*Annales Univ. Sci. Budapest., Sect. Comp.* 51 (2020) 59–68

# HYPERBOLIC GEOMETRY AND BLASCHKE-FUNCTIONS

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*Dedicated to Professor Antal Járai on his 70th birthday*

Communicated by Sándor Fridli

(Received June 16, 2020; accepted September 10, 2020)

**Abstract.** By the Fermat-principle, assuming a refractive index of $1/(1-|z|^2)$ $(z \in \mathbb{D})$, light traverses along hyperbolic lines. These lines can be described by Blaschke-functions and are also the solutions to a variational problem. In this paper we describe a generalization of the above problem, by taking a refractive index of $(1-|B_{\mathbf{a}}(z)|^2)/|B_{\mathbf{a}}^{'}(z)|$, where $B_{\mathbf{a}}$ denotes a Blaschke-product. We show that the solutions of the generalized problem are inverse images of Blaschke-products who induce an $n$-fold geometry on $\mathbb{D}$. In the special case of $n = 2$, we provide the explicit form of the solutions.

## 1. Introduction

In this work we introduce a generalization of the Poincaré model of Bolyai's geometry and consider its physical interpretations. The geometry of the model is described by Blaschke-functions. Hyperbolic lines (arcs of the disk $\mathbb{D}$ which intercept $\mathbb{T}$ perpendicularly) can be interpreted as the Blaschke-images of the

*Key words and phrases*: Blaschke-functions, Blaschke-products, Fermat-principle, hyperbolic geometry.

*2010 Mathematics Subject Classification*: Primary 30J10, Secondary 26C15, 30J99, 65T99.

EFOP-3.6.3-VEKOP-16-2017-00001: Talent Management in Autonomous Vehicle Control Technologies – The Project is supported by the Hungarian Government and co-financed by the European Social Fund.

https://doi.org/10.71352/ac.51.059

real interval $(-1,1)$. They can also be generated as the solution to the variational problem

$$\min_{\Gamma} \mathcal{J}(\Gamma),\ \mathcal{J}(\Gamma) := \mathcal{J}(\Phi) := \int_{t_1}^{t_2} \frac{|\Phi'(t)|}{1-|\Phi(t)|^2}\, dt, \tag{1.1}$$

where

$$\Gamma := \{\Phi(t) : t_1 \le t \le t_2\},\ \Phi(t_j) = z_j\ (j=1,2)$$

is a parametrization of the simple smooth curve $\Gamma$. The hyperbolic line segment connecting the points $z_1$ and $z_2$ is the solution $\Phi_0$ problem (1.1). As shown in section two, this is a simple consequence of the identity

$$\frac{|B_a'(z)|}{1-|B_a(z)|^2} = \frac{1}{1-|z|^2} \quad (a, z \in \mathbb{D}) \tag{1.2}$$

regarding Blaschke-functions.

According to the Fermat-principle, in a given substance, light traverses along the path which it can cover in the least amount of time. In section two we show that for a refractive index of $1/(1-|z|^2)$ $(z \in \mathbb{D})$, the path of light coincides with the solution to (1.1).

In this paper we propose a generalization of the variatonal problem (1.1) and describe its solutions. Instead of $\dfrac{1}{1-|z|^2}$ $(z \in \mathbb{D})$, let us take functions of the form

$$\frac{|B'(z)|}{1-|B(z)|^2} \quad (z \in \mathbb{D})$$

where

$$B(z) := \prod_{j=0}^{n-1} B_{a_j}(z) \quad (\mathbf{a} = (a_0, \dots, a_{n-1}) \in \mathbb{D}^n, z \in \overline{\mathbb{D}}) \tag{1.3}$$

denotes a Blaschke-product rather than an individual Blaschke-function

In this paper we will consider only the special case $n=2$. We will provide the inverse of (1.3) in an explicit form. Denoting the $j$-th $(j=0,1)$ branch of the inverse function of a Blaschke-product by $B_j^{-1}$, we introduce a geometric structure on the set set $\mathbb{D}^\diamond := \mathrm{B}_0^{-1}(\mathbb{D}) \cup \mathrm{B}_1^{-1}(\mathbb{D})$ (see (4.1)). Let $\mathcal{G}$ be the set of simple smooth curves on $\mathbb{D}$. The functions $B_j^{-1}$ map the curves of $\mathcal{G}$ to the curves in $\mathbb{D}^\diamond$.

