

# HOMOGENEOUS SYMMETRIC POLYNOMIAL-LIKE BOOLEAN FUNCTIONS

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*Dedicated to the 70th birthday of Professor Antal Járαι*

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**Abstract.** Polynomial-like Boolean functions form a class of the Boolean functions invariant with respect to a special transform of the linear space of the two-valued logical functions. Another special set of the Boolean functions are the set of the symmetric functions. In an earlier article we introduced the class of the symmetric polynomial-like Boolean functions and investigated some elementary properties of such functions. In the present article we deal with the special case of the homogeneous symmetric polynomial-like Boolean functions.

In this article disjunction and logical sum, conjunction and logical product, exclusive or and modulo two sum, as well as complementation and negation are used in the same sense and they are denoted respectively by  $\vee$ ,  $\wedge$ ,  $\oplus$  and  $\bar{\phantom{x}}$ . The elements of the field with two elements and the elements of the Boolean algebra with two elements are denoted by the same signs, namely by 0 and 1;  $\mathbb{N}$  denotes the non-negative integers, and  $\mathbb{N}^+$  the positive ones.

## 1. Introduction

Logical functions and especially the two-valued ones have important role in our everyday life, so it is easy to understand why they are widely investigated. A scope of the investigations is the representations of these functions and the transforms from one representation to another ([3], [4], [5]). Another area of

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the examinations is the search of special classes of the set of these functions. Post determined the closed classes of the switching functions [11], but there are a lot of another classes of the Boolean functions invariant with respect to some property. Such properties can be for example linear transforms. In [6] the author of the present paper introduced a class of the Boolean functions invariant under a special linear transform. The functions of that class are called polynomial-like Boolean functions.

### 1.1. Representations of a Boolean function

It is well-known that an arbitrary two-valued logical function of  $n$  variables can be written in the uniquely determined canonical disjunctive normal form, i.e. as a logical sum whose members are pairwise distinct logical products of  $n$  factors, where each of such logical products contains every logical variable exactly once, either negated or not negated exclusively. Clearly, there exist exactly  $2^n$  such products. Supposing that the variables are indexed by the integers  $0 \leq j < n$  and the variable indexed by  $j$  is denoted by  $x_j$ , these products can be numbered by the numbers  $0 \leq i < 2^n$  in such a way that we consider the non-negative integer containing 0 in the  $j$ -th position of its binary expansion if the  $j$ -th variable of the given product is negated, and 1 in the other case. Of course, this is a one to one correspondence between the  $2^n$  distinct products and the integers of the interval  $[0..2^n - 1]$ , and if  $i = \sum_{j=0}^{n-1} a_j^{(i)} 2^j$ , where  $a_j^{(i)}$  is either 0 or 1, then the product corresponding to it is

$$(1.1) \quad m_i^{(n)} = \bigwedge_{j=0}^{n-1} x_j^{(a_j^{(i)})},$$

where  $x^{(0)} = \bar{x} = \bar{0} \oplus x$  and  $x^{(1)} = x = \bar{1} \oplus x$ . Such a product is called *minterm* (of  $n$  variables).

With the numbering given above we numbered the Boolean functions of  $n$  variables, too. A Boolean function is uniquely determined by the minterms contained in its canonical disjunctive normal form, so a Boolean function is uniquely determined by a  $2^n$  long sequence of 0-s and 1-s, where a 0 in the  $j$ -th position (now  $0 \leq j < 2^n$ ) means that  $m_j^{(n)}$  doesn't occur in that function, and 1 means that the canonical disjunctive normal form of the function contains the minterm of the index  $j$  (this sequence is the spectrum of the canonical disjunctive normal form of the function, and similarly will be defined the spectra with respect to other representations of the function), i.e. for  $l = \sum_{i=0}^{2^n-1} \alpha_i^{(l)} 2^i$  with  $\alpha_i^{(l)} \in \{0, 1\}$

$$(1.2) \quad f_l^{(n)} = \bigvee_{i=0}^{2^n-1} \left( \alpha_i^{(l)} \wedge m_i^{(n)} \right).$$

Now  $f_l^{(n)}$  denotes the  $l$ -th Boolean function of  $n$  variables.

