < +\infty \iff f \in L\log^+ L[a,b] \qquad (3)$$

and (see Proposition 2)

$$\|Mf\|_1 < +\infty \Longrightarrow \lim_{y\to z}((F(y)-F(z))\cdot \ln(|y-z|)) = 0 \qquad (z\in[a,b]).$$

In other words the next consequence holds:

Proposition 3. *If* $f \in L^1[a,b]$ *and there is a point* $z\in[a,b]$ *such that*

$$\limsup_{y\to z}(|F(y)-F(z)|\cdot|\ln(|y-z|)|) > 0,$$

then $\|Mf\|_1 = +\infty$.

Now, we consider an example which shows that the assumption $\|Mf\|_1 = = +\infty$ is not enough for

$$\limsup_{y\to z}(|F(y)-F(z)|\cdot|\ln(|y-z|)|) > 0.$$

Proposition 4. *There exists* $f\in L^1[a,b]$ *such that* $\|Mf\|_1 = +\infty$, *but*

$$\limsup_{y\to z}(|F(y)-F(z)|\cdot|\ln(|y-z|)|) = 0$$

for all $z\in[a,b]$.

Proof. Indeed, let

$$c_n := q_n\cdot 2^n \qquad (n\in\mathbb{N}),$$

where the numbers $q_n > 0$ $(n\in\mathbb{N})$ satisfy the conditions

$$\sum_{n=0}^{\infty} q_n < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} nq_n = +\infty$$

and

$$n\cdot\sum_{k=n}^{\infty} q_n \to 0 \qquad (n\to\infty).$$

For example the numbers

$$q_n := \frac{1}{(n+1)^2\cdot\ln(n+2)} \qquad (n\in\mathbb{N})$$

fulfil these requirements. By means of c_n's we define the function

$$f = f_1 : [0,1] \to (0,+\infty)$$

as follows:

$$f_1(x) := c_n \qquad (x\in[2^{-n-1},2^{-n}],\ n\in\mathbb{N}).$$

Then

$$\int_0^1 |f_1| = \sum_{n=0}^{\infty} c_n \cdot 2^{-n-1} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} q_n < +\infty,$$

i.e. $f_1 \in L^1[0,1]$. However, with a suitable $N \in \mathbb{N}$ it follows that

$$\int_0^1 |f_1| \cdot (\log^+ \circ |f|) = \frac{1}{2} \cdot \sum_{n=0}^{\infty} q_n \cdot \log^+(c_n) = \frac{1}{2} \cdot \sum_{n=N}^{\infty} q_n \cdot \ln c_n =$$

$$= \frac{1}{2} \cdot \sum_{n=N}^{\infty} q_n (n \cdot \ln 2 - (2 \cdot \ln(n+1) + \ln(\ln(n+2)).$$

Here we can assume also that

$$2 \cdot \ln(n+1) + \ln(\ln(n+2)) < \frac{n \cdot \ln 2}{2} \qquad (N \leq n \in \mathbb{N}).$$

So

$$\int_0^1 |f_1| \cdot (\log^+ \circ |f|) \geq \frac{\ln 2}{4} \cdot \sum_{n=N}^{\infty} n q_n = +\infty.$$

This means that $f_1 \notin L \log^+ L[0,1]$, i.e. $\|Mf\|_1 = +\infty$. Furthermore, let $n \in \mathbb{N}$ be given, then

$$\int_0^{2^{-n}} |f_1| = \sum_{k=n}^{\infty} c_n \cdot 2^{-k-1} = \frac{1}{2} \cdot \sum_{k=n}^{\infty} q_k$$

and we get

$$|\ln 2^{-n}| \cdot \int_0^{2^{-n}} |f_1| = \frac{n \cdot \ln 2}{2} \cdot \sum_{k=n}^{\infty} q_k \to 0 \qquad (n \to \infty).$$

Finally, if $0 < z \in [0,1]$, then f_1 is bounded on $[z/2, 1]$ and this implies

$$\lim_{n\to\infty} |I_n| = 0 \Longrightarrow \lim_{n\to\infty} \Big(\ln(|I_n|) \cdot \int_{I_n} |f| \Big) = 0$$

for all intervals $z \in I_n \subset [0,1]$ $(n \in \mathbb{N})$. ■

Thus for the set

$$S[a,b] := \{ f \in L^1[a,b] : \lim_{y\to z} ((F(y) - F(z)) \cdot \ln(|y-z|)) = 0 \ \ (z \in [a,b]) \}$$

the inclusions

$$L\log^+ L[a,b] \subset S[a,b] \subset L^1[a,b]$$

are strong:

$$f_1 \in S[a,b] \setminus L\log^+ L[a,b] \quad \text{and } f_0 \in L^1[a,b] \setminus S[a,b].$$