In the second section we describe the most important properties of Blaschke-functions and give a simple proof of the minimum property of hyperbolic line segments based on (1.2). This will serve as an example for the general case.

In the third section we provide the inverse images of Blaschke-products for $n = 2$. We show that the functions $B_j^{-1}$ $(j = 0, 1)$ can be interpreted as generalizations of the square root function. We also describe the relationship between (1.1) and the solutions of the generalized variational problems. Moreover, we show that these solutions coincide with the transformed images of hyperbolic line segments.

Finally in the fourth section we provide a proof of the explicit form of inverse Blaschke-products.

## 2. Hyperbolic arc length, the path of light on a hyperbolic plane

In this section we show that for the refractive index

$$\frac{1}{1-|z|^2} \quad (z \in \mathbb{D})$$

a light-beam traverses from point $z_1$ to $z_2$ on a hyperbolic line segment. This well-known statement is a direct consequence of (1.2). As this result serves as the basis of our generalizations, we describe it in detail here.

First we remark some important properties of Blaschke-functions and related theorems. It is well-known (see [5], [6]), that

$$B_{\mathfrak{a}}(z) :=\epsilon B_a(z) := \epsilon\frac{z-a}{1-\overline{a}z}, \;\; \eta_{\mathfrak{a}}(z) := \epsilon\frac{1-z\overline{a}}{1-\overline{z}a}. \quad (\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{B})$$
$$(\mathfrak{a} := (a, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \;\; z \in \overline{\mathbb{D}})$$

Blaschke-functions are bijections on the disk and on the torus. In addition they form a group with respect to function composition:

$$B_{\mathfrak{a}}^{-1} = B_{\mathfrak{a}^-}, \quad B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2},$$
$$\mathfrak{a}^- = (-\epsilon a, \overline{\epsilon}), \;\; a_1 \circ a_2 = (B_{\mathfrak{a}_2^-}(a_1), \epsilon_1 \eta_{\mathfrak{a}_2^-}(a_1)).$$

This group can be identified as the group of congruence transformations in the Poincaré model. The lines in the model consist of circle arcs, and Euclidean lines inside the disk, which are perpendicular to the torus. These can be given by:

$$B_{\mathfrak{a}} : \overline{\mathbb{I}} := [-1, 1] \to \overline{\mathbb{D}} \quad (\mathfrak{a} \in \mathbb{B}).$$

The points $B_{\mathfrak{a}}(1)$, $B_{\mathfrak{a}}(-1) \in \mathbb{T}$ are called ideal points. We denote a hyperbolic line segment connecting the points $z_1$ and $z_2$ by $[z_1, z_2]$. Hyperbolic line segments can be parametrized by Blaschke-functions (see [5], [6]): for any $[z_1, z_2]$ segment there exists an interval $[t_1, t_2] \subset \mathbb{I}$ and $\mathfrak{c} \in \mathbb{B}$ such that

$$[z_1, z_2] = \{B_{\mathfrak{c}}(t) : t_1 \le t \le t_2\}.$$

We note that this parameterization of hyperbolic line segments is not unique, furthermore if $z_1$ and $z_2$ fall on a radius of $\mathbb{D}$, the segment $[z_1, z_2]$ coincides with the Euclidean segment connecting the points.