Another possibility for giving a Boolean function is the so-called Zhegalkin-polynomial. Let  $S_i^{(n)} = \bigwedge_{j=0}^{n-1} x_j^{a_j^{(i)}}$ , where  $x^0 = 1 = \bar{0} \vee x$ ,  $x^1 = x = \bar{1} \vee x$  and  $i = \sum_{j=0}^{n-1} a_j^{(i)} 2^j$  again. This product contains only non-negated variables, and the  $j$ -th variable is contained in it if and only if the  $j$ -th digit is 1 in the binary expansion of  $i$ . There exist exactly  $2^n$  such products which are pairwise distinct. Now any Boolean function of  $n$  variables can be written as a modulo two sum of such terms, and the members occurring in the sum are uniquely determined by the function. That means that we can give the function by a  $2^n$ -long 0 - 1 sequence, and if the  $i$ -th member of such a sequence is  $k_i$  then

$$(1.3) \quad f^{(n)} = \bigoplus_{i=0}^{2^n-1} \left( k_i \wedge S_i^{(n)} \right).$$

But this polynomial can be considered as a polynomial over the field of two elements, and in this case we write the polynomial in the following form:

$$(1.4) \quad f^{(n)} = \sum_{i=0}^{2^n-1} k_i S_i^{(n)}.$$

where now  $S_i^{(n)} = \prod_{j=0}^{n-1} x_j^{a_j^{(i)}}$ , and the sum, the product and the exponentiation are the operations of the field.

Between the first and the second representation of the same Boolean function there is a very simple linear algebraic transform. Considering the coefficients of the canonical disjunctive normal form of a Boolean function of  $n$  variables and the coefficients of the Zhegalkin polynomial of a function of  $n$  variables, respectively, as the components of an element of a  $2^n$ -dimensional linear space over the field of two elements, denoted by  $\mathbb{F}_2$ , the relation between the vectors belonging to the two representations of the same Boolean function of  $n$  variables can be given by  $\underline{k} = \mathbf{A}^{(n)} \underline{\alpha}$ . Here  $\underline{k}$  is the vector containing the components of the Zhegalkin polynomial,  $\underline{\alpha}$  is the vector, composed of the coefficients of the disjunctive representation of the given function, and  $\mathbf{A}^{(n)}$  is the matrix of the transform in the natural basis.

For the matrix of the transform it is true that

$$(1.5) \quad \mathbf{A}^{(n)} = \begin{cases} (1) & \text{if } n = 0 \\ \left( \begin{array}{cc} \mathbf{A}^{(n-1)} & \mathbf{0}^{(n-1)} \\ \mathbf{A}^{(n-1)} & \mathbf{A}^{(n-1)} \end{array} \right) & \text{if } n \in \mathbb{N}^+ \end{cases}$$

(this form of the matrix shows that for every  $n \in \mathbb{N}$ ,  $\mathbf{A}^{(n)}$  is the  $n$ -th power of the two-order  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  regular quadratic matrix, if the operation is the Kronecker-product).

From the previous results immediately follows that

$$(1.6) \quad \begin{aligned} \left(\mathbf{A}^{(n+1)}\right)^2 &= \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} = \\ &= \begin{pmatrix} \left(\mathbf{A}^{(n)}\right)^2 & \mathbf{0}^{(n)} \\ \mathbf{0}^{(n)} & \left(\mathbf{A}^{(n)}\right)^2 \end{pmatrix} \end{aligned}$$

and as  $\left(\mathbf{A}^{(0)}\right)^2 = (1)$ , so we get by induction that

$$(1.7) \quad \left(\mathbf{A}^{(n+1)}\right)^2 = \mathbf{I}^{(n+1)},$$

where  $\mathbf{I}^{(n)}$  denotes the  $n$ -order identity matrix.

## 1.2. Polynomial-like Boolean functions

Let us consider again the transform between the canonical disjunctive normal form and the Zhegalkin polynomial of the same function. If  $\underline{\alpha}$  is the spectrum of the canonical disjunctive normal form of the function, and  $\underline{k}$  is the spectrum of the Zhegalkin polynomial of the function, then  $\underline{k} = \mathbf{A}^{(n)}\underline{\alpha}$ . In the special case when  $\underline{\alpha} = \underline{k}$ , the corresponding function is a *polynomial-like Boolean function* [6]. As  $\mathbf{A}^{(0)} = (1)$ , so each of the two zero variable Boolean functions is polynomial-like. Now let  $\underline{u} = \underline{u}_0\underline{u}_1$  be the spectrum of the canonical disjunctive normal form of a Boolean function  $f$  of  $n + 1$  variables, where  $n$  is a nonnegative integer. Then

$$(1.8) \quad \begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} \begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \end{pmatrix}$$