3. Integrability

In connection of the equivalence (3) Stein [4, p. 23] says in his famous book (without proof) that for all functions $f \in L^1[a,b]$ and for all numbers $\rho > 0$

$$\int_a^b Mf\cdot (\log^+ \circ (Mf))^\rho < +\infty \iff \int_a^b |f|\cdot (\log^+ \circ |f|)^{\rho+1} < +\infty$$

holds. We remark, that a proof for $\Longleftarrow$ can be found in the books of Weisz and Bennet–Sharpley [5], [1], resp. Next we shall generalize this Stein's equivalence. As special case, a common proof for the (L^p, L^p) and $(L\log^+ L, L^1)$-boundedness follows as well.

To this end let the functions

$$\psi,\ \Lambda : [0,+\infty] \to [0,+\infty]$$

be given and assume that the next properties hold: $\psi,\ \Lambda$ are absolute continuous on all compact intervals $I \subset [0,+\infty)$, are nondecreasing,

$$\lim_{x\to+\infty} \psi(x) = \lim_{x\to+\infty} \Lambda(x) = +\infty$$

and $\psi(0) = \Lambda(0) = 0$. We consider the functions

$$\Phi(y) := y\cdot \psi(y) \qquad (y \geq 0)$$

and

$$\Theta(y) := y\cdot \Lambda(y) \qquad (y \geq 0).$$

It will be assumed also that there are numbers $r, c > 0$ such that

$$c\cdot y\cdot \psi'(y) \leq \int_0^y t^{-1}\cdot \psi(t)\,dt \leq \Lambda(y) + y\cdot \psi'(y) \qquad (\text{a.e. } y > r)$$

and

$$\Theta'(y) = \int_0^y t^{-1}\cdot \Phi'(t)\,dt \qquad (\text{a.e. } y > 0).$$

Theorem. *For all* $f \in L^1[a,b]$ *we have*

$$\int_a^b \Phi \circ (Mf) < +\infty \iff \int_a^b \Theta \circ |f| < +\infty.$$

It is clear that in the case $\Theta \le \Phi$ the part $\Longrightarrow$ follows simple from $|f| \le Mf$ a.e.

Proof of Theorem. First of all, we remark that by the assumptions there is $\alpha > r$ such that $\psi(\alpha) > 0$ and $\Lambda(\alpha) > 0$. If $g : [0,+\infty] \to [0,+\infty]$ is Lebesgue-measurable and a.e. finite, then

$$\int_a^b g\cdot (\psi \circ g) = \int_{\{g \le \alpha\}} g\cdot (\psi \circ g) + \int_{\{g > \alpha\}} g\cdot (\psi \circ g),$$

where

$$\int_{\{g \le \alpha\}} g\cdot (\psi \circ g) \le (b-a)\cdot \alpha \cdot \psi(\alpha) < +\infty.$$

Furthermore

$$\int_{\{g > \alpha\}} g\cdot (\psi \circ g) = \int_a^b (\Phi \circ g)\cdot \chi_{\{g>\alpha\}} = \int_0^{+\infty} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > y\}|\, dy =$$

$$= \int_0^{\Phi(\alpha)} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > y\}|\, dy + \int_{\Phi(\alpha)}^{+\infty} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > y\}|\, dy =: A + B.$$

Here

$$A = \int_0^{\Phi(\alpha)} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > y\}|\, dy \le (b-a)\cdot \Phi(\alpha) < +\infty$$

and (taking into account that Φ is increasing on $[\alpha, +\infty)$ and absolute continuous on $[\alpha, z]$ for all $z > \alpha$)

$$B = \int_{\Phi(\alpha)}^{+\infty} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > y\}|\, dy =$$

$$= \int_\alpha^{+\infty} |\{(\Phi \circ g)\cdot \chi_{\{g>\alpha\}} > \Phi(y)\}|\cdot \Phi'(y)\, dy = \int_\alpha^{+\infty} |\{g > y\}|\cdot \Phi'(y)\, dy.$$

Therefore

$$\int_a^b g\cdot(\psi\circ g)<+\infty \iff \int_\alpha^{+\infty} |\{g>y\}|\cdot \Phi'(y)\,dy<+\infty.$$