The mapping

$$\rho_0(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z}_2 z_1|} = |B_{z_2}(z_1)| \quad (z_1, z_2 \in \overline{\mathbb{D}})$$

is a metric on the set $\overline{\mathbb{D}}$. This metric has the following invariant property:

$$\rho_0(B_{\mathfrak{a}}(z_1), B_{\mathfrak{a}}(z_2)) = \rho_0(z_1, z_2) \quad (z_1, z_2 \in \overline{\mathbb{D}}, \mathfrak{a} \in \mathbb{B}).$$

$\rho_0$ is called *pseudo-hyperbolic metric*. In addition to $\rho_0$, it is useful to introduce the *hyperbolic metric* (see [4], [5], [6]):

$$\rho_1(z_1, z_2) := \operatorname{ath}(\rho_0(z_1, z_2)) = \frac{1}{2} \log \frac{1 + \rho_0(z_1, z_2)}{1 - \rho_0(z_1, z_2)} \quad (z_1, z_2 \in \mathbb{D}).$$

The metric $\rho_1$ is additive on hyperbolic lines:

$$t_0 \leq t_1 \leq t_2 \ \Rightarrow \ \rho_1(t_0, t_1) + \rho_1(t_1, t_2) = \rho_1(t_0, t_2),$$

therefore the following holds for the points $z_j := B_{\mathfrak{a}}(t_j)$ $(j = 0, 1, 2)$ of the line $\{B_{\mathfrak{a}}(t) : t \in \mathbb{I}\}$:

$$\rho_1(z_0, z_1) + \rho_1(z_1, z_2) = \rho_1(z_0, z_2).$$

We will make us of the following identities (see [4], [5]):

(2.1)

$$1 - |B_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2} = |B_a'(z)|(1 - |z|^2) \quad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}),$$

$$B_{B_{\mathfrak{a}}(z_1)}(B_{\mathfrak{a}}(z_2)) = \overline{\eta}_{\mathfrak{a}}(z_2) B_{z_1}(z_2), \quad \eta_{B_{\mathfrak{a}}(z_1)}(B_{\mathfrak{a}}(z_2)) = \overline{\eta}_{\mathfrak{a}}(z_1)\overline{\eta}_{\mathfrak{a}}(z_2)\eta_{z_1}(z_2).$$

Let $\Gamma \subset \mathbb{D}$ be a *simple smooth curve* and $\gamma : [t_1, t_2] \to \Gamma$ be a parametrization of $\Gamma$. Then the number

$$\mathcal{J}(\gamma) := \int_{t_1}^{t_2} \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}\, dt$$

is referred to as the the *hyperbolic arc length* of the curve.

We say that the parametrizations $\gamma$ and $\gamma_1$ are equivalent if there exists an $s \in C^1[\tau_1, \tau_2], s : [\tau_1, \tau_2] \to [t_1, t_2]$ bijection for which $\gamma_1 = \gamma \circ s$. The hyperbolic arc length is independent from (equivalent) parametrizations: $\mathcal{J}(\gamma_1) = \mathcal{J}(\gamma)$.

We now show that *hyperbolic arc length is invariant to congruent transformations.* Indeed, if $\gamma_1 = B_{\mathfrak{a}} \circ \gamma$, then by (2.1) we obtain

$$\begin{aligned}\mathcal{J}(\gamma_1) &= \int_{t_1}^{t_2} \frac{|\gamma_1'(t)|}{1-|\gamma_1(t)|^2}\,dt = \int_{t_1}^{t_2} \frac{|B_{\mathfrak{a}}'(\gamma(t))||\gamma'(t)|}{1-|B_{\mathfrak{a}}(\gamma(t))|^2}\,dt = \\ &= \int_{t_1}^{t_2} \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt = \mathcal{J}(\gamma).\end{aligned}$$

Using the above statement we can calculate the hyperbolic arc lengths of hyperbolic line segments. Any hyperbolic line segment connecting the points $z_1$ and $z_2$ can be transformed onto the interval $[0,t]$ by a Blaschke-function, where $t = \rho_0(z_1, z_2)$. The hyperbolic arc length of the transformed segment is given as

$$\int_0^t \frac{d\tau}{1-\tau^2} = \operatorname{ath}(t) = \operatorname{ath}(\rho_0(z_1,z_2)) = \rho_1(z_1,z_2). \tag{2.2}$$