if and only if  $\underline{u}_0 = \mathbf{A}^{(n)}\underline{u}_0$  and  $\underline{u}_1 = \mathbf{A}^{(n)}\underline{u}_0 + \mathbf{A}^{(n)}\underline{u}_1 = \underline{u}_0 + \mathbf{A}^{(n)}\underline{u}_1$ , that is  $f$  is polynomial-like if and only if  $\underline{u}_0 = \left(\mathbf{A}^{(n)} + \mathbf{I}^{(n)}\right)\underline{u}_1$ , where  $\underline{u}_1$  is the spectrum of the canonical disjunctive normal form of an arbitrary Boolean function of  $n$  variables. As a consequence we get that the number of the  $n + 1$  variable polynomial-like Boolean functions is equal to  $2^{2^n}$ . It is easy to see, too, that the spectra of the canonical disjunctive normal forms of the polynomial-like Boolean functions of  $n + 1$  variables make up a  $2^n$ -dimensional subspace of the  $2^{n+1}$ -dimensional linear space of the spectra of the canonical disjunctive normal forms of all of the  $n + 1$  variable Boolean functions. This space is spanned by the columns of the following matrix:

$$(1.9) \quad \begin{pmatrix} \mathbf{A}^{(n)} + \mathbf{I}^{(n)} \\ \mathbf{I}^{(n)} \end{pmatrix}.$$

### 1.3. Symmetric functions and symmetric polynomials

**Definition 1.1.** Let  $n \in \mathbb{N}$ , let  $X$  and  $Y$  be sets,  $f : X^n \rightarrow Y$  and  $\pi$  an arbitrary element of the symmetric group  $S_n$ . The function  $f$  is symmetric, if for any  $(u_0, \dots, u_i, \dots, u_{n-1}) \in X^n$

$$(1.10) \quad f(u_0, \dots, u_i, \dots, u_{n-1}) = f(u_{\pi(0)}, \dots, u_{\pi(i)}, \dots, u_{\pi(n-1)}).$$

If  $\mathcal{K}$  is a field, and  $p \in K[x_0, \dots, x_i, \dots, x_{n-1}]$ , then  $p$  is a symmetric polynomial over  $\mathcal{K}$ , if

$$(1.11) \quad p = p \circ (x_{\pi(0)}, \dots, x_{\pi(i)}, \dots, x_{\pi(n-1)}),$$

where  $\circ$  denotes the composition.

**Theorem 1.1.** *The Boolean function  $f$  is symmetric if and only if its Zhegalkin-polynomial is symmetric.*

**Proof.** Let  $n$  be a nonnegative integer and  $p$  a symmetric polynomial in  $n$  indeterminates over the field  $\mathcal{K}$ , furthermore let  $\hat{p}$  be the polynomial function belonging to  $p$ . If  $\pi$  is a permutation of the set  $\{0, \dots, n-1\}$ , then for any element  $u_0 \cdots u_{n-1}$  of the set  $K^n$

$$(1.12) \quad \begin{aligned} \hat{p}(u_0, \dots, u_{n-1}) &= p \circ (u_0, \dots, u_{n-1}) = \\ &= (p \circ (x_0, \dots, x_{n-1})) \circ (u_0, \dots, u_{n-1}) = \\ &= (p \circ (x_{\pi^{-1}(0)}, \dots, x_{\pi^{-1}(n-1)})) \circ (u_0, \dots, u_{n-1}) = \\ &= p \circ ((x_{\pi^{-1}(0)}, \dots, x_{\pi^{-1}(n-1)}) \circ (u_0, \dots, u_{n-1})) = \\ &= p \circ ((x_0, \dots, x_{n-1}) \circ (u_{\pi(0)}, \dots, u_{\pi(n-1)})) = \\ &= p \circ (u_{\pi(0)}, \dots, u_{\pi(n-1)}) = \\ &= \hat{p}(u_{\pi(0)}, \dots, u_{\pi(n-1)}), \end{aligned}$$

that is, if the polynomial  $p$  is symmetric, then so is the polynomial function determined by  $p$ , too.

Now let  $\mathcal{K}$  be a field of  $q$  elements and  $\varphi : K^n \rightarrow K$  a symmetric function. Then there exists one and only one polynomial  $p$  of degree at most  $q-1$  in every indeterminates over that field that  $\hat{p} = \varphi$ , namely

$$(1.13) \quad p = \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi(u_0, \dots, u_{n-1}) \prod_{i=0}^{n-1} (e - (x_i - u_i)^{q-1}).$$

Then

$$\begin{aligned}
 p[x_0, \dots, x_{n-1}] &= \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi(u_0, \dots, u_{n-1}) \prod_{i=0}^{n-1} \left( e - (x_i - u_i)^{q-1} \right) = \\
 &= \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi(u_0, \dots, u_{n-1}) \prod_{i=0}^{n-1} \left( e - (x_{\pi(i)} - u_{\pi(i)})^{q-1} \right) = \\
 (1.14) \quad &= \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi(u_{\pi(0)}, \dots, u_{\pi(n-1)}) \times \\
 &\quad \times \prod_{i=0}^{n-1} \left( e - (x_{\pi(i)} - u_{\pi(i)})^{q-1} \right) = \\
 &= \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi(u_0, \dots, u_{n-1}) \prod_{i=0}^{n-1} \left( e - (x_{\pi(i)} - u_i)^{q-1} \right) = \\
 &= p[x_{\pi(0)}, \dots, x_{\pi(n-1)}], \cdot
 \end{aligned}$$

so also the polynomial  $p$  is symmetric.