Now we assume that

$$\int_a^b \Theta\circ|f| = \int_a^b |f|\cdot(\Lambda\circ|f|)<+\infty.$$

Then

$$\int_\alpha^{+\infty} |\{Mf>y\}|\cdot\Phi'(y)\,dy \le C\cdot \int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_{\{|f|>y\}} |f|\Big)\cdot\Phi'(y)\,dy =$$

$$= C\cdot \int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_a^b |f|\cdot\chi_{\{|f|>y\}}\Big)\cdot\Phi'(y)\,dy =$$

$$= C\cdot \int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_0^{+\infty} |\{|f|\cdot\chi_{\{|f|>y\}}>t\}|\,dt\Big)\cdot\Phi'(y)\,dy =$$

$$= C\cdot\Bigg(\int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_0^{y} |\{|f|>y\}|\,dt + \int_y^{+\infty} |\{|f|>t\}|\,dt\Big)\cdot\Phi'(y)\,dy\Bigg) =$$

$$= C\cdot\Bigg(\int_\alpha^{+\infty} |\{|f|>y\}|\cdot\Phi'(y)\,dy + \int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_y^{+\infty} |\{|f|>t\}|\,dt\Big)\cdot\Phi'(y)\,dy\Bigg).$$

We obtain

$$\int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_y^{+\infty} |\{|f|>t\}|\,dt\Big)\cdot\Phi'(y)\,dy =$$

$$= \int_\alpha^{+\infty} y^{-1}\cdot\Big(\int_\alpha^{+\infty} |\{|f|>t\}|\cdot\chi_{[y,+\infty)}(t)\,dt\Big)\cdot\Phi'(y)\,dy =$$

$$= \int_\alpha^{+\infty} |\{|f|>t\}|\cdot\Big(\int_\alpha^{t} y^{-1}\cdot\Phi'(y)\,dy\Big)\,dt = \int_\alpha^{+\infty} |\{|f|>t\}|\cdot\Theta'(t)\,dt.$$

So by our previous remark (with $|f|$ and φ instead of g and ψ, resp.) the boundedness

$$\int_a^b |f|\cdot(\Lambda\circ|f|) < +\infty$$

implies

$$\int_\alpha^{+\infty} |\{|f|>t\}|\cdot\Theta'(t)\,dt < +\infty.$$

By

$$c\cdot\Phi'(y) = c\cdot(\psi(y)+y\psi'(y)) \le \Theta'(y) =$$

$$= \int_0^y z^{-1}\cdot\Phi'(z)\,dz = \int_0^y (\psi'(z)+z^{-1}\cdot\psi(z))\,dz =$$

$$= \psi(y) + \int_0^y z^{-1}\cdot\psi(z)\,dz \qquad (\text{a.e. } y>\alpha)$$

we get

$$\int_\alpha^{+\infty} |\{|f|>y\}|\cdot\Phi'(y)\,dy \le c^{-1}\cdot\int_\alpha^{+\infty} |\{|f|>y\}|\cdot\Theta'(y)\,dy < +\infty.$$

Summarizing the above facts it follows that

$$\int_a^b \Phi\circ(Mf) = \int_a^b Mf\cdot(\psi\circ(Mf)) < +\infty.$$

Next we assume the last boundedness. Then

$$\int_a^b |f|\cdot(\Lambda\circ|f|) = \int_{\{|f|\le\alpha\}} |f|\cdot(\Lambda\circ|f|) + \int_{\{|f|>\alpha\}} |f|\cdot(\Lambda\circ|f|) \le$$

$$\le (b-a)\cdot\alpha\cdot\Lambda(\alpha) + \int_{\{|f|>\alpha\}} |f|\cdot(\Lambda\circ|f|) =: C+D.$$

Since $C<+\infty$, it is anough to investigate the second expression:

$$D = \int_{\{|f|>\alpha\}} |f|\cdot(\Lambda\circ|f|) = \int_{\{|f|>\alpha\}} |f(x)|\cdot\Big(\int_0^{|f(x)|} \varphi(y)\,dy\Big),$$

where the function φ is defined by means of the equality

$$\Lambda(y) = \int_0^y \varphi(z)\,dz \qquad (y \geq 0).$$

Thus

$$\int_{\{|f|>\alpha\}} |f|\cdot (\Lambda \circ |f|) = \int_{\{|f|>\alpha\}} |f(x)|\cdot \Big(\int_0^{+\infty} \varphi(y)\cdot \chi_{[0,|f(x)|)}(y)\,dy\Big)\,dx =$$

$$= \int_0^{+\infty} \varphi(y)\cdot \Big(\int_{\{|f|>\alpha\}} |f(x)|\cdot \chi_{[0,|f(x)|)}(y)\,dx\Big)\,dy =$$