By (2.2) we showed that *the arc length of a hyperbolic line segment equals the* $\rho_1$ *distance between its endpoints.*

Similarly it can be shown that *among the curves connecting two points, the hyperbolic line segment has the minimal hyperbolic arc length.* Indeed, let $\phi : [t_1, t_2] \to \mathbb{D}$ be a parametrization of the smooth curve connecting the points $z_j = \phi(t_j)$ $(j = 1, 2)$. Denote by $B_{\mathfrak{c}}$ a transformation for which $B_{\mathfrak{c}}(z_1) = 0$, $B_{\mathfrak{c}}(z_2) = t = \rho_0(z_1, z_2)$ holds. Then, the transformed curve $\phi_1 := B_{\mathfrak{c}} \circ \phi$ has the following properties:

$$\begin{aligned}&\phi_1(t_1) = 0,\ \phi_1(t_2) = t = \rho_0(z_1,z_2),\\ &\mathcal{J}(\phi) = \mathcal{J}(\phi_1) = \int_{t_1}^{t_2} \frac{|\phi_1'(t)|}{1-|\phi_1(t)|^2}\,dt.\end{aligned}$$

Let $F(t) = \operatorname{ath}(|\phi_1(t)|),\ t_1 \le t \le t_2$. Then

$$F'(t) = \frac{\langle \phi_1'(t), \phi_1(t)\rangle / |\phi_1(t)|}{1-|\phi_1(t)|^2}$$

and

$$\mathcal{J}(\phi) = \mathcal{J}(\phi_1) \ge \left| \int_{t_1}^{t_2} F'(t)\ dt \right| = |F(t_2) - F(t_1)| = \operatorname{ath}(t) = \rho_1(z_1,z_2).$$

Hence we showed that *the arc length of any curve connecting two points is not less than the the arc length of the hyperbolic line segment connecting the same points.*

Since the geometric and optical minimum tasks are equivalent, the following statement holds. *Assuming a refractive index of* $1/(1-|z|^2)$ $(z \in \mathbb{D})$, *light traverses along hyperbolic lines.*

## 3. Blaschke-products of two factors

In this section we discuss the inverses of two-factor Blaschke-products:

$$B = B_{a_0}B_{a_1} \quad (a_0,\ a_1 \in \mathbb{D})$$

and consider their physical interpretations. We denote the half-disks by

$$\mathbb{D}^0 := \{z \in \mathbb{D} : \Re z > 0\} \cup \mathbb{J}i,\ \mathbb{J} := \{-t : 0 \le t < 1\}, \quad \mathbb{D}^1 := -\mathbb{D}^0.$$

These half-disks provide a partitioning of $\mathbb{D}$, for which $\mathbb{D}^0 \cap \mathbb{D}^1 = \{0\}$. The function $W \to \sqrt{W}$ will be defined as

$$\mathrm{sqrt}(W) := \sqrt{W} := \rho^{1/2}e^{it/2} \quad (W = \rho e^{it} \in \mathbb{D}, -\pi \le t < \pi).$$

Then the mapping $\mathrm{sqrt} : \mathbb{D} \to \mathbb{D}^0$ is a bijection which is differentiable on $\mathbb{D}$ with the exception of points which fall inside $\mathbb{J}$, where sqrt has a jump of $\pi$.