Both parts of the above proof obviously apply even when  $\mathcal{K}$  is a field of two elements, and then  $\varphi = \hat{p}$  is a Boolean function and  $p$  is the corresponding Zhegalkin polynomial.  $\blacksquare$

**Remark 1.1.** The polynomial function for a symmetric polynomial is a symmetric function, but the converse is not necessarily true. There are infinitely many polynomials with the same polynomial function over a finite field, most of which are not symmetric even when the corresponding polynomial function is symmetric. For example, the polynomial function for the polynomials  $p^{(1)} = x_0 x_1^2$  and  $p^{(2)} = x_0 x_1$  over the field of two elements is the symmetric function  $\hat{p} = x_0 x_1$ , but  $p^{(1)}$  is not a symmetric polynomial, since  $p^{(1)} = x_0 x_1^2 \neq x_0^2 x_1 = p^{(3)}$ .

It is worth to mention the following fact.

**Theorem 1.2.** *Let  $k$  be a nonnegative integer and  $k \leq n \in \mathbb{N}$ . The  $k$ -degree homogeneous symmetric Zhegalkin-polynomial in  $n$  indeterminates is the  $k$ -degree elementary symmetric polynomial in  $n$  indeterminates over  $\mathbb{F}_2$ .*

**Proof.** A Zhegalkin-polynomial is a polynomial of degree at most 1 in every indeterminates over  $\mathbb{F}_2$ , so every monomial of such a polynomial is a product of some distinct indeterminates of the polynomial. That means that the degree of a term is equal to the number of the indeterminates occurrence in that term. If the polynomial is homogeneous and symmetric, and the degree of one of its terms is  $k$  then the polynomial is the sum of the  $k$ -degree monomials and only

of these monomials. But now each of the  $k$ -degree monomial is a product of  $k$  distinct indeterminates and the polynomial is the sum of every such term, and this is the  $k$ -degree elementary symmetric polynomial in  $n$  indeterminates. ■

By this result we can determine a homogeneous symmetric Zhegalkin-polynomial by fixing the number of the indeterminates and the degree of the polynomial, that is by the ordered pair of  $(n; k)$  where  $n$  is the number of the indeterminates and  $k$  is the degree of the monomials occurring in the polynomial. Similarly, if  $A$  is a set of nonnegative integers not greater than  $n$  then  $(n; A)$  determines a symmetric Zhegalkin-polynomial containing the  $k$ -degree monomial if and only if  $k \in A$ .

Let  $n$  be a nonnegative integer,  $n \geq k \in \mathbb{N}$  and  $A$  is a subset of the nonnegative integers not greater than  $n$ . Then  $p^{(n;k)}$  is the  $k$ -degree homogeneous symmetric Zhegalkin-polynomial in  $n$  indeterminates and  $p^{(n;A)} = \sum_{k \in A} p^{(n;k)}$ .

The condition that  $k \leq n$  is not necessary, if we consider  $p^{(n;k)}$  as the zero-polynomial in the case when  $k$  is not a nonnegative integer not greater than  $n$ .

As  $2a = 0$  for any  $a \in \mathbb{F}_2$ , so  $p^{(n;A_1)} + p^{(n;A_2)} = p^{(n;A_1 \Delta A_2)}$ , where  $\Delta$  denotes the symmetric difference, that is,  $A_1 \Delta A_2 = (A_1 \cap \bar{A}_2) \cup (\bar{A}_1 \cap A_2)$ .

If  $p^{(n;A)}$  is polynomial-like, then the Boolean-function  $f$  belonging to that polynomial is the logical sum of the minterms containing exactly  $n - k$  negated variables, as the spectra of the function and the polynomial are identical.

#### 1.4. Earlier results

In this subsection we summarize the results contained in [8] relevant to the present paper.