$$= \int_0^{\alpha} \varphi(y)\cdot \Big(\int_{\{|f|>\alpha\}} |f|\Big)\,dy + \int_{\alpha}^{+\infty} \varphi(y)\cdot \Big(\int_{\{|f|>y\}} |f|\Big)\,dy \leq$$

$$\leq \|f\|_1\cdot \int_0^{\alpha} \varphi(y)\,dy + \int_{\alpha}^{+\infty} \varphi(y)\cdot \Big(\int_{\{|f|>y\}} |f|\Big)\,dy,$$

where

$$\|f\|_1\cdot \int_0^{\alpha} \varphi(y)\,dy = \|f\|_1\cdot \Lambda(\alpha) < +\infty.$$

We recall [3] that with a suitable $y_0 > 0$

$$\int_{\{|f|>y\}} |f| \leq 2y\cdot |\{Mf > y\}| \qquad (y \geq y_0).$$

Let $\beta := \max\{y_0, \alpha\}$, then

$$\int_{\alpha}^{\beta} \varphi(y)\cdot \Big(\int_{\{|f|>y\}} |f|\Big)\,dy \leq \|f\|_1\cdot \int_{\alpha}^{\beta} \varphi(y)\,dy = \|f\|_1\cdot (\Lambda(\beta) - \Lambda(\alpha)) < +\infty$$

and

$$\int_{\beta}^{+\infty} \varphi(y)\cdot \Big(\int_{\{|f|>y\}} |f|\Big)\,dy \leq 2\cdot \int_{\beta}^{+\infty} |\{Mf > y\}|\cdot y\cdot \varphi(y)\,dy.$$

From assumptions on ψ, Λ it follows that

$$y\cdot \varphi(y) = y\cdot \Lambda'(y) = \Theta'(y) - \Lambda(y) = \int_0^y t^{-1}\cdot \Phi'(t)\,dt - \Lambda(y) =$$

$$= \psi(y) + \int_0^y t^{-1}\cdot \psi(t)\,dt - \Lambda(y) \le \psi(y) + y\cdot \psi'(y) = \Phi'(y) \qquad (\text{a.e. } y > \alpha).$$

Therefore

$$\int_\beta^{+\infty} |\{Mf > y\}|\cdot y\cdot \varphi(y)\,dy \le \int_\alpha^{+\infty} |\{Mf > y\}|\cdot \Phi'(y)\,dy \le$$

$$\le \int_a^b Mf\cdot (\psi \circ (Mf)) < +\infty,$$

which proves the boundedness

$$\int_a^b \Theta \circ |f| = \int_a^b |f|\cdot (\Lambda \circ |f|) < +\infty. \qquad \blacksquare$$

4. Special cases

a) First let $\rho > 0$ and

$$\psi(x) := (\log^+ x)^\rho \qquad (x \ge 0).$$

Then

$$\psi'(y) = \begin{cases} 0 & (0 \le y < 1) \\ \rho\cdot y^{-1}\cdot (\ln y)^{\rho-1} & (y > 1) \end{cases}$$

and

$$\int_0^y t^{-1}\cdot \psi(t)\,dt = \begin{cases} 0 & (0 \le y < 1) \\ (\rho+1)^{-1}\cdot (\ln y)^{\rho+1} & (y > 1). \end{cases}$$

If $r := e^{\sqrt{\rho(\rho+1)}}$, then

$$y\cdot \psi'(y) = \rho\cdot (\ln y)^{\rho-1} \le \int_1^y t^{-1}\cdot \psi(t)\,dt = \frac{1}{\rho+1}\cdot (\ln y)^{\rho+1} \qquad (y \ge r).$$

It can be simply veryfied that with the function

$$\Lambda(y) := \frac{1}{\rho+1}\cdot(\log^+ y)^{\rho+1} \qquad (y \geq 0),$$

we get

$$\Theta'(y) = \begin{cases} 0 & (0 \leq y < 1) \\ (\rho+1)^{-1}\cdot(\ln y)^{\rho+1} + (\ln y)^{\rho} & (y > 1) \end{cases}$$

and

$$\int_0^y t^{-1}\cdot\psi'(t)\,dt =$$

$$= \int_0^y t^{-1}\cdot\psi(t)\,dt + \psi(y) = \begin{cases} 0 & (0 \leq y < 1) \\ (\rho+1)^{-1}\cdot(\ln y)^{\rho+1} + (\ln y)^{\rho} & (y > 1). \end{cases}$$

