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ON q-MULTIPLICATIVE FUNCTIONS WITH SPECIAL PROPERTIES

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. Let $q \in \{2, \dots, 50\}$. If g is any q-multiplicative function and

$$g(p) = 1 \quad \text{for every prime} \quad p,$$

then

$$g(qm) = 1 \quad \text{for every} \quad m \in \mathbb{N}$$

and

$$g(qm + r) = g(r) \quad \text{for every} \quad m \in \mathbb{N},\ r \in \{1, 2, \cdots, q-1\},$$

furthermore

$$g(n) = 1 \quad \text{for every} \quad n \in \mathcal{P} \cup \{n \in \mathbb{N} \mid (n, q) = 1\}.$$

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{C}$ be the set of positive integers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers.

For some integer $q \geq 2$ let

$$\mathbb{A}_q := \{0, 1, \dots, q-1\}.$$

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Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \mathbb{A}_q$$

and $a_r(n) = 0$ if $q^r > n$. Let $\mathcal{A}_q$ and $\mathcal{M}_q$ be the sets of q-additive and q-multiplicative functions, respectively. We say that $f \in \mathcal{A}_q$ ($g \in \mathcal{M}_q$), if $f : \mathbb{N}_0 \to \mathbb{C}$ ($g : \mathbb{N} \to \mathbb{C}$) satisfy the following conditions

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(a_r(n) q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}$$

and

$$g(0) = 1 \quad \text{and} \quad g(n) = \prod_{r=0}^{\infty} g\Big(a_r(n) q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}.$$

It is easy to see that $f \in \mathcal{A}_q$ ($g \in \mathcal{M}_q$) if and only if $f(aq^r + b) = f(aq^r) + f(b)$ $\big(g(aq^r + b) = g(aq^r)g(b)\big)$ for all $a \in \mathbb{N}_0,\ r \in \mathbb{N},\ 0 \le b < q^r$.

In the following let I_q be the set defined as:

$$I_q := \mathcal{P} \cup \{n \in \mathbb{N} | (n, q) = 1\}.$$

I. Kátai and B. M. Phong stated the following conjecture:

Conjecture 1. *Let $q \ge 2$ be an integer and $g \in \mathcal{M}_q$. If*

$$g(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P},$$

then

$$g(qm) = 1 \quad \textit{for every} \quad m \in \mathbb{N}$$

and

$$g(qm + r) = g(r) \quad \textit{for every} \quad m \in \mathbb{N}, r \in \{1, \cdots, q-1\},$$

furthermore

$$g(n) = 1 \quad \textit{for every} \quad n \in I_q.$$

In this paper, we prove the following

Theorem 1. *Conjecture 1 is true for every $q \in \{2, \cdots, 50\}$.*

Corollary 1. *If $q \in \{2, \cdots, 50\}$ and $q \in \mathcal{P}$, then the condition*

$$g(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P}$$

implies that

$$g(n) = 1 \quad \textit{for every} \quad n \in \mathbb{N}.$$

Corollary 2. *Let $q \in \{2, \cdots, 50\}$. If the conditions*

$$g(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P}$$

and

$$g(m) = 1 \quad \textit{for every} \quad m \in \{1, \cdots, q-1\} \setminus I_q$$

hold, then

$$g(n) = 1 \quad \textit{for every} \quad n \in \mathbb{N}.$$

2. Lemmas

We list some lemmas which are used in the proofs of Theorem 1. In the proof of our theorem we use the existence of prime in interval $[kn, (k+1)n], k, n \in \mathbb{N}$. For $k = 1$, Bertrands postulate is stated as: For $n > 1$ there is a prime number between n and $2n$. This was ultimately solved by P. Chebyshev in 1850, see [4]. Other nice proofs were given by Ramanujan in 1919 [9] and Erdős in 1932 (see Erdős and Surányi [6, p. 171–173]). For $k \geq 2$, El Bachraoui [1] proved in 2006 that every interval $[2n, 3n]$ contains a prime, while A. Loo [7] proved the same statement for every interval $[3n, 4n]$. In 1952, J. Nagura [8] proved that there is always a prime between $5n$ and $6n$ for $n \geq 5$. In 2017 K. D. Balliet [2] proved that there is a prime in the interval $[4n, 5n]$.