In section four we show that the inverse branches of the twofold mapping $W = B(w)$ can be given as

$$w_j = B_{\mathfrak{c}}((-1)^j w),\ w := \sqrt{v},\ v = B_{-r^2}(\overline{\kappa}W) \tag{3.1}$$

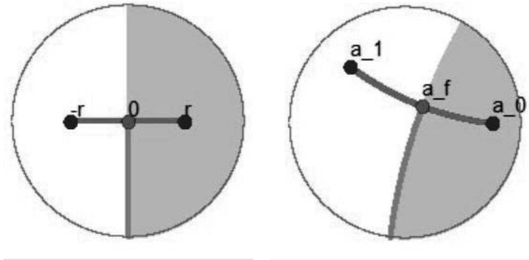
where $\kappa \in \mathbb{T}, \mathfrak{c} \in \mathbb{B}$ and $r = \rho_0(a_0, a_1)/2$. The parameter $\mathfrak{c} \in \mathbb{B}$ is determined by the condition

$$B_{\mathfrak{c}}(r) = a_0,\ B_{\mathfrak{c}}(-r) = a_1.$$

In other words $B_{\mathfrak{c}}$ is a hyperbolic congruence transformation which maps the interval $[-r, r]$ onto the segment $[a_1, a_0]$. It follows that $a_f := B_{\mathfrak{c}}(0)$ is the midpoint of the hyperbolic line segment. By (3.1) we get that $W \to w_0$ is a bijection between the sets $\mathbb{D}$ and $\mathbb{D}^0_{\mathfrak{c}} := B_{\mathfrak{c}}(\mathbb{D}^0)$, moreover $\mathrm{B}(w_j) = W$ $(j = 0, 1)$ holds.

In geometric terms, the first equation of (3.1) means that the point $w_0$ is the hyperbolic reflection of $w_1$ with respect to $a_f = B_{\mathfrak{c}}(0)$. It is therefore sufficient to consider the inverse branch

$$W \to w_0 =: B_0^{-1}(W)\ (W \in \mathbb{D}).$$

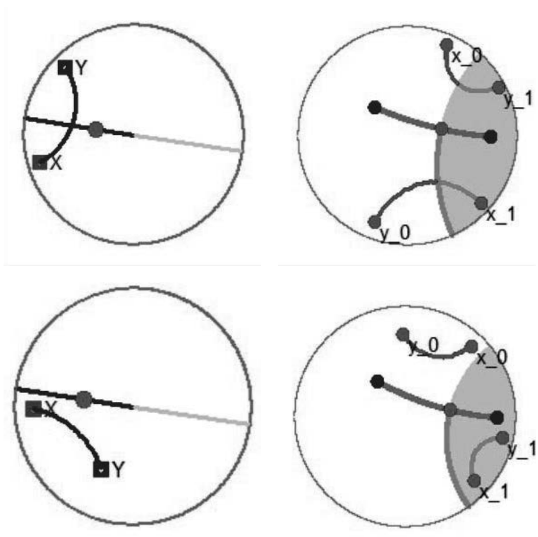



Sets $\mathbb{D}^j$ and $B_j^{-1}$ $(j = 0, 1)$

The mapping $B_0^{-1}$ is differentiable in any points $v \notin \mathbb{J}$. On the other hand, by (3.1), for the singular points of $B_0^{-1}$ we get

$$B_{-r^2}(\overline{\kappa} W) \in \mathbb{J} \;\Leftrightarrow\; W \in \kappa B_{r^2}(\mathbb{J}) = \kappa(-1, -r^2]. \tag{3.2}$$

With (3.2) we showed that the singular points of $B_0^{-1}$ (and thus those of $B_1^{-1}$) consist of the interval $\mathcal{C} = \mathcal{C}_{\mathbf{a}} = \kappa(-1, -r^2]$, which can interpreted as the rotation of the interval $(-1, -r^2]$. Next we determine the the Blaschke-product image of the endpoint of interval $\mathcal{C}$, which can be given as $W_1 := -\kappa r^2$. By (3.1) and $B_{-r^2}(\overline{\kappa} W_1) = 0$ we get $B_0^{-1}(W_1) = B_{\mathfrak{c}}(0) = a_f$, therefore the point $A_f := \mathrm{B}(a_f)$ is an endpoint of the interval $\mathcal{C}$.




*Left:* Inverse Blaschke-product images of the hyperbolic line segments which either avoid (top), or cross (bottom) the segment $[A_f, S]$.
*Right:* Hyperbolic line segments on the disk, and the segment $\mathcal{C}$.

