AROUND THE WEAK LIMIT OF ITERATES OF SOME RANDOM–VALUED FUNCTIONS

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a real and separable Banach space X, a linear and continuous $\Lambda : X \to X$ with $\|\Lambda\| < 1$, and an $\mathcal{A}$–measurable and integrable $\xi : \Omega \to X$ with the Fourier transform $\gamma : X^* \to \mathbb{C}$ we characterize the weak limit of *iterates of the random–valued* function $f : X \times \Omega \to X$ given by $f(x, \omega) = \Lambda x + \xi(\omega)$ with the aid of the functional equation

$$\varphi(x^*) = \gamma(x^*)\varphi(x^* \circ \Lambda).$$

Then, making use of this characterization, given a probability Borel measure μ on X we examine continuous at zero solutions $\varphi : X^* \to \mathbb{C}$ of the equation

$$\varphi(x^*) = \hat{\mu}(x^*)\varphi(x^* \circ \Lambda).$$

1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a real and separable Banach space X.

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Given a linear and continuous $\Lambda : X \to X$ and an $\mathcal{A}$–measurable $\xi : \Omega \to X$ consider the function $f : X \times \Omega \to X$ given by

$$f(x,\omega) = \Lambda x + \xi(\omega) \tag{1.1}$$

and its *iterates* defined by (see [7, 1.4])

$$f^0(x,\omega_1,\omega_2,\dots) = x, \quad f^n(x,\omega_1,\omega_2,\dots) = f\left(f^{n-1}(x,\omega_1,\omega_2,\dots),\omega_n\right)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1,\omega_2,\dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. It follows from [6, Corollary 5.6 and Lemma 3.1] (see also [1, Theorem 3.1]) that if $\|\Lambda\| < 1$ and $\xi : \Omega \to X$ is integrable, then the sequence $(f^n(x,\cdot))_{n\in\mathbb{N}}$ of random variables on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ converges in law to a random variable independent of $x \in X$, i.e., for every $x \in X$ the sequence $\left(\pi_n^f(x,\cdot)\right)_{n\in\mathbb{N}}$ of the distributions of $(f^n(x,\cdot))_{n\in\mathbb{N}}$ converges weakly to a probability Borel measure π^f on X; additionally

$$\int_X \|x\|\pi^f(dx) < \infty.$$

In [2] we characterized this limit distribution in the case where X is a real and separable Hilbert space. It turns out that this characterization works also in our case, in fact with the same proof.

2. A characterization of the limit distribution

The following theorem provides a characterization of π^f via the functional equation

$$\varphi(x^*) = \gamma(x^*)\varphi(x^* \circ \Lambda) \tag{2.1}$$

for its Fourier transform $\varphi^f : X^* \to \mathbb{C}$,

$$\varphi^f(x^*) = \int_X e^{ix^*x}\pi^f(dx),$$

where γ stands for the Fourier transform of ξ,

$$\gamma(x^*) = \int_\Omega e^{ix^*\xi(\omega)}P(d\omega) \quad \text{for } x^* \in X^*.$$

Note that any two probability Borel measures on X with the same Fourier transform are equal, see [5, p. 36].

Theorem 2.1. *Assume X is a real and separable Banach space, $\Lambda : X \to X$ is linear and continuous, $\xi : \Omega \to X$ is $\mathcal{A}$–measurable, and $f : X \times \Omega \to X$ is given by (1.1). If*

$$\|\Lambda\| < 1 \quad \text{and} \quad \int_\Omega \|\xi(\omega)\| P(d\omega) < \infty,$$

then:

(i) *the Fourier transform of π^f is the only solution $\varphi : X^* \to \mathbb{C}$ of (2.1) which is continuous at zero and fulfils $\varphi(0) = 1$;*

(ii) *if $\varphi : X^* \to \mathbb{C}$ is a continuous at zero solution of (2.1) and $\varphi(0) = 0$, then $\varphi = 0$.*

Proof. For $n \in \mathbb{N}$ define $\xi_n : \Omega^\infty \to X$ by $\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n)$ and note that ξ_n, $n \in \mathbb{N}$, are identically distributed: Denoting by ρ the distribution of ξ we have

$$P^\infty(\xi_n \in B) = P(\xi \in B) = \rho(B)$$

for $n \in \mathbb{N}$ and Borel $B \subset X$. Since

$$f^n(x, \omega) = \Lambda f^{n-1}(x, \omega) + \xi_n(\omega) \quad \text{for } x \in X,\ \omega \in \Omega^\infty,$$

and the random variables $\Lambda \circ f^{n-1}(x, \cdot)$, ξ_n are independent, we see that

$$\pi_n^f(x, \cdot) = \left(\pi_{n-1}^f(x, \cdot) \circ \Lambda^{-1}\right) * \rho \quad \text{for } n \in \mathbb{N},\ x \in X.$$