Now we state these in the following lemmas:

Lemma 1. (Bertand's Postulate) *For any positive integer $n > 1$ there is a prime number between n and $2n$.*

Lemma 2. (M. El Bachraoui [1]) *For any positive integer $n > 1$ there is a prime number between $2n$ and $3n$.*

Lemma 3. (A. Loo [7]) *For any positive integer $n > 1$ there is a prime number between $3n$ and $4n$.*

Lemma 4. (K. D. Balliet [2]) *For any positive integer $n > 2$ there is a prime number between $4n$ and $5n$*

Lemma 5. (J. Nagura [8]) *For any positive integer $n > 1$ there is a prime number between $5n$ and $6n$.*

Lemma 6. (K. D. Balliet [3], Theorem 2.3.4) *For $k, n \in \mathbb{N}$ with $n \geq k$ and $519 \geq k \geq 2$, there is at least one prime number between kn and $(k+1)n$.*

We note that by using the result of P. Dusart [5] which states that if $x \geq 3275$ then there is a prime p such that

$$x < p < \left(1 + \frac{1}{2\ln^2 x}\right) x,$$

one can prove Lemma 6 for other $k > 519$.

Lemma 7. *Assume that $2 \leq q \leq 519$ and $g \in \mathcal{M}_q$ satisfies*

$$g(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P}. \tag{2.1}$$

If

$$g(r) = 1 \quad \textit{for every} \quad (r, q) = 1, r \in \{1, \cdots, q-1\}, \tag{2.2}$$

then

$$g(n) = 1 \quad \textit{for every} \quad n \in I_q \quad (= \mathcal{P} \cup \{n \in \mathbb{N} | (n, q) = 1\}). \tag{2.3}$$

Proof. Assume that $2 \leq q \leq 519$ and $g \in \mathcal{M}_q$. Since (2.1), we shall prove that

$$g(n) = 1 \quad \text{for every} \quad n \in \mathbb{N}, n \in I_q \setminus \mathcal{P}.$$

By using (2.2), we assume that

$$g(n) = 1 \quad \text{for every} \quad n < kq^\alpha, k \leq q^\alpha, (n, q) = 1, \alpha, k \in \mathbb{N}. \tag{2.4}$$

We shall prove that

$$g(n) = 1 \quad \text{for every} \quad n < (k+1)q^\alpha, (n, q) = 1, \tag{2.5}$$

which proves that $g(n) = 1$ for every $n < q^{\alpha+1}$, $(n, q) = 1$. This implies that if $g(n) = 1$ for every $n < q^\alpha$, $(n, q) = 1$, then $g(n) = 1$ for evey $n < q^{\alpha+1}$, $(n, q) = 1$, which with (2.2) proves the assertion (2.3) and Lemma 7.

From Lemma 1 - Lemma 6, for fixed $k \in \{1, \cdots, q\}$ there is a prime p_k such that

$$kq^\alpha < p_k < (k+1)q^\alpha.$$

Let $n_k := p_k - kq^\alpha$, and so

$$p_k := kq^\alpha + n_k, n_k < q^\alpha, (n_k, q) = 1.$$

From our assumptions we have $g(n_k) = 1$, therefore (2.1) implies that

$$1 = g(p_k) = g(kq^\alpha + n_k) = g(kq^\alpha)g(n_k) = g(kq^\alpha), g(kq^\alpha) = 1.$$

Consequently

$$g(kq^\alpha + n) = g(kq^\alpha)g(n) = g(n) = 1 \quad \text{for every} \quad n < q^\alpha, (n, q) = 1,$$

which proves that $g(t) = 1$ for every $kq^\alpha \le t < kq^\alpha + q^\alpha = (k+1)q^\alpha, (n, q) = 1$. This with (2.4) proves (2.5), and the proof of Lemma 7 is completes. ■

Lemma 8. *Assume that* $2 \le q \le 519$ *and* $g \in \mathcal{M}_q$ *satisfies*

$$g(p) = 1 \quad \textit{for every} \quad p \in \mathcal{P}. \tag{2.6}$$

If

$$g(n) = 1 \quad \textit{for every} \quad n < q, (n, q) = 1, \tag{2.7}$$

then

$$g(qm) = 1 \quad \textit{for every} \quad m \in \mathbb{N}, \tag{2.8}$$

and

$$g(qm + r) = g(r) \quad \textit{for every} \quad m \in \mathbb{N},\ r \in \{1, \dots, q-1\}.$$

Proof. Assume that $2 \le q \le 519$ and $g \in \mathcal{M}_q$ satisfies (2.6) and (2.7).