By Proposition 7. in [6] if  $f(x_0, \dots, x_{n-1})$  is an  $n$ -variable polynomial-like Boolean function, and  $\pi$  is in  $S_n$ , that is in the symmetric group of  $n$  elements, then  $f(x_{\pi(0)}, \dots, x_{\pi(n-1)})$  is also a polynomial-like Boolean function. As the modulo two sum of polynomial-like Boolean functions is again a polynomial-like Boolean function, so if  $f$  is an  $n$ -variable polynomial-like Boolean function, then

$$(1.15) \quad g(x_0, \dots, x_{n-1}) = \bigoplus_{\pi \in S_n} f(x_{\pi(0)}, \dots, x_{\pi(n-1)})$$

is evidently a symmetric polynomial-like Boolean function. By this result the following theorem is not very surprising.

**Theorem 1.3.** *For every nonnegative integer  $n$  there are symmetric polynomial-like Boolean functions.*

**Proof.** The 0-function of  $n$  variables is polynomial-like and symmetric for every nonnegative integer  $n$ , and this is true for the Boolean function belonging to the monomial of  $p = \prod_{i=0}^{n-1} x_i$  that is for the  $n$ -variable  $f = \bigwedge_{i=0}^{n-1} x_i$  AND-function. ■

**Theorem 1.4.** *For any  $n \in \mathbb{N}$  the spectra of the symmetric polynomial-like Boolean functions of  $n$ -variables form a linear space.*

**Proof.** For a given nonnegative integer  $n$  the sum of symmetric functions of  $n$  variables is a symmetric function, and over the field of two elements this is enough to be a linear space. Similarly, the set of the polynomial-like Boolean functions of  $n$  variables is a linear space, but then the intersection of the two spaces is a linear space, too. ■

**Theorem 1.5.** *If  $n$  is a positive integer, then the negated function of a symmetric polynomial-like Boolean function of  $n$  variables is not a symmetric polynomial-like Boolean function.*

**Proof.** The index of a polynomial-like Boolean function of at least one variable is an even number (see [7], Proposition 3.), so the negated function can not be polynomial-like, as then  $2^{2^n} - 1 - k$  is an odd number. ■

**Corollary 1.1.** *If  $n$  is a positive integer then at most half of the  $n$ -variable symmetric functions is polynomial-like, so the space of the symmetric polynomial-like Boolean functions of  $n$ -variables is a proper subspace of the space of the  $n$ -variable symmetric Boolean functions.*

**Proof.** The complement of a symmetric Boolean function is symmetric, and it is different from the original one. ■

**Theorem 1.6.** *For any  $3 \leq n \in \mathbb{N}$  the collection of the symmetric polynomial-like Boolean functions is a proper subspace of the space of the polynomial-like Boolean functions.*

**Proof.** The number of the  $n$ -variable symmetric Boolean functions is equal to  $2^{n+1}$ , and exactly half of the indices of that functions are even, so for a positive integer  $n$  the number of the symmetric polynomial-like Boolean functions is at most  $2^n$ . For that  $n$  the cardinality of the set of the polynomial-like Boolean functions of  $n$  variables is equal to  $2^{2^{n-1}}$ . If  $n \geq 3$  then  $2^{n-1} > n$ , and then  $2^{2^{n-1}} > 2^n$ , the statement is true. ■

## 2. New results

In the following, unless we say otherwise, polynomial means a polynomial over  $\mathbb{F}_2$  of degree at most one in every indeterminate, and if  $p$  is the Zhegalkin polynomial of a polynomial-like Boolean function then also the polynomial itself is called as polynomial-like.

In this article the zero polynomial has no degree.

The following statement is almost obvious.

**Theorem 2.1.** *Let  $n$  be a nonnegative integer. The degree of the Zhegalkin polynomial of an  $n$ -variable  $f \neq 0$  Boolean function is at most  $n$ .*

**Proof.** The Zhegalkin polynomial of an  $n$ -variable Boolean function is a polynomial in  $n$  indeterminates. The degree of a nonzero polynomial is the maximum of the degrees of its terms. A Zhegalkin polynomial is a polynomial of degree at most one in every indeterminate, so every monomial of such a polynomial is a product of certain distinct indeterminates and then the degree of the monomial is the number of the indeterminates contained by that term, that means not greater than  $n$ . ■

**Theorem 2.2.** *For every nonnegative integer  $n$  the Boolean function belonging to the homogeneous Zhegalkin polynomial of degree  $n$  in  $n$  indeterminates is polynomial-like.*

**Proof.** The homogeneous Zhegalkin polynomial of degree  $n$  in  $n$  indeterminates is a monomial that is the product of the  $n$  indeterminates and this polynomial is the Zhegalkin polynomial of the  $n$ -variable AND-function which is a polynomial-like Boolean function. ■