Hence, passing to the limit (cf. [8, Ch. III, Th. 1.1]),

$$\pi^f = (\pi^f \circ \Lambda^{-1}) * \rho.$$

Consequently, see also [8, p. 58], for $x^* \in X^*$,

$$\begin{aligned}
\varphi^f(x^*) &= \int_X e^{ix^*x} \left((\pi^f \circ \Lambda^{-1}) * \rho\right)(dx) = \\
&= \int_{X \times X} e^{ix^*(x+z)} \left((\pi^f \circ \Lambda^{-1}) \times \rho\right)(d(x, z)) = \\
&= \int_X \left(\int_X e^{ix^*x} \cdot e^{ix^*z} (\pi^f \circ \Lambda^{-1})(dx) \right) \rho(dz) = \\
&= \left(\int_X e^{ix^*x} (\pi^f \circ \Lambda^{-1})(dx) \right) \left(\int_X e^{ix^*z} \rho(dz) \right) = \\
&= \left(\int_X e^{ix^*\Lambda x} \pi^f(dx) \right) \gamma(x^*) = \varphi^f(x^* \circ \Lambda)\gamma(x^*).
\end{aligned}$$

To prove the uniqueness consider a continuous at zero solution $\varphi : X^* \to \mathbb{C}$ of (2.1). Then

$$\varphi(x^*) = \varphi\left(x^* \circ \Lambda^n\right) \prod_{k=0}^{n-1} \gamma\left(x^* \circ \Lambda^k\right) \quad \text{for } n \in \mathbb{N},\ x^* \in X^*,$$

$\lim_{n\to\infty} \Lambda^n = 0$ and

$$|\gamma(x^* \circ \Lambda^k)| \leq 1 \quad \text{for } k \in \mathbb{N} \cup \{0\} \text{ and } x^* \in X^*.$$

Hence, if $\varphi(0) = 1$, then

$$\varphi(x^*) = \prod_{n=0}^{\infty} \gamma\left(x^* \circ \Lambda^n\right) \quad \text{for } x^* \in X^*,$$

and if $\varphi(0) = 0$, then $\varphi = 0$. ■

Remind that a probability Borel measure μ on a real and separable Banach space X is called *Gaussian* (see [4, 1.3 and C.1], [5, p. 37]) if for every $x^* \in X^*$ the measure $\mu \circ x^{*-1}$ is either a Dirac measure or a Gauss distribution on the real line.

Note that by the Fernique theorem (see [5, Theorem 2.6]) every Gaussian measure has finite moments.

Example 2.2. If X is a real and separable Banach space, $\Lambda : X \to X$ is linear and continuous with $\|\Lambda\| < 1$, $\xi : \Omega \to X$ is $\mathcal{A}$–measurable and Gaussian, and $f : X \times \Omega \to X$ is given by (1.1), then π^f is Gaussian.

Proof. By [4, 1.8] the Fourier transform γ of ξ has the form

$$\gamma(x^*) = e^{iL(x^*) - \frac{1}{2}B(x^*,x^*)} \quad \text{for } x^* \in X^*,$$

where $L : X^* \to \mathbb{R}$ is continuous and linear and $B : X^* \times X^* \to \mathbb{R}$ is continuous, bilinear and symmetric with $B(x^*, x^*) \geq 0$ for $x^* \in X^*$. Since

$$|L(x^* \circ \Lambda^n)| \leq \|L\|\|x^*\|\|\Lambda\|^n \quad \text{for } x^* \in X^*,\ n \in \mathbb{N},$$

$$|B(x^* \circ \Lambda^n, z^* \circ \Lambda^n)| \leq \|B\|\|x^*\|\|z^*\|\|\Lambda\|^{2n} \quad \text{for } x^*, z^* \in X^* \text{ and } n \in \mathbb{N},$$

the formulas

$$l(x^*) = \sum_{n=0}^{\infty} L(x^* \circ \Lambda^n) \quad \text{for } x^* \in X^*,$$

$$b(x^*, z^*) = \sum_{n=0}^{\infty} B(x^* \circ \Lambda^n, z^* \circ \Lambda^n) \quad \text{for } x^*, z^* \in X^*,$$

define a continuous and linear function $l : X^* \to \mathbb{R}$ and a bilinear, symmetric and continuous function $b : X^* \times X^* \to \mathbb{R}$ such that $b(x^*, x^*) \geq 0$ for $x^* \in X^*$. In particular, the function $\varphi : X^* \to \mathbb{R}$ given by

$$\varphi(x^*) = e^{il(x^*) - \frac{1}{2}b(x^*,x^*)}$$

is continuous. It is easy to check that

$$l(x^*) = L(x^*) + l(x^* \circ \Lambda) \quad \text{and} \qquad b(x^*, x^*) = B(x^*, x^*) + b(x^* \circ \Lambda, x^* \circ \Lambda)$$

for $x^* \in X^*$ which shows that φ solves (2.1). By Theorem 2.1(i) and [4, 1.8] we infer that φ is the Fourier transform of π^f and π^f is Gaussian. ■