Let $m \in \mathbb{N}$. By using Lemma 7, it follows from (2.6) and (2.7) that $g(qm + 1) = g(1) = 1$. Consequently

$$g(qm) = g(qm)g(1) = g(qm + 1) = 1,$$

thus (2.8) holds.

Finally, it is clear from (2.8) that

$$g(qm + r) = g(qm)g(r) = g(r) \quad \text{for every} \quad m \in \mathbb{N},\ r \in \{1, \dots, q-1\}.$$

The proof of Lemma 8 is completes. ■

3. Proof of Theorem 1.

Let $q \in \{2, \cdots, 50\}$. Assume that $g \in \mathcal{M}_q$ satisfies

$$g(p) = 1 \quad \text{for every} \quad p \in \mathcal{P}. \tag{3.1}$$

In order to prove Theorem 1, using Lemma 7 and Lemma 8, we need prove that (2.2) holds, i. e

$$g(n) = 1 \quad \text{for every} \quad n < q, (n, q) = 1.$$

Let

$$J_q = \{n < q | n \notin \mathcal{P}, (n, q) = 1\}.$$

Since $g(p) = 1$ for every $p \in \mathcal{P}$, we shall prove that

$$g(N) = 1 \quad \text{for every} \quad N \in J_q.$$

For the cases $q \in \{2, 3\}$, we have $J_2 = J_3 = \{1\}$. We prove $g(1) = 1$ in each cases.

• Let $q = 2$. From (2.6), we have $g(2) = 1, g(3) = 1$ and so $g(3) = g(2+1) = = g(2)g(1) = g(1)$, consequently $g(1) = 1$. Thus $g(1) = 1$ holds for $q = 2$.

• Now we consider the case when $q = 3$. From (2.6), we have $g(2) = 1$, $g(3) = 1$, $g(11) = 1$ and $g(13) = 1$. Since $11 = 9+2 \in \mathcal{P}$ and $13 = 9+3+1 \in \mathcal{P}$, we infer from the fact $g \in \mathcal{M}_q$ that

$$g(9) = g(9)g(2) = g(9 + 2) = g(11) = 1$$

and

$$g(1) = g(9)g(3)g(1) = g(9 + 3 + 1) = g(13) = 1.$$

Therefore $g(1) = 1$ holds for $q = 3$.

The following we assume that $q \in \{4, 5, \cdots, 50\}$. For each $N \in J_q$, with the help of computer, we find a number $m \in \mathbb{N}$ such that

$$qm + N \in \mathcal{P} \quad \text{and} \quad qm + \pi \in \mathcal{P}, \text{for some} \ \ \pi \in \mathcal{P}, \pi < q. \tag{3.2}$$

If $m \in \mathbb{N}$ satisfies (3.2), then we obtain from the following two relations

$$g(qm) = g(qm)g(\pi) = g(qm + \pi) = 1$$

and

$$g(N) = g(qm)g(N) = g(qm + N) = 1$$

that $g(N) = 1$.

For $q \in \{4, 5, \cdots, 50\}$ and $N \in J_q$ we give values of π and m which are given in the following tables:

q	N	m	π	$qm+N$	$qm+\pi$
4	1	1	3	5	7
5	1	2	3	11	13
5	4	3	2	19	17
6	1	1	5	7	11
7	1	4	3	29	31
7	4	15	2	109	107
7	6	5	2	41	37
8	1	2	3	17	19
9	1	2	5	19	23
9	4	1	2	13	11
9	8	1	2	17	11
10	1	1	3	11	13
10	9	1	3	19	13
11	1	2	7	23	29
11	4	9	2	103	101
11	6	1	2	17	13
11	8	1	2	19	13
11	9	2	7	31	29
11	10	9	2	109	101
12	1	1	5	13	17
13	1	4	7	53	59
13	4	3	2	43	41
13	6	5	2	71	67
13	8	3	2	47	41
13	9	4	7	61	59
13	10	33	2	439	431
13	12	17	2	233	223
14	1	2	3	29	31
14	9	1	3	23	17
15	1	2	7	31	37
15	4	1	2	19	17
15	8	1	2	23	17
15	14	1	2	29	17
16	1	1	3	17	19
16	9	4	3	73	67
16	15	1	3	31	19
17	1	6	5	103	107
17	4	27	2	463	461