We denote by $\mathcal{G}_0$ and $\mathcal{G}_1$ subsets of the simple smooth curves $\Gamma \subset \mathbb{D}$ which have 0 and 1 intersection points with $\mathcal{C}$ respectively. We note that hyperbolic line segments have this property, or equivalently $[z_1, z_2] \in \mathcal{G} := \mathcal{G}_0 \cup \mathcal{G}_1$. Consider the case $\Gamma \in \mathcal{G}_1, \Gamma \cap \mathcal{C} = \{P\}$. Then we denote the partitions of $\Gamma$ that depend on $P$ by $\Gamma_0, \Gamma_1$. We now introduce the images $\Gamma_j^\diamond$ $(j = 0, 1)$ of any simple smooth curve $\Gamma \in \mathcal{G}$ as

$$\text{(3.3)} \quad \Gamma_j^\diamond := B_j^{-1}(\Gamma) \ (\Gamma \in \mathcal{G}_0), \quad \Gamma_j^\diamond := B_j^{-1}(\Gamma_0) \cup B_{1-j}^{-1}(\Gamma_1) \ (\Gamma \in \mathcal{G}_1), j = 0, 1.$$

The images $\Gamma_j^\diamond$ $(j = 0, 1)$ of the smooth curves $\Gamma \in \mathcal{G}$ are simple smooth curves of $\mathbb{D}^\diamond$. In particular, we call inverse Blaschke-product images of hyperbolic line segments the segments of plane $\mathbb{D}^\diamond$.

We now establish a relationship between the weighted arc lengths

$$\mathcal{J}(\Gamma) := \int_{t_1}^{t_2} \sigma_0(\Phi(t))|\Phi'(t)|\, dt, \quad \mathcal{J}^\diamond(\Gamma_j^\diamond) := \int_{t_1}^{t_2} \sigma(\phi_j(t))|\phi_j'(t)|\, dt,$$

where

$$\sigma_0(z) := \frac{1}{1 - |z|^2}, \ \sigma(z) := \frac{|\mathrm{B}'(z)|}{1 - |\mathrm{B}(z)|^2}$$

and $\Gamma = \{\Phi(t) : t_1 \le t \le t_2\}$, $\Gamma_j^\diamond = \{\phi_j(t) : t_1 \le t \le t_2\}$ are parameterizations of $\Gamma$ and $\Gamma_j^\diamond$, respectively. We proceed to show

$$\text{(3.4)} \qquad \mathcal{J}(\Gamma) = \mathcal{J}^\diamond(\Gamma_j^\diamond) \ \ (\Gamma \in \mathcal{G}, j = 0, 1).$$

Based on (3.3), the following relationship holds between the parameterizations of $\Gamma$ and $\Gamma_j^\diamond$:

$$\Gamma \in \mathcal{G}_0 \ \Rightarrow \ \phi_j(t) = B_j^{-1}(\Phi(t)) \ \ (t_1 \le t \le t_2),$$
$$\Gamma \in \mathcal{G}_1 \ \Rightarrow \ \phi_j(t) = B_j^{-1}(\Phi(t)) \ \ (t_1 \le t \le t^*), \ \phi_j(t) = B_{1-j}^{-1}(\Phi(t)) \ \ (t^* \le t \le t_2),$$

where $\Phi(t^*) \in \mathcal{C}$. Since

$$\frac{|\mathrm{B}'(\phi_j(t))||\phi_j'(t)|}{1 - |\mathrm{B}(\phi_j(t)|^2} = \frac{|\frac{d}{dt}\mathrm{B}(\phi_j(t))|}{1 - |\mathrm{B}(\phi_j(t)|^2},$$
$$\mathrm{B}(B_j^{-1}(\Phi(t))) = \Phi(t), \ \mathrm{B}(B_{1-j}^{-1}(\Phi(t))) = \Phi(t),$$

then (3.4) indeed holds. From this and results from section two, the solutions to the variational problem

$$\min_{\Gamma^\diamond \in \mathcal{G}^\diamond} \mathcal{J}^\diamond(\Gamma^\diamond)$$

are the transformations $[z_1, z_2]^\diamond$ of the hyperbolic line segments $[z_1, z_2]$.