Since

$$\int_0^y t^{-1}\cdot\psi(t)\,dt \leq \Lambda(y) \qquad (y \geq 0),$$

the second assumption for ψ is also fulfilled. Therefore for all $f \in L^1[a,b]$ the equivalence

$$\int_a^b Mf\cdot(\log^+ \circ(Mf))^{\rho} < +\infty \iff \int_a^b |f|\cdot(\log^+ \circ|f|)^{\rho+1} < +\infty$$

holds, which is the above mentioned Stein‘s equivalence.

b) Now, let $s > 0$ be given and define the function ψ as follows:

$$\psi(x) := x^s \qquad (x \geq 0).$$

Then

$$\Phi(x) := x^{s+1} \qquad (x \geq 0)$$

and

$$y\cdot\psi'(y) = s\cdot y^s \qquad (y \geq 0).$$

Furthermore

$$\int_0^y t^{-1}\cdot\psi(t)\,dt = \frac{y^s}{s} \qquad (y \geq 0)$$

and

$$\Theta'(y) = (s+1)\cdot \int_0^y t^{s-1}\, dt = \frac{s+1}{s}\cdot y^s \qquad (y \geq 0).$$

In other words

$$\Theta(y) = \frac{y^{s+1}}{s} \qquad (y \geq 0)$$

and

$$\Lambda(y) = \frac{y^s}{s} \qquad (y \geq 0)$$

and our assumptions for ψ (with the constant $c := 1/s^2$) are true. Thus Theorem gives the next equivalence for all $1 < p < +\infty$:

$$\|Mf\|_p < +\infty \iff \|f\|_p < +\infty,$$

which is the Hardy–Littlewood‘s (L^p, L^p)-boundedness.

c) Finally, we take $s > 0$ and

$$\psi(x) := x^s\cdot \log^+ x \qquad (x \geq 0).$$

Then

$$cy\cdot \psi'(y) = c\cdot (sy^s\cdot \ln y + y^s) \leq \int_1^y t^{-1}\cdot \psi(t)\, dt =$$

$$= \frac{1}{s}\cdot y^s\cdot \ln y - \frac{1}{s^2}\cdot y^s + \frac{1}{s^2} \qquad (y \geq r)$$

with $c := 1/(2s^2)$ and with a suitable $r > 1$. Furthermore

$$\Theta'(y) = \frac{s+1}{s}\cdot y^s\cdot \ln y - \frac{1}{s^2}\cdot y^s + \frac{1}{s^2} \qquad (y \geq 1).$$

Therefore let

$$\Theta(y) := \begin{cases} 0 & (0 \leq y < 1) \\ \frac{1}{s}\cdot y^{s+1}\cdot \ln y - \frac{1}{s^2}\cdot y^{s+1} + \frac{1}{s^2}\cdot y & (y \geq 1), \end{cases}$$

and the second assumtion on ψ is automatically fulfiled. Thus

$$\int_a^b \Phi \circ (Mf) < +\infty \iff \int_a^b \Theta \circ |f| < +\infty.$$

It is easy to see that

$$\int_a^b \Theta \circ |f| < +\infty \iff \int_a^b |f|^{s+1} \cdot (\log^+ \circ |f|) < +\infty,$$

which implies the next equivalence for all $1 < p < +\infty$:

$$\int_a^b (Mf)^p \cdot (\log^+ \circ (Mf)) < +\infty \iff \int_a^b |f|^p \cdot (\log^+ \circ |f|) < +\infty.$$

(where $\Longrightarrow$ follows from $|f| \le Mf$ a.e. as well also for $p = +\infty$).

We remark that the last equivalence does not hold in the case $p = 1$. To this end we consider the function $f \in L^1[0,1]$ such that

$$f(x) := \frac{2^n}{(n+1)^3} \qquad (n \in \mathbb{N},\, 2^{-n-1} < x \le 2^{-n}).$$

Then

$$\int_0^1 |f| \cdot (\log^+ \circ |f|) < \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Since for $x \in (2^{-n-1}, 2^{-n})$ the inequality

$$Mf(x) \ge 2^n \cdot \int_0^{2^{-n}} f = 2^{n-1} \cdot \sum_{k=n}^{\infty} \frac{1}{(k+1)^3} > \frac{2^{n-2}}{(n+1)^2} \qquad (11 \le n \in \mathbb{N})$$

is true, so we get

$$\int_0^1 Mf \cdot (\log^+ \circ (Mf) \ge \sum_{n=10}^{\infty} \int_{2^{-n-1}}^{2^{-n}} Mf \cdot (\log^+ \circ (Mf) \ge \frac{1}{16} \cdot \sum_{n=11}^{\infty} \frac{1}{n+1} = +\infty.$$

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