q	N	m	π	$qm+N$	$qm+\pi$
17	6	1	2	23	19
17	8	3	2	59	53
17	9	2	3	43	37
17	10	3	2	61	53
17	12	1	2	29	19
17	14	1	2	31	19
17	15	4	3	83	71
17	16	3	2	67	53
18	1	1	5	19	23
19	1	10	3	191	193
19	4	3	2	61	59
19	6	5	2	101	97
19	8	5	2	103	97
19	9	2	3	47	41
19	10	3	2	67	59
19	12	5	2	107	97
19	14	3	2	71	59
19	15	2	3	53	41
19	16	3	2	73	59
19	18	5	2	113	97
20	1	2	3	41	43
20	9	1	3	29	23
21	1	2	5	43	47
21	4	5	2	109	107
21	8	1	2	29	23
21	10	1	2	31	23
21	16	1	2	37	23
21	20	1	2	41	23
22	1	1	7	23	29
22	9	1	7	31	29
22	15	1	7	37	29
22	21	1	7	431	29
23	1	2	7	47	53
23	4	3	2	73	71
23	6	7	2	167	163
23	8	15	2	353	347
23	9	4	5	101	97
23	10	3	2	79	71

q	N	m	π	$qm+N$	$qm+\pi$
23	12	7	2	173	163
23	14	3	2	83	71
23	15	2	7	61	53
23	16	87	2	2017	2003
23	18	7	2	179	163
23	20	3	2	89	71
23	21	2	7	67	53
23	22	15	2	367	347
24	1	3	7	73	79
25	1	4	3	101	103
25	4	9	2	229	227
25	6	5	2	131	127
25	8	9	2	233	227
25	9	2	3	59	53
25	12	5	2	137	127
25	14	5	2	139	127
25	16	9	2	241	227
25	18	11	2	293	277
25	21	2	3	71	53
25	22	39	2	997	977
25	24	5	2	149	127
26	1	2	7	53	59
26	9	2	7	61	59
26	15	1	3	41	29
26	21	1	3	47	29
26	25	3	5	103	83
27	1	4	5	109	113
27	4	1	2	31	29
27	8	3	2	89	83
27	10	1	2	37	29
27	14	1	2	41	29
27	16	1	2	43	29
27	20	1	2	47	29
27	22	3	2	103	83
27	25	2	5	79	59
27	26	1	2	53	29
28	1	1	3	29	31
28	9	1	3	37	31
28	15	1	3	43	31
28	25	1	3	53	31
28	27	2	3	83	59
29	1	2	3	59	61
29	4	51	2	1483	1481

q	N	m	π	$qm+N$	$qm+\pi$
29	6	13	2	383	379
29	8	1	2	37	31
29	9	2	3	67	61
29	10	3	2	97	89
29	12	1	2	41	31
29	14	1	2	43	31
29	15	2	3	73	61
29	16	3	2	103	89
29	18	1	2	47	31
29	20	3	2	107	89
29	21	2	3	79	61
29	22	3	2	109	89
29	24	1	2	53	31
29	25	2	3	83	61
29	26	3	2	113	89
29	27	10	3	317	293
29	28	69	2	2029	2003
30	1	1	7	31	37
31	1	10	3	311	313
31	4	9	2	283	281
31	6	35	2	1091	1087
31	8	5	2	163	157
31	9	2	5	71	67
31	10	21	2	661	653
31	12	5	2	167	157
31	14	9	2	293	281
31	15	4	3	139	127
31	16	27	2	853	839
31	18	5	2	173	157
31	20	27	2	857	839
31	21	2	5	83	67
31	22	15	2	487	467
31	24	5	2	179	157
31	25	4	3	149	127
31	26	5	2	181	157
31	27	2	5	89	67
31	28	9	2	307	281
31	30	47	2	1487	1459
32	1	3	5	97	101
32	9	1	5	41	37
32	15	1	5	47	37
32	21	1	5	53	37
32	25	2	3	89	67