**Theorem 2.3.** *Let  $k$  be a nonnegative integer and let  $k \leq n \in \mathbb{N}$ . If the Boolean function of a homogeneous symmetric Zhegalkin-polynomial of degree  $k$  in  $n$  indeterminates is not polynomial-like then the Boolean function of a homogeneous symmetric Zhegalkin-polynomial of degree  $k$  in  $n+1$  indeterminates is not polynomial-like, too.*

**Proof.** If the polynomial  $0 \neq p = p^{(0)} + x_n p^{(1)}$  in  $n+1$  indeterminates, where  $p^{(0)}$  and  $p^{(1)}$  are polynomial of the indeterminates  $x_0, \dots, x_{n-1}$ , is

- a polynomial of degree  $n \geq k$ , then  $p^{(0)}$  is a polynomial of degree at most  $k$ , as every term of  $p$  and then every term of  $p^{(0)}$ , too, is of degree maximum  $k$ ;

- homogeneous, then also  $p^{(0)}$  is homogeneous, because  $p^{(0)}$  and  $x_n p^{(1)}$  have no common terms;
- symmetric, then  $p^{(0)}$  is symmetric, too, since for any permutation of the indeterminates  $x_0, \dots, x_{n-1}$  there is no common term in  $p^{(0)}$  and  $x_n p^{(1)}$ ;
- the Zhegalkin polynomial of an  $n + 1$ -variable polynomial-like Boolean function, then  $p^{(0)}$  is the Zhegalkin polynomial of an  $n$ -variable polynomial-like Boolean function. Indeed, if the spectrum of  $p$  is  $\mathbf{u} = \begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{u}^{(1)} \end{pmatrix}$ , where  $\mathbf{u}^{(0)}$  is the vector composed of the first  $2^n$  components of  $\mathbf{u}$ , then  $\mathbf{u}^{(0)}$  and  $\mathbf{u}^{(1)}$  is the spectrum of  $p^{(0)}$  and  $p^{(1)}$ , respectively. If  $p$  is polynomial-like, then from  $\begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{u}^{(1)} \end{pmatrix} = \mathbf{u} = \mathbf{A}^{(n+1)} \mathbf{u} = \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} \begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{u}^{(1)} \end{pmatrix}$  follows that  $\mathbf{u}^{(0)} = \mathbf{A}^{(n)} \mathbf{u}^{(0)}$ , so  $p^{(0)}$  is polynomial-like.

Based on the above properties, if  $p$  is an  $n + 1$  indeterminate  $k$ -degree homogeneous symmetric polynomial which is the Zhegalkin polynomial of an  $n + 1$ -variable polynomial-like Boolean function, then  $p^{(0)}$  is a  $k$ -degree homogeneous symmetric polynomial in  $n$  indeterminates belonging to an  $n$ -variable polynomial-like Boolean function. From this follows that the theorem holds. ■

**Corollary 2.1.** *If for an integer  $n$  not less than the nonnegative integer  $k$  the Boolean function belonging to a  $k$ -degree homogeneous symmetric Zhegalkin-polynomial in  $n$  indeterminates is not polynomial-like and  $n \leq m$  is an integer then the Boolean function belonging to a  $k$ -degree homogeneous symmetric Zhegalkin-polynomial in  $m$  indeterminates is not polynomial-like, too.*

**Proof.** It follows immediately from the previous theorem by induction on  $m$ . ■

**Definition 2.1.** Let  $n \in \mathbb{N}$  and  $2^n > i \in \mathbb{N}$ .  $i$  can be written in a unique way in the form  $i = \sum_{l=0}^{n-1} i_l 2^l$  where for any nonnegative index  $l$  less than  $n$   $i_l \in \{0, 1\}$  and  $i_0 \cdots i_{n-1} = \mathbf{i} \in \{0, 1\}^n$ . Then  $w(\mathbf{i}) = w(i) = \sum_{l=0}^{n-1} i_l$  is the weight of  $i$  and  $\mathbf{i}$  covers the vector  $j_0 \cdots j_{n-1} = \mathbf{j} \in \{0, 1\}^n$  belonging to the nonnegative integer  $2^n > \sum_{l=0}^{n-1} j_l = j$ , if for each of the indices  $l$   $i_l \geq j_l$ . The fact that  $\mathbf{i}$  covers  $\mathbf{j}$  is denoted by  $\mathbf{i} \succcurlyeq \mathbf{j}$ . In the case of  $\mathbf{i} \succcurlyeq \mathbf{j}$  we also say that  $i$  covers  $j$ .