## 4. Explicit form of inverted Blaschke-products

Finally we prove the explicit form of the mapping $W \to w_j$ given in (3.1). In the special case, when the two inverse poles of the Blaschke product are $r$ and $-r$, i.e. $\mathbf{a} = \mathbf{r} = (r, -r)$ $(r \in [0, 1))$

$$B_{\mathbf{r}}(z) = \frac{z-r}{1-rz}\frac{z+r}{1+rz} = B_{r^2}(z^2) \quad (z \in \overline{\mathbb{D}}),$$

therefore the inverse of twofold mapping $W = B_{\mathbf{r}}(w)$ can be formally given as

$$w = B_{\mathbf{r}}^{-1}(W) = (B_{-r^2}(W))^{1/2} \quad (W \in \overline{\mathbb{D}}).$$

Then the two solutions of $B_{\mathbf{r}}(w) = W$ are

$$w_j = v_j, v_j = (-1)^j\sqrt{v}, \; v := B_{-r^2}(W) \quad (W \in \mathbb{D}).$$

We note that the function $v \to \sqrt{v}$ is analytic on $\overline{\mathbb{D}} \setminus [-1, 0]$ and has a jump of $\pi$ on the points of the interval $(-1, 0]$. The inverse of the bijection $B_{-r^2} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is given as $B_{r^2}$, therefore the discontinuity jumps of $w \to \sqrt{B_{-r^2}(w)}$ occur at the points

$$\mathcal{C}_r := B_{r^2}((-1, 0]) = (-1, B_{r^2}(0) = (-1, -r^2].$$

The function $z_0 = \sqrt{v}, \; v = B_{-r^2}(W)$ $(W \in \mathbb{D})$ maps $\mathbb{D}$ onto the half-disk $\mathbb{D}^0$.

In the general case let $B_{\mathfrak{c}}$ $(\mathfrak{c} = (c, \epsilon) \in \mathbb{B})$ be a function that maps the interval $[-r, r]$ onto the hyperbolic line segment connecting the points $a_0$ and $a_1$: $B_{\mathfrak{c}}(-r) = a_0, B_{\mathfrak{c}}(r) = a_1$.

Then,

$$B_{\mathfrak{c}^-}(a_0) = -r, \; B_{\mathfrak{c}^-}(a_1) = r$$

holds for the inverse $\mathfrak{c}^-$ of $\mathfrak{c} \in \mathbb{B}$, furthermore by (2.1) choosing appropriate $\kappa_0, \kappa_1, \kappa = \kappa_{\mathbf{a}} := \kappa_1\kappa_2 \in \mathbb{T}$ numbers yields the equation

$$B_{\mathbf{a}} \circ B_{\mathfrak{c}} = (B_{a_0} \circ B_{\mathfrak{c}})\; (B_{a_1} \circ B_{\mathfrak{c}}) = \kappa_0 B_{-r}\; \kappa_1 B_r = \kappa B_{-r} B_r$$

where taking the function values at 1 we get $\kappa = B_{\mathbf{a}}(B_{\mathfrak{c}}(1))$. Using this and the equation

$$W = B_{\mathbf{a}}(w) = (B_{\mathbf{a}} \circ B_{\mathfrak{c}})(B_{\mathfrak{c}^-}(w)) = \kappa B_{r^2}(B_{\mathfrak{c}^-}^2(w))$$

we can give an explicit formula for $w$ as a two-valued function of $W$:

$$(4.1) \qquad w_j := \mathrm{B}_j^{-1}(W) := B_{\mathfrak{c}}(v_j), \; v_j := (-1)^j\sqrt{v}, \; v = B_{-r^2}(\overline{\kappa} w) \; (j = 0, 1),$$

which proves (3.1).

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