q	N	m	π	$qm+N$	$qm+\pi$
32	27	1	5	59	37
33	1	2	5	67	71
33	4	3	2	103	101
33	8	3	2	107	101
33	10	3	2	109	101
33	14	3	2	113	101
33	16	5	2	181	167
33	20	7	2	251	233
33	25	4	5	157	137
33	26	5	2	191	167
33	28	3	2	127	101
33	32	3	2	131	101
34	1	3	5	103	107
34	9	1	3	43	37
34	15	2	3	83	71
34	21	2	3	89	71
34	25	1	3	59	37
34	27	1	3	61	37
34	33	1	3	67	37
35	1	2	3	71	73
35	4	3	2	109	107
35	6	1	2	41	37
35	8	1	2	43	37
35	9	2	3	79	73
35	12	1	2	47	37
35	16	9	2	331	317
35	18	1	2	53	37
35	22	3	2	127	107
35	24	1	2	59	37
35	26	1	2	61	37
35	27	2	3	97	73
35	32	1	2	67	37
35	33	2	3	103	73
35	34	3	2	139	107
36	1	1	5	37	41
36	25	1	5	61	41
36	35	1	5	71	41
37	1	4	3	149	151
37	4	45	2	1669	1667
37	6	23	2	857	853
37	8	15	2	563	557
37	9	2	5	83	79
37	10	33	2	1231	1223
37	12	11	2	419	409
37	14	11	2	421	409
37	15	2	5	89	79
37	16	3	2	127	113
37	18	17	2	647	631
37	20	3	2	131	113
37	21	8	11	317	307
37	22	15	2	577	557
37	24	11	2	431	409
37	25	4	3	173	151
37	26	3	2	137	113
37	27	10	3	397	373
37	28	3	2	139	113
37	30	17	2	659	631
37	32	11	2	439	409
37	33	2	5	107	79
37	34	45	2	1699	1667
37	35	2	5	109	79
37	36	11	2	443	409
38	1	5	3	191	193
38	9	1	3	47	41
38	15	1	3	53	41
38	21	1	3	59	41
38	25	2	3	101	79
38	27	2	3	103	79
38	33	1	3	71	41
38	35	1	3	73	41
39	1	2	5	79	83
39	4	1	2	43	41
39	8	1	2	47	41
39	10	11	2	439	431
39	14	1	2	53	41
39	16	5	2	211	197
39	20	1	2	59	41
39	22	1	2	61	41
39	25	2	5	103	83
39	28	1	2	67	41
39	32	1	2	71	41
39	34	1	2	73	41
39	35	2	5	113	83
39	38	5	2	233	197
40	1	1	3	41	43
40	9	2	3	89	83