**Remark 2.1.** If  $n \in \mathbb{N}$ ,  $2^n > i \in \mathbb{N}$ ,  $2^n > j \in \mathbb{N}$  and  $\mathbf{i} \succcurlyeq \mathbf{j}$ , then  $i \geq j$  and  $w(i) \geq w(j)$ , since under the given conditions  $i = \sum_{l=0}^{n-1} i_l 2^l \geq \sum_{l=0}^{n-1} j_l 2^l = j$  and  $w(i) = \sum_{l=0}^{n-1} i_l \geq \sum_{l=0}^{n-1} j_l = w(j)$ .

**Theorem 2.4.** *Let  $n$  be a nonnegative integer and let  $n \geq k \in \mathbb{N}$ . Then  $p^{(n;k)}$  is polynomial-like if and only if  $\binom{w}{k}$  is an even number for every integer  $k < w \leq n$ .*

**Proof.**  $p^{(n;k)}$  is polynomial-like if and only if  $\mathbf{u} = \mathbf{A}\mathbf{u}$ , where  $\mathbf{u}$  is the spectrum of the polynomial, that is, if  $u_i = v_i = (\mathbf{A}\mathbf{u})_i = \sum_{j=0}^{2^n-1} a_{i,j}u_j = \sum_{j=0}^i a_{i,j}u_j$  for every nonnegative integer  $i$  less than  $2^n$ . The  $i$ -th component  $u_i$  of  $\mathbf{u}$  belonging to  $p^{(n;k)}$  is equal to 1 if and only if  $w(i) = k$  and  $a_{i,j} = 1$  exactly in the case when  $i$  covers  $j$  (see the Corollary in [5]). By the Remark above  $i \geq j$  if  $\mathbf{i} \succcurlyeq \mathbf{j}$ . Depending from the weight of  $i$  we have to distinguish three cases:

1. if  $w_i = w(i) < k$ , then  $u_i = 0$ , furthermore if  $u_j = 1$  then  $a_{i,j} = 0$ , so in that case  $a_{i,j}u_j = 0$  for every  $0 \leq j \leq i$ , and then  $v_i = \sum_{j=0}^i a_{i,j}u_j = 0 = u_i$ ;

2. if  $w_i = w(i) = k$  then there exists one and only one nonnegative integer  $j$  not greater than  $i$  that  $u_j = 1$  and  $a_{i,j} = 1$ , namely  $j = i$ , so now in the sum  $\sum_{j=0}^i a_{i,j}u_j$  there is exactly one nonzero member,  $a_{i,i}u_i$ , therefore  $v_i = \sum_{j=0}^i a_{i,j}u_j = a_{i,i}u_i = 1 = u_i$ ;

3. finally let  $i$  be such an integer that  $w_i = w(i) > k$ . Then there are  $\binom{w_i}{k}$  such index  $j$ , that  $\mathbf{i} \succcurlyeq \mathbf{j}$  and  $w_j = w(j) = k$ , and then  $v_i = \sum_{j=0}^i a_{i,j}u_j = \sum_{\mathbf{i} \succcurlyeq \mathbf{j}} u_j = \sum_{\mathbf{i} \succcurlyeq \mathbf{j}} 1 = 0 = u_i$  if and only if  $\binom{w_i}{k}$  is an even number. ■

**Corollary 2.2.** *Let  $n$  be a nonnegative integer. Then*

1.  $p^{(n;0)}$  is polynomial-like if and only if  $n = 0$ ;
2.  $p^{(n;n)}$  is polynomial-like for every nonnegative integer  $n$ ;
3.  $p^{(n+1;n)}$  is polynomial-like if and only if  $n$  is an odd number.

**Proof.** 1.  $w_i = w(i) = 0$  if and only if  $i = 0$ . Now there is no such  $i$ , that  $w(i) < 0$ , and if  $n > 0$ , then for instance  $\binom{n}{0} = 1$  is not an even number;

2. the weight of the index of a term of a Zhegalkin-polynomial is equal to the number of the indeterminates occurring in that term so an  $n$ -indeterminate Zhegalkin polynomial has no member with index  $2^n > i \in \mathbb{N}$  such that  $w_i = w(i) > n$ ;

3. in this case it is only true for  $n + 1$  that  $w(i) > n$ , and  $\binom{n+1}{n} = \binom{n+1}{1} = n + 1$  is even exactly in the case when  $n$  is odd. ■

**Remark 2.2.** The first two cases are not new results. On the one hand the  $n$ -variable constant 1-function is polynomial-like if and only if  $n = 0$  (see Proposition 3 and the Remark after Proposition 6 in [7] and  $p^{(n;0)}$  is the Zhegalkin polynomial of that function. On the other hand the second statement says the same as Theorem 2.2 in this article.