3. A functional equation

Assuming now that X is a real and separable Banach space, $X \neq \{0\}$, $\Lambda : X \to X$ is linear and continuous with $\|\Lambda\| < 1$, and μ is a probability Borel measure on X, consider, following [3], the equation

$$\varphi(x^*) = \hat{\mu}(x^*)\varphi(x^* \circ \Lambda), \tag{3.1}$$

where $\hat{\mu}$ denotes the Fourier transform of μ,

$$\hat{\mu}(x^*) = \int_X e^{ix^*x}\mu(dx) \quad \text{for } x^* \in X^*.$$

Clearly, $\hat{\mu} : X^* \to \mathbb{C}$ is continuous, and if additionally μ has a finite first moment, i.e., if $\int_X \|x\|\mu(dx)$ is finite, then $\hat{\mu}$ is of class C^1,

$$\hat{\mu}'(x^*)z^* = i\int_X (z^*x)e^{ix^*x}\mu(dx) \quad \text{for } x^*, z^* \in X^*,$$

and

$$|\hat{\mu}(x^*) - \hat{\mu}(z^*)| \leq \left(\int_X \|x\|\mu(dx)\right)\|x^* - z^*\| \quad \text{for } x^*, z^* \in X^*.$$

Theorem 3.1. *If μ has a finite first moment, then there exists a probability Borel measure ν on X with a finite first moment such that $\hat{\nu}$ solves (3.1), and for any continuous at zero solution $\varphi : X^* \to \mathbb{C}$ of (3.1) we have*

$$\varphi = \varphi(0)\hat{\nu};$$

in particular, every continuous at zero solution $\varphi : X^* \to \mathbb{C}$ of (3.1) is of class C^1 and Lipschitz.

We shall prove Theorem 3.1 letter on, together with the next one and with the following remark.

Remark 3.1. If μ has a finite first moment and $\Lambda(X) = X$, then for every $c \in \mathbb{C}$ the set of all discontinuous at zero solutions $\varphi : X^* \to \mathbb{C}$ of (3.1) such that $\varphi(0) = c$ and $\varphi\,|_{X^*\setminus\{0\}}$ is of class C^1 and Lipschitz has the cardinality at least that of the continuum.

Theorem 3.1 implies that for every Borel and integrable with respect to μ function $\xi : X \to X$ the equation

$$\varphi(x^*) = \varphi(x^* \circ \Lambda) \int_X e^{ix^*\xi(x)} \mu(dx) \tag{3.2}$$

has exactly one continuous at zero solution $\varphi^\xi : X^* \to \mathbb{C}$ such that $\varphi^\xi(0) = 1$, and it is of class C^1 and Lipschitz. Consequently, we have the operator $\xi \mapsto$ $\mapsto \varphi^\xi$, $\xi \in L^1(\mu, X)$, and a kind of its continuity gives the following theorem.

Theorem 3.2. *If $\xi, \eta : X \to X$ are Borel and integrable with respect to μ, then*

$$\left|\varphi^\xi(x^*) - \varphi^\eta(x^*)\right| \le \frac{\|x^*\|}{1 - \|\Lambda\|} \int_X \|\xi(x) - \eta(x)\| \mu(dx) \quad \textit{for } x^* \in X^*.$$

Proof. Consider the σ–algebra $\mathcal{B}$ of all Borel subsets of X, the probability space $(X, \mathcal{B}, \mu)$ and, given Borel $\xi : X \to X$ integrable with respect to μ, the function $f : X \times X \to X$ defined by (1.1), as well as the limit distribution π^f. Put $\pi_\xi = \pi^f$. According to Theorem 2.1(i), $\hat{\pi_\xi}$ solves (3.2). Since the first moment of π_ξ is finite, $\hat{\pi_\xi}$ is of class C^1 and Lipschitz.

Putting $\nu = \pi_{id_X}$ and applying Theorem 2.1 we get Theorem 3.1.

To prove Theorem 3.2 it is enough to observe that since $\varphi^\xi = \hat{\pi_\xi}$, $\varphi^\eta = \hat{\pi_\eta}$ and

$$|e^{ix^*x_1} - e^{ix^*x_2}| \le \|x^*\|\|x_1 - x_2\| \quad \text{for } x^* \in X^* \text{ and } x_1, x_2 \in X,$$

by [3, Theorem 1] for every $x^* \in X^*$ we have

$$\begin{aligned}\left|\varphi^\xi(x^*) - \varphi^\eta(x^*)\right| &= \left|\int_X e^{ix^*x}\pi_\xi(dx) - \int_X e^{ix^*x}\pi_\eta(dx)\right| \le \\ &\le \frac{\|x^*\|}{1 - \|\Lambda\|} \int_X \|\xi(x) - \eta(x)\|\, \mu(dx).\end{aligned}$$

To verify the Remark given $c \in \mathbb{C}$ for every $a \in \mathbb{C} \setminus \{c\}$ define $\varphi_a : X^* \to \mathbb{C}$ by

$$\varphi_a(x^*) = a\hat{\nu}(x^*) \quad \text{for } x^* \in X^* \setminus \{0\}, \quad \varphi_a(0) = c,$$

and note that it solves (3.1), it is discontinuous at zero and $\varphi_a\,|_{X^* \setminus \{0\}}$ is of class C^1 and Lipschitz. ■

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