q	N	m	π	$qm+N$	$qm+\pi$
40	21	1	3	61	43
40	27	1	3	67	43
40	33	1	3	73	43
40	39	1	3	79	43
41	1	2	7	83	89
41	4	15	2	619	617
41	6	1	2	47	43
41	8	39	2	1607	1601
41	9	4	3	173	167
41	10	27	2	1117	1109
41	12	1	2	53	43
41	14	39	2	1613	1601
41	15	2	7	97	89
41	16	15	2	631	617
41	18	1	2	59	43
41	20	1	2	61	43
41	21	2	7	103	89
41	22	21	2	883	863
41	24	127	2	5231	5209
41	25	2	7	107	89
41	26	1	2	67	43
41	27	2	7	109	89
41	28	15	2	643	617
41	30	1	2	71	43
41	32	1	2	73	43
41	33	4	3	197	167
41	34	45	2	1879	1847
41	35	4	3	199	167
41	36	91	2	3767	3733
41	38	1	2	79	43
41	39	8	3	367	331
41	40	57	2	2377	2339
42	1	1	5	43	47
42	25	1	5	67	47
43	1	4	7	173	179
43	4	63	2	2713	2711
43	6	59	2	2543	2539
43	8	3	2	137	131
43	9	4	7	181	179
43	10	3	2	139	131
43	12	17	2	743	733
43	14	9	2	401	389
43	15	2	3	101	89
43	16	15	2	661	647
43	18	53	2	2297	2281
43	20	3	2	149	131
43	21	2	3	107	89
43	22	3	2	151	131
43	24	23	2	1013	991
43	25	4	7	197	179
43	26	17	2	757	733
43	27	2	3	113	89
43	28	3	2	157	131
43	30	17	2	761	733
43	32	9	2	419	389
43	33	10	3	463	433
43	34	3	2	163	131
43	35	6	5	293	263
43	36	29	2	1283	1249
43	38	3	2	167	131
43	39	4	7	211	179
43	40	27	2	1201	1163
43	42	17	2	773	733
44	1	2	13	89	101
44	9	1	3	53	47
44	15	1	3	59	47
44	21	2	13	109	101
44	25	2	13	113	101
44	27	1	3	71	47
44	35	1	3	79	47
44	39	1	3	83	47
45	1	4	11	181	191
45	4	3	2	139	137
45	8	1	2	53	47
45	14	1	2	59	47
45	16	1	2	61	47
45	22	1	2	67	47
45	26	1	2	71	47
45	28	1	2	73	47
45	32	3	2	167	137
45	34	1	2	79	47
45	38	1	2	83	47
45	44	1	2	89	47
46	1	1	7	47	53
46	9	2	5	101	97
46	15	1	7	61	53
46	21	1	7	67	53
46	25	1	7	71	53
46	27	1	7	73	53
46	33	1	7	79	53

q	N	m	π	$qm+N$	$qm+\pi$
46	35	2	5	127	97
46	39	2	5	131	97
46	45	2	5	137	97
47	1	6	11	283	293
47	4	75	2	3529	3527
47	6	13	2	617	613
47	8	7	2	337	331
47	9	2	3	103	97
47	10	63	2	2971	2963
47	12	115	2	5417	5407
47	14	31	2	1471	1459
47	15	2	3	109	97
47	16	33	2	1567	1553
47	18	7	2	347	331
47	20	7	2	349	331
47	21	8	3	397	379
47	22	75	2	3547	3527
47	24	7	2	353	331
47	25	6	11	307	293
47	26	31	2	1483	1459
47	27	20	7	967	947
47	28	33	2	1579	1553
47	30	7	2	359	331
47	32	13	2	643	613
47	33	2	3	127	97
47	34	75	2	3559	3527
47	35	4	3	223	191
47	36	13	2	647	613
47	38	7	2	367	331
47	39	4	3	227	191
47	40	51	2	2437	2399
47	42	13	2	653	613
47	44	7	2	373	331
47	45	2	3	139	97
47	46	33	2	1597	1553
48	1	2	5	97	101
48	25	1	5	73	53

q	N	m	π	$qm+N$	$qm+\pi$
48	35	1	5	83	53
49	1	4	3	197	199
49	4	3	2	151	149
49	6	29	2	1427	1423
49	8	9	2	449	443
49	9	2	3	107	101
49	10	3	2	157	149
49	12	29	2	1433	1423
49	15	2	3	113	101
49	16	3	2	163	149
49	18	11	2	557	541
49	20	3	2	167	149
49	22	9	2	463	443
49	24	11	2	563	541
49	25	16	3	809	787
49	26	3	2	173	149
49	27	4	3	223	199
49	30	11	2	569	541
49	32	3	2	179	149
49	33	2	3	131	101
49	34	3	2	181	149
49	36	23	2	1163	1129
49	38	9	2	479	443
49	39	2	3	137	101
49	40	21	2	1069	1031
49	44	3	2	191	149
49	45	4	3	241	199
49	46	3	2	193	149
49	48	11	2	587	541
50	1	2	3	101	103
50	9	1	3	59	53
50	21	1	3	71	53
50	27	2	3	127	103
50	33	1	3	83	53
50	39	1	3	89	53
50	49	2	3	149	103

By using these tables, it follows that $g(N) = 1$ for every $q \in \{4, 5, \cdots, 50\}$ and for every $N \in J_q$. The proof of Theorem 1 is completed. ■

The corollaries are direct consequences of Theorem 1.

Final remark. By using the computer, in same way as above, one can prove that Theorem 1 holds for every $q \in \{2, \cdots, 591\}$.

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