**Theorem 2.5.** *Let  $l$  be a positive and  $t$  a nonnegative integer and let  $k = 2^l(2t+1) - 1$ . Then  $p^{(n;k)}$  is polynomial-like, if  $k \leq n < k + 2^l$ , and  $p^{(k+2^l;k)}$  is not polynomial-like.*

**Proof.** By the previous theorem the only thing to prove is that  $\binom{w}{k}$  is an even number if  $k < w \leq n < k + 2^l$  but  $\binom{k+2^l}{k}$  is an odd number.

Let  $2^l \geq r = 2^u(2v+1)$  be a positive integer.  $r$  uniquely determines the nonnegative integers  $u$  and  $v$ .  $2^l \geq 2^u(2v+1)$  and  $2v+1 \geq 1$  imply that  $u \leq l$ , and  $u = l$  is equivalent to the case that  $r = 2^l$ . If  $w = k + r$  then

$$(2.1) \quad \begin{aligned} \binom{w}{k} &= \binom{k+r}{k} = \binom{k+r}{r} = \\ &= \frac{\prod_{i=1}^r (k+i)}{\prod_{i=1}^r i} = \frac{\prod_{i=0}^{r-1} ((k+1)+i)}{\prod_{i=1}^r i} = \frac{k+1}{r} \prod_{i=1}^{r-1} \frac{(k+1)+i}{i}. \end{aligned}$$

Let us write the positive integer  $i$  less than  $r$  in the form  $r = 2^{p_i}(2q_i+1)$ . Now  $l > p_i \in \mathbb{N}$ . Then

$$(2.2) \quad \frac{k+1}{r} = 2^{l-u} \frac{2t+1}{2v+1},$$

$$(2.3) \quad \begin{aligned} \frac{(k+1)+i}{i} &= \frac{2^l(2t+1) + 2^{p_i}(2q_i+1)}{2^{p_i}(2q_i+1)} = \\ &= \frac{2(2^{l-p_i-1}(2t+1) + q_i) + 1}{2q_i+1} = \frac{2w_i+1}{2q_i+1}, \end{aligned}$$

so

$$(2.4) \quad \binom{w}{k} = 2^{l-u} \frac{2t+1}{2v+1} \prod_{i=1}^{r-1} \frac{2w_i+1}{2q_i+1} = 2^{l-u} \frac{(2t+1) \prod_{i=1}^{r-1} (2w_i+1)}{(2v+1) \prod_{i=1}^{r-1} (2q_i+1)}.$$

Both the numerator and the denominator of the fraction are products of odd integers, so both the numerator and the denominator are odd integers. The binomial coefficient is an integer and the denominator of the fraction is relative prime to  $2^{l-u}$ , hence itself the fraction is an integer. From this follows that  $\binom{w}{k}$  is odd for the given  $w$  and  $k$  if and only if  $u = l$ , that is, when  $r = 2^l$ . ■

**Theorem 2.6.** *Let  $p$  be a Zhegalkin polynomial of  $x_0, \dots, x_{n-1}, x_n$  and let  $p = p^{(0)} + x_n p^{(1)}$  where  $p^{(0)}$  and  $p^{(1)}$  are Zhegalkin polynomials of the indeterminates  $x_0, \dots, x_{n-1}$ . Then  $p = p^{(n+1;k)}$  if and only if  $k = 0$ ,  $k = n + 1$ , or  $0 < k \leq n$  and  $p^{(0)} = p^{(n;k)}$  and  $p^{(1)} = p^{(n;k-1)}$ .*

**Proof.** With the cases  $k = 0$  and  $k = n + 1$  we dealt earlier. Let  $0 < k \leq n$  and  $p = p^{(n+1;k)}$ . We know that in this case  $p^{(0)} = p^{(n;k)}$ . Now  $p^{(n+1;k)}$  is the sum of the  $k$ -degree monomials of the  $n + 1$  indeterminates not containing any other terms. These monomials either contain  $x_n$  or not in a mutually exclusive manner. The sum of the latter is the set of the  $k$ -degree monomials of the indeterminates  $x_0, \dots, x_{n-1}$ , and their sum is  $p^{(n;k)}$ . Each of the other monomials is the product of  $x_n$  and one and only one monomial of degree  $k - 1$  of the other indeterminates and all of these monomials are in  $p$ , so their sum is  $x_n p^{(n;k-1)}$ . Conversely, if  $p = p^{(n;k)} + x_n p^{(n;k-1)}$ , then each member of  $p^{(n;k)}$  and  $x_n p^{(n;k-1)}$  is a  $k$ -degree monomial of the  $n + 1$  indeterminates, and each of such monomial is a member of one of the two preceding polynomials, so  $p = p^{(n+1;k)}$ . ■

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